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Inelastic scattering and pair breaking in anisotropic and isotropic superconductors

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The pair-breaking effect of inelastic scattering of electrons off boson fluctuations is examined for anisotropic and isotropic superconductors. We show that the ratio g of the couplings of the boson fluctuations to the pairing and normal electron self-energies is an important parameter. Phonon-mediated s-wave superconductivity corresponds to a value of $g=1$, and spin-fluctuation-mediated d-wave superconductivity to $g<1$. For $g<1$, there is a critical frequency $\omega_c \sim T_c e^{1/g}$ in the boson spectrum: bosons at $\omega<\omega_c$ are pair breaking. We give an approximate expression for the pair-breaking effect of low-lying bosons. We also study the dependence of T_c upon g in an Einstein model in which the boson spectral weight is concentrated at a frequency ω_E. We show that for fixed electron-boson coupling, T_c/ω_E decreases rapidly as g decreases from 1, and for $g\neq 1$ T_c/ω_E saturates at a low value as the coupling tends to infinity. Applications to heavy-fermion superconductors and to the new high-T_c materials are discussed.

I. INTRODUCTION

The recent interest in superconductivity in the heavy fermion compounds[1] and the new "high-T_c" oxide superconductors[2] has reopened the question of the interplay between normal state self-energy effects and the superconducting gap equation, and in particular of the possible pair-breaking effects of inelastic scattering.

It now appears that the superconductivity in the heavy-fermion materials UPt_3, UBe_{13}, $CeCu_2Si_2$ is spin singlet and anisotropic[3] and is probably mediated by exchange of antiferromagnetic spin fluctuations.[4–6] However, the resistivity near the superconducting transition temperature T_c in $CeCu_2Si_2$ and UBe_{13} is large, temperature dependent, and presumably due to inelastic scattering of conduction electrons off spin fluctuations. For these materials the mean-free path one would estimate from the resistivity at T_c is of the order of coherence length.[7] It is known that elastic scattering from nonmagnetic impurities is pair breaking for anisotropic superconductors.[8] We therefore wish to investigate the extent to which inelastic scattering involving the bosons which cause pairing is pair breaking in anisotropic superconductors.

The high-T_c oxide superconductors have a resistivity which is large, and increases linearly with temperature for temperatures higher than the superconducting T_c.[2,9] (Note the temperature dependence of the resistivity is quite different from that of the heavy-fermion superconductors.) Lee and Read[10] have pointed out that the temperature-dependent resistivity implies strong inelastic scattering, and they have also asserted that this inelastic scattering would substantially reduce T_c from its value in a hypothetical material in which the inelastic scattering were absent, but would not affect the magnitude of the zero-temperature gap Δ_0. Thus, they expect for the high-T_c materials a value of T_c which is correlated with the magnitude of the temperature-dependent part of the resistivity, and a value of Δ_0/T_c which is much higher than the BCS value.

Lee and Read based their conclusions upon the following line of reasoning (also used elsewhere in the literature[11]): inelastic scattering leads to an imaginary part of the electron self-energy, which increases the lower cutoff in the logarithmic divergence of the Cooper pair propagator and therefore suppresses T_c. This procedure, however, treats self-energy and vertex corrections at an unequal footing and is therefore dangerous. For the case of the *particle-hole* channel in dirty metals it was shown by Castellani *et al.*[12] that proper inclusion of the vertex corrections leads to a cancellation of the cutoff term in the particle-hole propagator. Using a method which does not explicitly use the Cooper pair propagator, it has in fact been shown by Bergmann and Rainer[13] that inelastic scattering due to electron-phonon interactions does not lead to a decrease in T_c for conventional s-wave superconductivity. Low-lying phonons lead to a pairing interaction in addition to inelastic scattering; the first effect always overcomes the second and the effects exactly cancel at zero phonon frequency. This last result can be understood as a consequence of Anderson's theorem[14] because in some respects very low-frequency phonons mimic static lattice distortions. Thus the argument of Lee and Read[10] that inelastic scattering *per se* is pair breaking is not generally correct, and it is of interest to investigate the circumstances under which it might be so.

We will consider the effect on spin singlet superconductivity (both s wave and anisotropic, e.g., "d wave") of low-lying Bose excitations. We use a general form of the familiar Eliashberg equations. We note that to derive these equations one must make approximations which are justified by Migdal's theorem[15] in the electron-phonon problem, but which have not been proven correct in heavy-fermion or high-T_c materials. The Bose excitations will be characterized by a spectral weight $A(\omega)$ which represents the combination of the boson propagator and an electron boson-coupling constant squared. We assume that $A(\omega)$ does not change when the electrons become superconducting. In the case of phonons $A(\omega)$ is the well-known[16,17] function $\alpha^2F(\omega)$. To study spin fluctuations in s-wave superconductors or anisotropic superconductivity it is necessary to introduce a new coupling constant g. This coupling constant characterizes the difference in the boson coupling to the normal and anomalous parts of the electron self-energy. The constant g can be frequency dependent, but for simplicity we will only consider the frequency-independent case in this paper. It would not be difficult to extend our results to the more general case. For the case of phonons in isotropic superconductors the coupling constant g is identically one. In the models of anisotropic superconductors so far considered, $0<g<1$. Spin fluctuations in isotropic superconductors[18] lead to a value of $g=-1$.

The case $g=1$ has been previously studied by Bergmann and Rainer.[13] They showed that no frequency regime in $A(\omega)$ leads to a suppression in T_c. More precisely, increasing $A(\omega)$ in the neighborhood of *any* frequency ω leads to an increase in T_c. However, introducing large spectral weight in $A(\omega)$ at frequencies $\omega<T$ will obviously lead to strong inelastic scattering of electrons of energy $\varepsilon \sim T$ in the normal phase. The analysis in Ref. 13 shows that this increase in inelastic scattering will *not* lead to a decrease in T_c. In this paper we extend this analysis to $g<1$. We find that for $g<1$ there exists a critical frequency $\omega_c \sim T_c e^{1/g}$ such that for $\omega<\omega_c$, an increase in $A(\omega)$ leads to a decrease in T_c. We estimate the amount by which T_c decreases per unit of Bose fluctuation at $\omega<\omega_c$ and show how to infer a pair-breaking parameter from the resistivity at T_c. We argue that a reasonable measure of the low-frequency boson spectral weight is the mean-free path of an electron at the Fermi surface due to electron-boson scattering; we show for d-wave superconductivity with $g<1$ that spin fluctuations at frequencies $\omega<\omega_c$ have essentially the same pair-breaking effect per unit mean-free path as do nonmagnetic or magnetic impurities. Because the mean-free path of electrons at the Fermi surface due to scattering off of low-frequency Bose fluctuations is temperature dependent, the pair-breaking parameter due to a fixed distribution of low-frequency bosons has a different temperature dependence than that due to impurities. If this temperature dependence is taken into account then the suppression of T_c and the change in the temperature dependence of the gap by low-lying spin fluctuations is well described by the standard theory,[19-21] provided the pair breaking is weak in a sense defined more precisely in Sec. V.

We also study the dependence of the superconducting T_c upon the parameter g. For fixed electron-boson coupling and a boson energy scale we find that the superconducting T_c decreases rapidly as g decreases from unity. Surprisingly, for $g<1$ we find that the superconducting T_c saturates at a relatively low value as the electron-boson coupling constant increases.

The outline of this paper is as follows. In Sec. II we give the necessary formalism. In Sec. III we review the results of Bergmann and Rainer[13] for $g=1$ and the results of Ramakrishnan and Varma[18] for spin fluctuations in a conventional superconductor. In Sec. IV we study the Eliashberg equations arising from a simple model for anisotropic singlet superconductivity which has previously appeared in the literature.[4-6]

In Sec. V we present and justify by numerical calculations expressions for the pair-breaking effect on d-wave superconductivity of conventional and magnetic impurities and low-lying boson modes and discuss the application of our results to the heavy-fermion materials UPt_3 and UBe_{13}. There is a brief conclusion and an appendix, in which the numerical methods are outlined.

II. FORMALISM

We begin by establishing notation. The electron Green function is written in standard notation[20]

$$G(k,\omega_n)=\frac{iZ(k,\omega_n)\omega_n+\varepsilon_k\tau_3+W(k,\omega_n)\tau_2\sigma_2}{[Z(k,\omega_n)]^2\omega_n^2+\varepsilon_k^2+[W(k,\omega_n)]^2} . \qquad (2.1)$$

Here $\omega_n=(2n+1)\pi T$ is a Matsubara frequency, ε_k is the electron energy measured from the Fermi surface, $i\omega_n[1-Z(k,\omega_n)]$ is the normal part of the electron self-energy, and $W(k,\omega_n)$ is the anomalous part. The gap function $\Delta(\omega_n)=W(\omega_n)/Z(\omega_n)$, τ_3 and τ_2 are Pauli matrices acting on the particle-hole degrees of freedom, while σ_2 acts on the spin degrees of freedom. In writing Eq. (2.1) we have assumed singlet superconductivity and that the interactions renormalize only the frequency scale and not the momentum scale. Thus we assume Z and W depend only on the direction of k and not on its magnitude.

We represent the electron-boson interactions by the effective interaction Hamiltonian

$$H_i=H_d+H_s , \qquad (2.2)$$

where the density (d) and spin (s) channel interactions are

$$H_d=-\tfrac{1}{2}\sum_{k,\omega_n} n(k,\omega_n)\alpha^2D(k,\omega_n)n(-k,-\omega_n) , \qquad (2.3a)$$

$$H_s=-\tfrac{1}{2}\sum_{\substack{k,\omega_n\\ ij}} S_i(k,\omega_n)I^2\chi_{ij}(k,\omega_n)S_j(-k,-\omega_n) . \qquad (2.3b)$$

Here $D(k,\omega_n)$ is the propagator and α the coupling constant for the bosons such as phonons which couple to electron density; χ_{ij} and I pertain to the spin interaction. We write the electron density n and spin density S in Nambu notation as

$$n(k,\omega_n)=\sum_{p,\Omega_n}\psi^\dagger_{p+k}(\omega_n+\Omega_n)\tau_3\psi_p(\Omega_n)\ , \tag{2.4a}$$

$$S_i(k,\omega_n)=\sum_{p,\Omega_n}\psi^\dagger_{p+k}(\omega_n+\Omega_n)\alpha^i\psi_p(\Omega_n)\ , \tag{2.4b}$$

where the spin matrix α^i is given by[20]

$$\alpha^i=\tfrac{1}{2}(1+\tau_3)\sigma^i-\tfrac{1}{2}(1-\tau_3)(\sigma^i)^T\ . \tag{2.5}$$

Here σ^i is one of the usual Pauli spin matrices and T denotes the transpose.

In the standard superconductivity theory[17] D is the phonon propagator, α^2 is the electron-phonon coupling constant, and $I=0$. In the theories believed to describe heavy-Fermion superconductivity[4–6] χ_{ij} is related to the electron spin susceptibility, I is a phenomenological interaction constant, and $\alpha=0$

We also include nonmagnetic and magnetic impurity scattering in the standard way.[20] We denote the nonmagnetic impurity potential $v_i(p)$ and the magnetic impurity potential $v_s(p)$.

From Eqs. (2.1)–(2.4) one may easily derive the self-consistent one-loop equations for the self-energies,

$$[1-Z(k,\omega_n)]i\omega_n=-\int\frac{d^2S_p}{8\pi^3v_p}\pi T\sum_{\Omega_n}\frac{i\Omega_nZ(p,\Omega_n)}{\sqrt{[\Omega_nZ(p,\Omega_n)]^2+[W(p,\omega_n)]^2}}\times\left\{\alpha^2D(k-p,\omega_n-\Omega_n)+I^2\chi(k-p,\omega_n-\Omega_n)+\frac{\delta_{mn}}{2\pi T}[\,|v_i(k-p)|^2+|v_s(k-p)|^2]\right\}\ , \tag{2.6a}$$

$$W(k,\omega_n)=\int\frac{d^2S_p}{8\pi^3v_p}\pi T\sum_{\Omega_n}\frac{W(p,\Omega_n)}{\sqrt{[\Omega_nZ(p\Omega_n)]^2+W(p,\Omega_n)^2}}\times\left[\alpha^2D(k-p,\omega_n-\Omega_n)-I^2\chi(k-p,\omega_n-\Omega_n)+\frac{\delta_{nm}}{2\pi T}[\,|v_i(k-p)|^2-|v_s(k-p)|^2]\right]\ . \tag{2.6b}$$

Here $\chi(q,\omega_n)=\mathrm{Tr}\chi_{ij}(q,\omega_n)$. The p integral is over the Fermi surface, $v_p=\partial\varepsilon_p/\partial p$ is the unrenormalized velocity.

If one assumes s-wave superconductivity (so W is isotropic) and $I=0$ one recovers the familiar Eliashberg equations. If one assumes d-wave superconductivity and $\alpha=0$ one obtains equations previously considered in the context of heavy-fermion superconductivity.[4–6] If one assumes $\alpha\neq0$, $I\neq0$ but W is isotropic and takes the limit $W\to0$ one obtained the equations previously considered by Ramakrishnan and Varma[18] in their study of T_c reduction by inelastic spin-flip scattering in conventional superconductors.

To simplify the subsequent analysis without affecting the essential conclusions we assume that D is independent of momentum while χ has the separable form[22]

$$\chi(q,\omega_n)=\chi_0(q)\Phi(\omega_n)\ . \tag{2.7}$$

Further, it will be convenient to write D and Φ in spectral representation,

$$N_0\alpha^2D(\omega_n)=\frac{2}{\pi}\int_0^\infty\frac{\omega A_1(\omega)d\omega}{\omega_n^2+\omega^2}\ , \tag{2.8a}$$

$$N_0I^2\Phi(\omega_n)=\frac{2}{\pi}\int_0^\infty\frac{\omega A_2(\omega)d\omega}{\omega_n^2+\omega^2}\ . \tag{2.8b}$$

$N_0=\int(dS_p/8\pi^3v_p)$, is the electron density of states.

To conclude this section we show that for $T>T_c$. Eq. (2.6a) gives the standard expression[23] for the scattering rate of electrons scattering off bosons. The rate is given by the imaginary part of the self-energy. By combining (2.6a) (2.7), and (2.8) and neglecting the impurity contributions v_i and v_s we easily find

$$\tau^{-1}(\omega,T)=-2\,\mathrm{Im}\Sigma(\omega,T)=\pi\int_0^\infty\frac{2d\omega'}{\pi}[A_1(\omega')+A_2(\omega')]\left[\coth\frac{\omega'}{2T}-\tfrac{1}{2}\tanh\frac{\omega'+\omega}{2T}-\tfrac{1}{2}\tanh\frac{\omega'-\omega}{2T}\right]\ . \tag{2.9}$$

Suppose $A_1(\omega)+A_2(\omega)=0$ for $\omega>\omega_{\max}$. Then for $T>\omega_{\max}$, $\tau^{-1}(\omega,T)\sim T$ while for $\omega>\omega_{\max}$ and $T<\omega_{\max}$, $\tau^{-1}\sim$const. This frequency and temperature scattering rate will lead to a temperature-dependent resistivity. But if $A_1(\omega)+A_2(\omega)=0$ for $\omega<\omega_{\min}$, then for $T,\omega<\omega_{\min}$, $\tau^{-1}=0$. Low-lying (compared to T) boson spectral weight therefore gives rise to electron-boson scattering. Note that the frequency dependence of τ^{-1} is always small.

By substituting this form for τ^{-1} into standard expressions[24] one finds that for temperatures $T\gtrsim\omega_{\max}/4$,

$$\tau^{-1}(T)=2\pi\lambda T , \tag{2.10}$$

where

$$\lambda=2\int_0^\infty \frac{d\omega}{\pi}\frac{A_1(\omega)+A_2(\omega)}{\omega} . \tag{2.11}$$

However, for $T=0$ and $\omega>\omega_{\max}$,

$$\tau^{-1}(\omega)=2\pi\int_0^\infty \frac{d\omega}{\pi}A_1(\omega)+A_2(\omega) . \tag{2.12}$$

In the rest of the paper we shall assume λ is an appropriate measure of the magnitude of the lifetime of an electron at the Fermi surface, and also of the contribution of electron-boson scattering to the resistivity. Indeed, if static impurity scattering and umklapp scattering are important, one may write the resistivity ρ as

$$\rho^{-1}=\frac{Ae^2S_F l}{12\pi^3\hbar} , \tag{2.13}$$

where S_F is the area of the Fermi surface ($S_F=4\pi k_F^2$ for a spherical Fermi surface). A is a numerical factor of order unity. It is, e.g., equal to one in the case of a spherical Fermi surface and scattering from pointlike impurities. l, the mean-free path of an electron at the Fermi surface, is given by

$$l=v_p\tau(T) . \tag{2.14}$$

Note that l, τ, and v_p are all "renormalized" quantities which do not depend on the renormalization factor Z. This follows from our convention for the Green function, Eq. (2.1).

III. ISOTROPIC SUPERCONDUCTIVITY

In this section we use arguments devised by Bergmann and Rainer[13] to study the effect on T_c of various functional forms of $A_1(\omega)$ and $A_2(\omega)$. We begin with the s-wave case. Beginning from Eq. (2.6), assuming W is isotropic, linearizing in W, and rearranging yields

$$\omega_n Z(\omega_n)=\omega_n+\pi T\sum_{\Omega_n}\mathrm{sgn}(\Omega_n)\int_0^\infty \frac{2d\omega'}{\pi}\frac{\omega'[A_1(\omega')+A_2(\omega')]}{(\omega_n-\Omega_n)^2+\omega'^2}+\frac{\mathrm{sgn}\omega_n}{2\tau_i}+\frac{\mathrm{sgn}\omega_n}{2\tau_s} , \tag{3.1}$$

$$W(\omega_n)=\pi T\sum_{\Omega_m}\frac{W(\Omega_m)}{|\Omega_m|Z(\Omega_m)}\left[\int_0^\infty \frac{2d\omega'}{\pi}\frac{\omega'[A_1(\omega')-A_2(\omega')]}{(\omega_n-\Omega_n)^2+\omega'^2}+\frac{\delta_{nm}}{2\pi T\tau_i}-\frac{\delta_{nm}}{2\pi T\tau_s}\right] . \tag{3.2}$$

Here, as usual, the impurity scattering rates are given by

$$\tau_{i(s)}^{-1}=n_{i(s)}\int \frac{d^2S_p}{8\pi^3 v_F}|v_{i(s)}(p)|^2 .$$

The superconducting order parameter $\Delta(\omega_n)=W(\omega_n)/Z(\omega_n)$.

Bergmann and Rainer have shown that analysis of this system can be considerably simplified if (3.2) is cast in the form of an eigenvalue equation for the eigenvector $\Delta(\omega_n)/|\omega_n|$,

$$\sum_{\Omega_m}\left[K(\omega_n,\Omega_m)-\frac{|\omega_n|Z(\omega_n)}{\pi T}\delta_{nm}\right]\left[\frac{\Delta(\Omega_m)}{|\Omega_m|}\right]=\Phi\left[\frac{\Delta(\omega_n)}{|\omega_n|}\right] , \tag{3.3}$$

where

$$K(\omega_n,\Omega_m)=2\int_0^\infty \frac{d\omega'}{\pi}\frac{\omega'[A_1(\omega')-A_2(\omega')]}{(\omega_n-\Omega_m)^2+\omega'^2}+\frac{\delta_{nm}}{2\pi T\tau_i}-\frac{\delta_{nm}}{2\pi T\tau_s} . \tag{3.4}$$

At high temperatures the eigenvalues of Eq. (3.3) are the negative odd integers. As the temperature is lowered the largest eigenvalue increases and crosses zero at T_c.

From Eqs. (3.1) and (3.3) one sees immediately that nonmagnetic impurity scattering cancels between K and Z while magnetic impurity scattering does not. By transferring the τ_s^{-1} term to the right-hand side of Eq. (3.3) one sees that magnetic impurity scattering makes an additive contribution to Φ. Thus it is clear that for $T<T_c$, the largest eigenvalue Φ corresponds to the amount of pair breaking necessary to reduce the critical temperature to T.

We are interested in changes in T_c arising from changes in the lifetime of electrons at the Fermi surface. This may be calculated from a consideration of the functional derivatives $\omega\delta T_c/\delta A_1(\omega)$, $\omega\delta T_c/\delta A_2(\omega)$. By the chain rule these may be represented in the form, e.g.,

$$\omega\frac{\delta T_c}{\delta A_{1,2}(\omega)}=-\omega\frac{\delta\Phi/\delta A_{1,2}(\omega)|_{T=T_c}}{d\Phi/dT|_{T=T_c}} . \tag{3.5}$$

As Φ is a pair-breaking parameter, $d\Phi/dT$ is negative. Therefore, we may determine whether increasing $A_1(\omega)$ or $A_2(\omega)$ at a certain frequency ω will increase or decrease T_c by determining if the functional derivative $\delta\Phi/\delta A_1(\omega)[\delta\Phi/\delta A_2(\omega)]$ is positive or negative. Apart from an unimportant normalization constant this functional derivative may easily be evaluated to yield

$$\delta\Phi=\pm\omega^2 \sum_{\substack{\omega_n\\ \omega_n>0}} \sum_{\substack{\omega_m\\ \omega_m>0}} \frac{\Delta(\omega_n)\Delta(\omega_m)}{|\omega_n|\;|\omega_m|}\left[\frac{1}{\omega^2+(\omega_n-\omega_m)^2}+\frac{1}{\omega^2+(\omega_n+\omega_m)^2} \mp \delta_{nm}\sum_{\substack{\omega_l\\ \omega_l>0}}\left[\frac{1}{\omega^2+(\omega_n-\omega_l)^2}-\frac{1}{\omega^2+(\omega_n+\omega_l)^2}\right]\right] . \quad (3.6)$$

Here the upper sign pertains to $\delta\Phi/\delta A_1$ and the lower sign to $\delta\Phi/\delta A_2$. Consider first the expression for $\delta\Phi/\delta A_1$. Bergmann and Rainer[13] proved that if all the $\Delta(\omega_n)>0$, and $\Delta(\omega_n)$ decreases with increasing $|\omega_n|$, then $\delta\Phi/\delta A_1>0$. They also proved that Δ has these properties in the pure phonon model where $A_2=0$, and by numerical solution of the equations showed $\delta\Phi/\delta A_1>0$ also when realistic values of the Coulomb pseudopotential μ^* are allowed for. Note that the first term in Eq. (3.6) comes from differentiating the interaction kernel which leads to pairing, while the second term comes from differentiating the self-energy. If all of the $\Delta(\omega_n)>0$, the first term makes a positive contribution to $\delta\Phi/\delta A_1$; this reflects the obvious fact that adding phonon spectral weight in any frequency range increases the pairing interaction. The second term makes a negative contribution to $\delta\Phi/\delta A_1$; this reflects the fact that adding phonon spectral weight changes the electron self-energy (both mass enhancement and scattering rate) in such a way as to decrease T_c. The argument presented in the introduction that inelastic scattering from low-lying bosons suppresses T_c is now seen to be incorrect in general: a low-lying phonon mode will contribute also to the pairing interaction and the net effect is to raise T_c.

It is instructive to examine (3.6) analytically using a simple BCS model in which $A_1(\omega)=(\pi\lambda\omega_D/2)\delta(\omega-\omega_D)$ and $A_2(\omega)=0$. Then, to leading logarithmic order one may write $Z(\omega_n)=(1+\lambda)\ \theta(\omega_D-\omega_n)$, $\Delta(\omega_n)=\Delta_0\theta(\omega_D-\omega_n)$, and $T_c=\omega_D\exp-(1+\lambda)/\lambda$. Turning now to Eq. (3.6) one finds for $\omega\gg\omega_D$

$$\delta\Phi/\delta A_1(\omega)\sim\Delta_0^2\frac{1+\lambda}{\lambda^2}>0 , \quad (3.7)$$

while for $\omega\to 0$ the leading terms in the positive first and negative second terms diverge as δ_{nm}/ω^2. These two divergences cancel, leaving

$$\delta\Phi/\delta A_1(\omega)\sim\omega^2\Delta_0^2\frac{1+\lambda}{\lambda}>0 . \quad (3.8)$$

This cancellation of the $1/\omega^2$ divergences is a special feature of the electron-phonon interaction. It may be understood as a consequence of Anderson's theorem[14] that nonmagnetic impurity scattering does not affect T_c because a very low-frequency phonon is in many ways equivalent to a static lattice distortion.

Now consider $\delta\Phi/\delta A_2(\omega)$. From (3.6) one easily sees that (provided the conditions on Δ specified above are satisfied) $\delta\Phi/\delta A_2<0$. Thus for s-wave superconductivity spin-flip scattering is always pair breaking. The effect on T_c of adding an infinitesimal amount of spin fluctuations is shown in Fig. 1. A more detailed discussion for the case of antiferromagnetically coupled spins was given by Ramakrishnan and Varma.[18] Our results are equivalent to theirs; however, we have normalized our changes in A_1 to changes in λ while they have normalized in such a way that a sum rule is preserved. Our results therefore differ from theirs by a frequency-dependent factor.

IV. ANISOTROPIC SUPERCONDUCTIVITY

In this section we will apply the Bergmann-Rainer analysis of the Eliashberg equations to the case of anisotropic even-parity singlet superconductivity. As pointed out by Emery,[4] Scalapino *et al.*,[5] and Miyake *et al.*[6] the antiferromagnetic correlations in a Fermi liquid near an antiferromagnetic instability can lead to anisotropic singlet superconductivity.

For concreteness we adopt the model of Ref. 6 but add impurity scattering; we thus study Eqs. (2.6)–(2.8) with $D(k,\omega_n)=0$ and

$$I^2\chi_0(q)=J_0-J_1\gamma_q , \quad (4.1a)$$

$$\gamma_q=2(\cos q_x a+\cos q_y a+\cos q_z a) . \quad (4.1b)$$

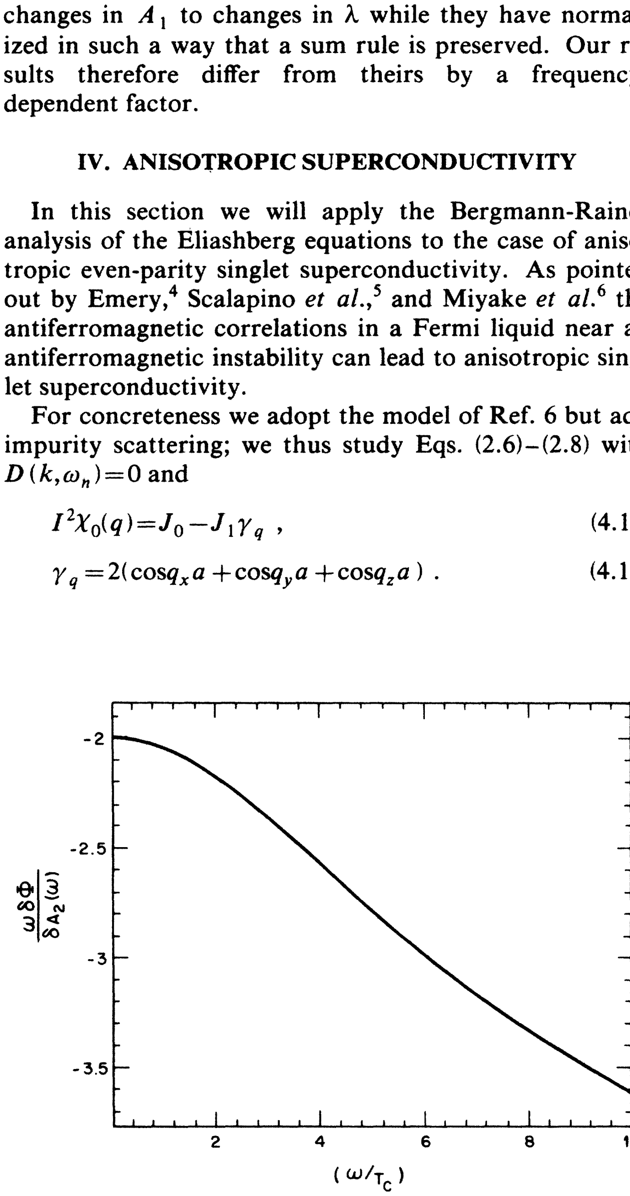

FIG. 1. The functional derivative of the critical temperature T_c for an s-wave superconductor with respect to the spin-fluctuation spectral weight $A_2(\omega)$ plotted against ω/T_c. The derivative has been normalized so that each frequency makes an equal contribution to the scattering rate defined in Eq. (2.10) of text.

When it is necessary to assume a form for the anomalous self-energy $W(p,\omega_n)$, we shall use

$$W(p,\omega_n)=W(\omega_n)\eta(p) \ . \tag{4.2}$$

The function $\eta(p)$ is a basis function for a representation of the crystal symmetry group and gives the angular dependence of the superconducting order parameter. One $\eta(p)$ considered in Ref. 6 is $\eta(p)=\sqrt{6}[\cos(p_x a)-\cos(p_y a)]$.

Combining Eqs. (2.6)–(2.8) and (3.1) and (3.2) and linearizing the Eliashberg equations one finds

$$\omega_n Z(\omega_n)=\omega_n+\pi T\sum_{\Omega_n}\mathrm{sgn}(\Omega_n)\times\int_0^\infty\frac{2d\omega}{\pi}\frac{\omega A(\omega)}{(\omega_n-\Omega_n)^2+\omega^2}+1/2\tau_i \ , \tag{4.3a}$$

$$W(\omega_n)=\pi T\sum_{\Omega_n}\left[\frac{W(\Omega_n)}{|\Omega_n Z(\Omega_n)|}\times g\int_0^\infty\frac{2d\omega}{\pi}\frac{\omega A(\omega)}{(\omega_n-\Omega_n)^2+\omega^2}\right]+\frac{g_I}{2\tau_i}\frac{W(\omega_n)}{|\omega_n Z(\omega_n)|} \ . \tag{4.3b}$$

Here we have introduced two new couplings and have dropped the subscript on $A_2(\omega)$ for convenience.

The coupling constant g is given by

$$g=\frac{\int dS_p dS_k\eta(p)\eta(k)I^2\chi_0(p-k)}{\int dS_p dS_k I^2\chi_0(p-k)} \ . \tag{4.4a}$$

According to our conventions, $\int dS_p\eta(p)^2=1$. If, as is usually assumed in the heavy-fermion problem, the interaction is repulsive for all q (i.e., $I^2\chi_0(q)>0$), then $g<1$. For the model of Ref. 6, $g<0.25$.

The coupling constant g_I is similarly given by

$$g_I=\frac{\int dS_p dS_k\eta(p)\eta(k)|v_i(k-p)|^2}{\int dS_p dS_k|v_i(k-p)|^2} \ . \tag{4.4b}$$

Note $-1<g_I<1$. For isotropic impurity scattering $g_I=0$; in general, one expects $|g_I|\ll 1$. By retracing the standard analysis[20] it is easy to see that the presence of impurity scattering leads to a pair-breaking parameter $\Phi=(1-g_I)/(2\pi T\tau_i)$. Magnetic impurities in an s-wave superconductor would lead to a pair-breaking parameter $\Phi=1/\pi T\tau_s$.

Before discussing the sensitivity of T_c to boson spectral weights at various frequencies, it is of interest to discuss the value of T_c predicted by Eqs. (4.3). For simplicity we assume an Einstein model, in which

$$A(\omega)=(\pi/2)J_0\omega_E\delta(\omega-\omega_E) \ ; \tag{4.5}$$

and we set $\tau_i=\infty$. In the model of Ref. 6 one has $g=J_1/J_0$. In the weak coupling ($J_0\ll 1$) limit, the BCS approximation is presumably valid and one has

$$Z(\omega_n)=1+J_0 \ , \tag{4.6}$$

$$T_c=\omega_E\exp-\frac{1+J_0}{gJ_0} \ . \tag{4.7}$$

From this weak-coupling analysis one infers that, because $g<1$, anisotropic superconductors have intrinsically much lower critical temperatures than s-wave superconductors with the same interaction scale and coupling constant. This inference is confirmed by numerical solution of the Eliashberg equations for various values of g and J_0; some typical results are shown in Table I. Very similar conclusions have been reached by Levin and Valls[25] in a study of the superfluid transition temperature of ^{3}He, which is a p-wave superfluid with (in our notation) $g\lesssim 0.33$.

It is also instructive to consider the strong coupling limit. In this limit the s-wave problem was solved by Allen and Dynes[26] who assumed $\Delta(\omega_n)=\Delta\delta_{\omega_n,\pi T_c}$ and solved the Eliashberg equations variationally. Extending their procedure to the case $g<1$ we find

$$T_c=0.183\omega_E\left[\frac{J_0(2g-1)-1}{1+J_0(1-g)}\right]^{1/2} \ . \tag{4.8}$$

Equation (4.8) is a variational lower bound for T_c.

TABLE I. Superconducting transition temperature, calculated by numerical solution of Eliashberg equations (4.3a) and (4.3b) of text, and compared to two approximate expressions. The Einstein form given in Eq. (4.5) is assumed and ω_E has been set equal to 1. BCS is the weak-coupling value $T_c=\exp-(1+J)/gJ$ and AD is the generalized Allen-Dynes value $T_c=0.183\sqrt{[(2g-1)J-1]/1+(1-g)J}$. β is related to pair breaking and is discussed in Sec. V. The values for $J=10$ and $J=100$ are of mathematical interest only; for these values of J we suspect the Eliashberg equations are not physically relevant.

g	T_c	BCS	AD	β
		$J=1$		
1.0	0.115	0.135		0.61
0.8	0.068	0.082		0.66
0.4	0.005	0.007		0.78
		$J=2$		
1.0	0.212	0.223		0.68
0.8	0.135	0.153		0.60
0.4	0.019	0.024		0.75
		$J=10$		
1.0	0.558	0.333	0.549	1.37
0.8	0.274	0.253	0.183	0.81
0.6	0.150	0.160	0.082	0.61
0.4	0.057	0.064		0.68
		$J=100$		
1.0	1.82	0.364	1.82	1.90
0.8	0.360	0.283	0.31	0.97
0.6	0.185	0.186	0.12	0.67
0.4	0.074	0.080		0.66

This equation has three noteworthy features: (i) for $g < 1$ it saturates,

$$T_c \to 0.183\omega_E \left[\frac{2g-1}{1-g}\right]^{1/2},$$

(ii) for large J_0 and g near 1, it shows a rapid decrease as g moves away from 1, and (iii) Eq. (4.8) predicts at fixed J_0, T_c goes to zero at $g = g_c = (1+J_0)/J_0 > 0.5$. Now Eq. (4.8) is a lower bound; the actual T_c is of course nonzero; however, as g decreases from unity, T_c rapidly becomes more accurately approximated by Eq. (4.7) than by Eq. (4.8). This analysis has been confirmed by numerical solution of the Eliashberg equations; typical results are given in Table I, where the numerical value of T_c is also compared to that predicted by Eqs. (4.7) and (4.8). Note that for the case $g < 0.25$, which, it has been argued, is relevant to heavy-fermion materials, Eq. (4.7) is always appropriate and T_c is limited by $\omega_E \exp(-1/g)$, no matter how large J_0 is.

The techniques discussed in Sec. III may be used to evaluate the sensitivity of T_c to the boson spectral weight. We cast (4.3b) in the form, (3.3), of an eigenvalue equation. The kernal $K(\Omega_n, \omega_n)$ is now given by

$$K(\omega_n, \Omega_n) = 2g \int_0^\infty \frac{d\omega'}{\pi} \frac{\omega' A(\omega')}{(\omega_n - \Omega_n)^2 + \omega'^2} + \frac{g_I \delta_{nm}}{2\pi T \tau_i} . \tag{4.9}$$

From (3.3) and 4.9 we then find

$$\frac{\omega\delta\Phi}{\delta A(\omega)} = 2\omega^2 \sum_{\substack{\omega_n \\ \omega_n > 0}} \sum_{\substack{\omega_m \\ \omega_m > 0}} \frac{\Delta(\omega_n)\Delta(\omega_m)}{|\omega_n|\,|\omega_m|} \left[\frac{g}{\omega^2 + (\omega_n - \omega_m)^2} + \frac{g}{\omega^2 + (\omega_n + \omega_m)^2} - \delta_{nm} \sum_{\substack{\omega_l \\ \omega_l > 0}} \left[\frac{1}{\omega^2 + (\omega_n - \omega_l)^2} - \frac{1}{\omega^2 + (\omega_n + \omega_l)^2} \right] \right], \tag{4.10}$$

where again we have dropped an unimportant normalization constant.

We now consider the high- and low-frequency limits of Eq. (4.10). Clearly as $\omega \to \infty$, the term in Eq. (4.9) proportional to δ_{nm} scales as ω^{-4} (because $\sum_n \Delta(\omega_n)/|\omega_n|$ converges as $\omega_n \to \infty$) while the positive terms go as ω^{-2}. Thus at large enough ω, $\delta\Phi/\delta A(\omega) > 0$. This is physically obvious; the superconductivity is supposed to be mediated by spin fluctuations.

However, as $\omega \to 0$ the situation is different. The ω^{-2} terms do not cancel and one finds

$$\frac{\delta\Phi}{\delta\lambda(\omega)} = \omega \frac{\delta\Phi}{\delta A(\omega)} = -(1-g) + \cdots . \tag{4.11}$$

Thus at low enough frequencies $\delta\Phi/\delta A(\omega)$ is negative, if $g < 1$, and so there must exist a critical frequency ω_c; spin fluctuations at $\omega > \omega_c$ promote pairing, spin fluctuations at $\omega < \omega_c$ inhibit it. The precise value of ω_c depends weakly on the spectrum $A(\omega)$ and strongly upon g. In the simple BCS approximation given above one finds analytically that for $\omega_0 > \omega > T_c$,

$$\delta T_c/\delta A(\omega) \sim (g \ln^2\omega/T_c - \ln\omega/T_c + \text{const})$$

so that $\omega_c \sim T_c \exp 1/g$. The g dependence of ω_c for a more general model may be easily computed numerically; the nearly exponential dependence of ω_c on $1/g$ is shown in Fig. 2 for an Einstein model of arbitrary J_0. It is perhaps surprising that ω_c is essentially independent of J_0. However, the only J dependence in Eq. (4.9) (apart from that of T_c) is via the gap function $\Delta(\omega_n)$, and we have verified that the form of $\Delta(\omega_n)$ is insensitive to J.

The value of $\omega\delta\Phi/\delta A(\omega)$ may be easily calculated numerically for general ω. Results for various values of g calculated using the Einstein model of Eq. (4.5) for $J_0 \equiv 2$ are shown in Fig. 3. The curves for $J = 1$ and $J = 10$ are very similar.

For $g \lesssim 0.5$ (and certainly for the $g < 0.25$ case which may be relevant for heavy fermions) one sees from the curve in Fig. 3 that $\omega\delta\Phi/\delta A(\omega)$ is constant for frequencies less than several times T_c. One infers from this that boson fluctuations in this frequency range have the same pair-breaking effect as static impurities.

V. PAIR BREAKING

In this section we give a quantitative analysis of the pair-breaking effects of low-frequency spin fluctuations

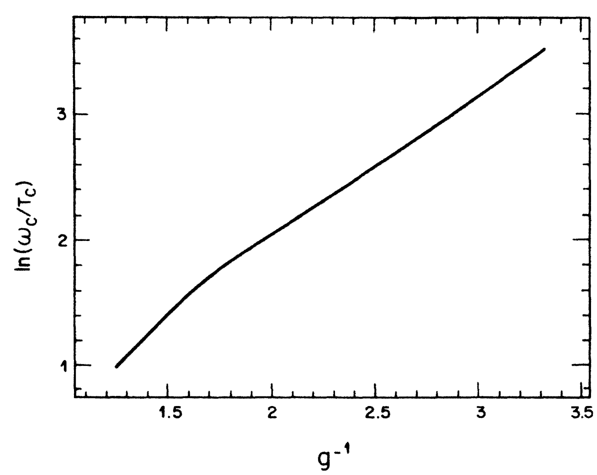

FIG. 2. The logarithm of the ratio (ω_c/T_c) plotted against the coupling constant $1/g$. ω_c is the frequency at which $\delta\Phi/\delta A(\omega)$ crosses zero in an Einstein model of a spin-fluctuation mediated d-wave superconductor.

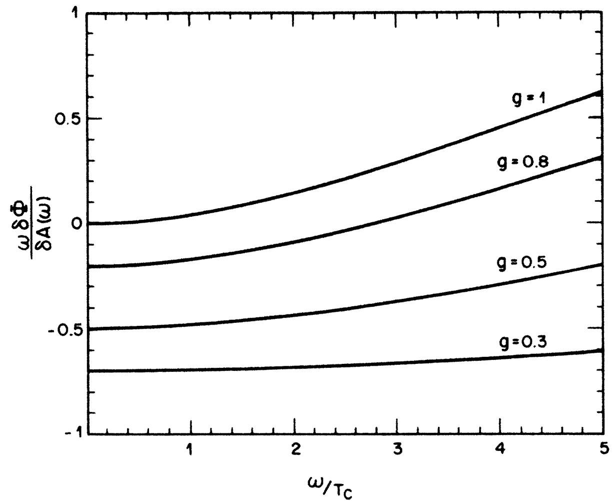

FIG. 3. The functional derivative of the critical temperature T_c for a d-wave superconductor with respect to the spin-fluctuation spectral weight $A(\omega)$, plotted against the ratio ω/T_c for various values of g. The Einstein model of Eq. (4.5) with $J_0=2$ has been used. The results for $J=1$ or 10 are very similar.

and static magnetic and nonmagnetic impurities upon d-wave superconductivity. We discuss the applicability of our results to UPt_3 and UBe_{13} and make a few remarks about the new high T_c materials. As far as possible, we shall formulate our results in terms of experimentally measurable quantities.

We define a coherence length ξ_0 by

$$\xi_0=\frac{v_p}{Z_0\pi T_c} . \tag{5.1}$$

The factor $Z_0=\lim_{\omega\to 0}Z(\omega)$ is the usual mass enhancement factor; v_p is the "unrenormalized" velocity. This definition is sensible if one is interested in temperatures and energies much less than a typical spin-fluctuation frequency ω_E. For $\omega\sim\omega_E$, $Z(\omega)$ depends on frequency and it is not clear how to define a length. Although we do not know whether the Eliashberg equations are valid in the very strong coupling limit, we present some results in this limit below. Throughout, we use Eq. (5.1) to define a coherence length.

We first ask the question: given a spin fluctuation mediated d-wave superconductor of given T_c and essentially infinite mean-free path for electrons of energy $\varepsilon\lesssim T_c$, how does T_c change when a small amount of scattering of electrons of energy $\varepsilon\lesssim T_c$ is added?

We begin by assuming that the scattering is due to static, nonmagnetic impurities. The physically relevant measure of scattering is the mean-free path, which may be inferred from the measured resistivity via Eq. (2.13). We therefore write, using (2.14), (3.3), (4.8a), and (4.9)

$$\frac{1}{T_c\pi}\frac{dT_c}{\xi_0 d(1/l)}=-(1-g_I)\beta , \tag{5.2}$$

where

$$\beta=\frac{Z_0}{T_c\pi}\frac{1}{|d\Phi/dT|} . \tag{5.3}$$

For comparison, we note that for a weak-coupling s-wave superconductor with an infinitesimal amount of magnetic impurity scattering one finds[20]

$$d\ln T_c/\pi\xi_0 d(1/l)=2\beta_M ,$$

with $\beta_M=\pi/4=0.785$. We have computed β numerically in the Einstein model of Eq. (4.8); some typical results are given in Table I. One sees that for $g<1$, β never differs by more than 20% from β_M. Surprisingly, we have also found numerically that within the Einstein model β depends only on the value T_c/ω_E, and not on g or J separately for $g<1$. As expected, β tends to the value β_M as T_c/ω_E tends to zero. The dependence of β on T_c/ω_E is shown in Table II.

Equations (5.2) and (5.3) show—as has been known for many years—that conventional impurities are pair breaking for d-wave superconductors. The explicit expressions are apparently new and show that the pair-breaking rate for conventional impurities in a d-wave superconductor is of the same order as that for magnetic impurities in an s-wave superconductor.

One may make a similar analysis for magnetic impurities, obtaining an equation analogous to (5.2) but with $(1+g_s)$ replacing $(1-g_I)$. The quantity g_s is defined by Eq. (4.4b), but with the spin-flip scattering amplitude v_s replacing v_I. Note $-1<g_s<1$ also. Thus magnetic impurities are also pair breaking for d-wave superconductors.

The mathematical similarity between the cases of conventional (or magnetic) impurities in d-wave superconductors and magnetic impurities shows that the standard[20,21] analysis of the suppression of T_c for finite impurity density or for the temperature dependence of the gap for $T<T_c$ carries over to the d-wave case provided one multiplies the pair-breaking parameter ρ defined in Ref. 20 by the factor $(1-g_I)/2$ or $(1+g_m)/2$ is appropri-

TABLE II. Pair-breaking rate β, calculated using the Einstein model of Eq. (4.5) with $\omega_E=1$, is shown to depend only on the ratio of the superconducting T_c to Einstein frequency ω_E.

λ	g	β
	$T_c=0.05$	
0.563	1.0	0.69
1.45	0.6	0.69
6.9	0.4	0.70
	$T_c=0.10$	
0.9	1.0	0.61
3.3	0.6	0.62
10.0	0.5	0.63
	$T_c=0.2$	
1.85	1.0	10.66
4.0	0.8	0.67
9.0	0.7	0.67

ate. Note that in the present work it is necessary to assume that the d-wave superconductor is in the clean limit, so that $\xi_0 = v_F p / T_c Z_0$; were this not the case, T_c would have been reduced to zero by pair breaking.

We now consider the case in which electrons scatter off spin fluctuations at $\omega < \omega_c$. The situation here is more subtle than in the impurity case. Consider for example the case $g \ll 1$; then $\omega_c \gg T_c$ so that adding spin-fluctuation spectral weight at, say, $\omega_0 = \omega_c/2$ will reduce T_c, yet will cause no change in the resistivity at T_c. Rather, this added spectral weight will change the mass enhancement of electrons of energy $\varepsilon < \omega_0$ and the scattering rate of electrons with $\varepsilon > \omega_0$. Both effects will presumably contribute to the decrease in T_c. This work was motivated, however, by the existence of superconductors with a large, temperature-dependent resistivity at T_c. Within the models we have considered such resistivity is produced by boson spectral weight at frequencies comparable to T_c. We therefore focus on the pair-breaking effect of such low-lying ($\omega \sim T_c$) Bose fluctuations in what follows. For such bosons mass enhancement is only changed for electrons of $\varepsilon < \omega$; these give a negligible contribution to the gap equation.

Note that for fixed temperature and λ, Eq. (2.10) is an upper bound for the scattering rate of electrons of energy $\varepsilon \sim T$. **Thus by combining Eqs. (2.10), (2.14), (3.3), (4.3a), and (4.9) we may write an inequality for the effect on T_c of introducing an infinitesimal quantity of Bose fluctuations at a frequency $\omega < \omega_c$:**

$$\frac{1}{\pi T_c}\left|\frac{dT_c}{\xi_0 d(1/l)}\right| > \left|\omega\frac{\delta\Phi}{\delta A(\omega)}\right|\beta\,. \tag{5.4}$$

Now $\omega\delta\Phi/\delta A(\omega)$ was discussed in Sec. IV. Its properties were shown to be very sensitive to the value of g; however, for $g \lesssim 0.8$, one sees that for $\omega \sim T_c$, it is essentially equal to its zero-temperature value $(1-g)$. From this one may conclude that, for the models so far proposed for d-wave superconductivity,[4-6] weak scattering is pair breaking regardless of whether it is due to interaction of electrons with static impurities or with spin fluctuations; further, the pair-breaking rate is comparable to that for magnetic impurities in an s-wave superconductor up to factors of order unity, provided the ratio of T_c to the pairing boson frequency is not too large. Recall also that $T_c/\omega_E \sim e^{-1/g}$ for $g < 0.8$.

The extension of these results to a finite boson spectral weight at $\omega < \omega_c$ and to $T < T_c$ is of interest; however, this extension also is more subtle than in the impurity case. Imagine adding boson spectral weight at a frequency $\omega_0 < \omega_c$ to a d-wave superconductor with a given T_c. As the low-frequency boson spectral weight is increased, T_c and ω_c will decrease. Eventually one will have $\omega_c \to \omega_0$ and the decrease in T_c will cease. The situation for $T < T_c$ is also problematic for two reasons. In the d-wave case the boson is believed[4-6] to be a spin fluctuation of the conduction electrons. When the conduction electrons become superconducting one expects the low-frequency, pair-breaking part of the spin-fluctuation spectrum to change, because of the superconducting gap. We have not investigated this issue in detail. The second reason, of more interest here, is that as $T \to 0$, the scattering rate for electrons of $\varepsilon > \omega_0$ off an Einstein mode at frequency ω_0 changes from $2\pi\lambda T$ to $\pi\lambda\omega_0$. (Again, if $\omega_0 \sim T_c$, electrons with energy $\varepsilon < \omega_0$ give a negligible contribution to the gap equation.) Thus, the effective pair-breaking parameter has a different temperature dependence than in the impurity case.

To illustrate this point we have studied the case $\omega_0 < \omega_c$ with $T_c \sim \omega_0$ and (assuming that the boson spectral function does not change when the electrons become superconducting) $T < T_c$. We use the model equations

$$\omega_n Z(\omega_n) = \omega_n + \pi T \sum_{\Omega_n} \frac{\Omega_n Z(\Omega_n)}{\sqrt{\Omega_n^2 Z^2(\Omega_n) + W^2(\Omega_n)}} \times \int_0^\infty \frac{2d\omega}{\pi} \frac{\omega A(\omega)}{(\omega_n - \Omega_n)^2 + \omega^2}\,, \tag{5.5a}$$

$$W(\omega_n) = g\pi T \sum_{\Omega_n} \frac{W(\Omega_n)}{\sqrt{\Omega_n^2 Z^2(\Omega_n) + W^2(\Omega_n)}} \times \int_0^\infty \frac{2d\omega}{\pi} \frac{\omega A(\omega)}{(\omega_n - \Omega_n)^2 + \omega^2}\,. \tag{5.5b}$$

These equations ignore the mixing of the anisotropic gap function with other representations of the cubic group and, as mentioned above, changes in the boson spectrum. We add a pair-breaking part to the Einstein model defined above, writing

$$A(\omega) = A_p(\omega) + A_E(\omega)\,, \tag{5.6}$$

where $A_E(\omega)$ is given by the right-hand side of Eq. (4.5) and

$$A_p(\omega) = \frac{\pi}{2} J_p \omega_p \delta(\omega - \omega_p)\,. \tag{5.7}$$

We assume $\omega_p < \omega_c$, $\omega_p \lesssim T_c$, where T_c is the critical temperature corresponding to the spectral weight $A(\omega)$ in (5.6).

We find that our results are to good approximation described by the standard analysis,[20] but with the pair-breaking parameter given by

$$\rho = \frac{(1-g)}{2}\frac{J_p\omega_p}{2}\coth\frac{\omega_p}{2T}\,. \tag{5.8}$$

To see why Eq. (5.8) is justified, consider the contribution of the pair-breaking part, $A_p(\omega)$, to the gap equation, (5.5b). One obtains

$$g\pi T \sum_{\Omega_n} \frac{W(\Omega_n)}{|\Omega_n Z(\Omega_n)|} \frac{J_p \omega_p^2}{(\Omega_n - \omega_n)^2 + \omega_p^2}\,.$$

If $\omega_p \ll T$ the second factor is sharply peaked about $\omega_n = \Omega_n$, while if pair breaking is not too strong, $\omega(\Omega_n)$ and $Z(\Omega_n)$ will vary with frequency only on the scale of $\omega_E \gg \omega_p$, T. Thus one may extract the ω/Z term and perform the sum, obtaining

$$\tfrac{1}{2}g\pi J_p\omega_p\coth\frac{\omega_p}{2T}\ .$$

Comparison of (5.9) and (4.3) shows that the low-frequency bosons enter the Eliashberg equations exactly as do impurities with a scattering rate of $\pi J_p\omega_p\coth\omega_p/2T$ and $g_I=g$. Eq. (5.8) thus follows.

We calculate the superconducting T_c and the ratio of the superconducting order parameter $\Delta_0(T)/T_c$. Here

$$\Delta_0=\lim_{\omega\to 0}\frac{W(\omega)}{\omega Z(\omega)}\ .$$

It is not the energy gap, which may or may not vanish.[20]

We choose parameters $\omega_E=1$, $g=0.67$, and $a_E=1.2$; in the absence of pair breaking these yield a superconducting transition temperature $T_{c0}=0.053$ and a critical frequency $\omega_c=0.22\cong 4T_{c0}$. We discuss three choices for $A_p(\omega)$: (a) $\omega_p(a)=0.0003$, $J_p(a)=10$—this yields $T_c=0.0069$; (b) $\omega_p(b)=0.01$, $J_p(b)=3$—this yields $T_c=0.019$; (c) $\omega_p(c)=0.03$, $J_p(c)=1$—this yields $T_c=0.035$. The results are plotted in Fig. 4. The outermost line represents the solution with $A_p(\omega)=0$. For cases (a), (b), and (c) the solid lines represent the solutions of the full equations (5.5), (5.6), and (5.7) while the dashed lines represent the solution to the same equations with $A(\omega)=A_e(\omega)$, and an additional pair-breaking parameter ρ given by Eq. (5.7). For case (a), the dashed and solid lines are superposed indicating that the approximation is essentially exact. In case (a) we have $\omega_p \ll T_c$, explaining the success of the approximation. For cases (b) and (c) we have $\omega_p\approx 1/2T_c$ and $\omega_p\approx T_c$, respectively. Despite this, the approximation [Eq. (5.8)] is valid to within a 10% accuracy throughout the entire temperature range.

We now consider applying these results to the heavy-fermion materials UBe_{13} and UPt_3. A serious difficulty is

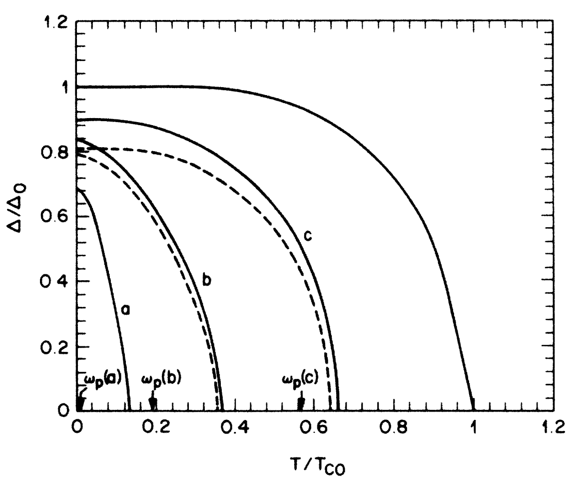

FIG. 4. The temperature dependence of the zero-temperature order parameter Δ. The outermost line represents the solution with no pair breaking $A(\omega)=A_E(\omega)$. Cases (a), (b), and (c) represent three values for the pair-breaking part $A_p(\omega)$. The solid lines represent the complete numerical diagonalizations of Eqs. (5.5), (5.6), and (5.9) while the dashed lines indicate the results obtained by introducing a pair-breaking parameter ρ given by Eq. (5.8).

the absence of a microscopic theory of heavy-fermion superconductivity so that the all important parameter g is unknown. It is even unknown whether the inequality $g<1$ need apply, much less whether the more stringent inequality $g<0.25$ which follows from the previously proposed models[4–6] is appropriate. Nevertheless, we assume in what follows that a spin-fluctuation model with $g<0.8$ applies to the heavy-fermion materials.

For UPt_3, the resistivity near T_c is low. The mean-free path may be estimated from the Fermi surface area,[27] the measured resistivity,[1] and Eq. (2.13) to be in excess of 2000 Å, while, using the measured quasiparticle velocity,[27] one finds $\xi_0\sim 150$ Å. Linearizing Eq. (5.4) and assuming $g=0.25$ and $\beta=\beta_M$ we estimate that the scattering has reduced the observed T_c of UPt_3 from the T_{c0} of a hypothetical material with no scattering by less than 10%,

$$\frac{T_{c0}-T_c}{T_{c0}}<0.1\ . \tag{5.9}$$

The pair breaking seems to be weak in UPt_3. However, the resistivity of UBe_{13} at T_c is ~ 50 times larger than that of UPt_3;[1] thus the ratio of the mean-free path to the coherence length for UBe_{13} must be larger than in UPt_3. Indeed an analysis[7] (using, it should be noted, a theory appropriate for s-wave superconductivity) of the upper critical field in UBe_{13} has yielded the values $\xi_0=150$ Å and $l=60$ Å. This difference of at least 30 in the ξ_0/l ratio implies that for UBe_{13} one has

$$\frac{T_{c0}-T_c}{T_{c0}}>1 \tag{5.10}$$

indicating that the pair breaking in UBe_{13} is strong. We therefore suggest that if UBe_{13} were a spin-fluctuation mediated d-wave superconductor, it would have a gapless region near T_c and a large value of the ratio of the zero-temperature gap to T_c. We note, however, that in Ref. 7 the specific-heat jump at T_c was shown to be in good agreement with a theoretical model which presumably included no pair breaking; this is perhaps evidence against gaplessness.

Of course, Eq. (5.11) was obtained by extrapolating Eq. (5.4) far beyond its region of validity. In a more detailed analysis, T_c could be computed for a boson spectrum extended to low frequencies and for arbitrary impurity scattering. However, what has been computed suffices for a rough estimate; in view of the sensitive dependence of our results upon the (uncertain) value of g, such an analysis does not seem worthwhile.

Were the high-T_c materials spin-fluctuation mediated d-wave superconductors, a similar analysis could be made. Again, the issue of the elasticity or otherwise of the scattering would be irrelevant if $g\lesssim 0.8$.

VI. CONCLUSION

We have examined in this paper the differences in the nature of pair breaking due to inelastic scattering of electrons off boson fluctuations in anisotropic and isotropic superconductors. We have also studied the superconducting transition temperatures predicted by an Eliashberg equation for d-wave superconductivity. We have identified a parameter g which measured the difference in the coupling of boson fluctuations to the normal and pairing self-energies. For conventional phonon superconductors $g=1$. For models with purely repulsive interactions, $g<1$ for d-wave superconductivity. For models which have appeared in the heavy-fermion literature, $g<0.25$. We know of no fundamental reason why models which yield d-wave superconductivity must have $g<1$, but we have only studied this case in this paper.

Our principal results are that for spin-fluctuation mediated d-wave superconductors: (i) there is a critical frequency $\omega_c \sim T_c e^{1/g}$ in the spin-fluctuation spectrum. Spin fluctuations with $\omega<\omega_c$ are detrimental to d-wave superconductivity; spin fluctuations with $\omega>\omega_c$ increase T_c. (ii) For fixed interaction strength and frequency T_c is a rapidly decreasing function of g. For $g<1$, T_c saturates, independent of the magnitude of the coupling constant, J. For $g<0.8$, T_c in an Einstein model is well fit by the form $T_c=\omega_E e^{-(1+J)/gJ}$ independent of the magnitude of J. (iii) Provided $g<0.8$ it is essentially correct to infer a pair-breaking parameter from the measured resistivity, whether the scattering is elastic or inelastic. Assuming UBe_{13} is a d-wave superconductor with $g<0.8$ we suggest it is in the strong-pair-breaking regime, and its T_c has been substantially reduced by the low-frequency spin fluctuations (or impurities) which produce the measured resistivity at T_c. Similar considerations would apply to the high-T_c materials, were they d-wave superconductors. However, because our results are so sensitive to the precise value of g, and because it seems difficult to determine the appropriatly value of g for UBe_{13}, it is difficult to draw firm conclusions.

Our results have important implications for any microscopic calculation of T_c in UBe_{13}. First, it is crucial that the microscopic model produce a reliable value of g. Second, the model must produce the observed resistivity at T_c.

Concerning the general question of the effect of inelastic scattering we have reiterated the result of Bergmann and Rainer that it is necessary to consider the effect of the inelastic scattering mechanism on the pairing interaction as well as on the normal-state self-energy, before one can draw conclusions about the variation of T_c.

ACKNOWLEDGMENTS

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APPENDIX

In the Appendix we outline the numerical methods used. From (3.3) and the ensuing discussion one immediately sees that to find T_c one must diagonalize a real symmetric matrix. We have searched until a temperature yielding an appropriately small largest eigenvalue is obtained. T_c and $d\Phi/dT$ are then calculated by quadratic interpolation using three temperatures bracketing T_c; the results are accurate to at least a part in 10^3.

A straightforward extension of this method to $T<T_c$ [Eqs. (5.5a) and (5.5b)] yield a matrix which depends self-consistently upon the gap and is nonsymmetric. To proceed it is convenient to define

$$\Gamma(\Omega_n)=\frac{W(\Omega_n)}{\sqrt{\Omega_n^2 Z^2(\Omega_n)+W^2(\Omega_n)}} . \tag{A1}$$

Then, proceeding in close analogy to the derivation of (3.3) from (3.1) and (3.2) one may write (5.5a) as an equation for $Z(\omega_n)$ in terms of $\sqrt{1-\Gamma^2(\Omega_n)}$ and (5.5b) as an eigenvalue equation for $\Gamma(\Omega_n)$ involving a matrix depending on Γ. We being with a guess for $\Gamma(\Omega_n)$, determine $Z(\omega_n)$, solve the matrix equation to find the eigenvector corresponding to eigenvalue $\Phi=0$, and iterate. In practice we find that a solution accurate to a part in 10^3 is obtained in five or six iterations.

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