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Dilute Bose gas in two dimensions

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The weakly interacting Bose gas in two dimensions is considered in the dilute limit $n^{1/2}a \ll 1$, where n is the particle density and a is the range of the potential. The standard many-body perturbation theory for this system has two separate divergences: the first, associated with classical phase fluctuations, is responsible for the vanishing of the long-range order; the second is quantum mechanical and is connected with the vanishing of the scattering t matrix at long wavelengths and low energies. An earlier diagrammatic theory of Popov, which provides a consistent description of the system in the dilute limit, is rederived heuristically from a quasiparticle picture, and also using the renormalization group. It is shown that the superfluid transition temperature is $T_c \approx 4\pi(\hbar^2/2m)n/[\ln\ln(1/na^2)]$, and the condition of validity of the dilute limit is $\ln\ln(1/na^2) \gg 1$. The connection to the dilute Bose gas in dimensions $d>2$ and the universal behavior beyond the extreme asymptotic low-density domain are also discussed.

I. INTRODUCTION

It is well known[1] that Bose-Einstein condensation occurs in the ideal Bose gas for any dimension $d>2$. A weak interparticle interaction[2] only changes the behavior significantly at low temperatures and long wavelengths[2,3] on the one hand, and near the transition temperature[4] on the other. Both of these effects are confined to narrow regions of temperature in the *dilute limit*

$$n^{1/d}a \ll 1 , \tag{1.1}$$

where n is the particle density and a is a length scale associated with the potential, e.g., a scattering length or the radius of a hard sphere. In *two dimensions*, Bose condensation does not occur in either the ideal[1] or the interacting[5] system, but a phase transition to a superfluid state is expected in the latter case.[6] The purpose of the present paper is to discuss the dilute limit (1.1) in two dimensions, where both the Bose-Einstein condensation temperature *and* the scattering length for binary collisions[7,8] vanish. These two singularities, the first of classical origin and the second purely quantum mechanical, imply that the usual dilute gas expansion[3,7] must be modified in an essential way.

This problem was in fact solved some time ago by Popov,[9] using a diagrammatic formalism based on functional integrals. Popov did not, however, explicitly ask what the diluteness condition analogous to (1.1) should be, though his theory can be used to show that (1.1) is replaced by

$$\ln\ln(1/\gamma) \gg 1 , \tag{1.2a}$$

where

$$\gamma \equiv na^2 , \tag{1.2b}$$

and a is now the range of the interaction (to be defined more precisely below).

The purpose of the present paper is to explain Popov's theory[9] in a heuristic way, based on Bogoliubov quasiparticles, and to rederive it using a modern renormalization-group argument[10,4] which clearly exhibits the origin of the double logarithm appearing in (1.2). We show that when (1.2) is satisfied the critical temperature is well approximated by its value in the Bogoliubov theory

$$T_B \approx \frac{4\pi\hbar^2 n}{2m\,\ln\ln(1/\gamma)} , \tag{1.3}$$

and over most of the temperature range $0 \lesssim T \lesssim T_B$ the superfluid density ρ_s has a "free-particle" form expected from the Landau-Bogoliubov quasiparticle theory[11]

$$\rho_s/\rho = 1 - T/T_B . \tag{1.4}$$

The low-temperature phonon corrections to (1.4),

$$\rho_s/\rho = 1 - T^3/T_B\mu^2(0) , \tag{1.5}$$

appear only for $T < \mu(0) \approx T_B/\ln(1/\gamma)$, where $\mu(0)$ is the chemical potential at $T=0$. The fluctuation corrections associated with the Kosterlitz-Thouless transition,[6] on the other hand, are confined to a critical region of order

$$(T - T_B)/T_B \lesssim 1/\ln\ln(1/\gamma) . \tag{1.6}$$

In Sec. II the dilute Bose gas in dimensions $d>2$ is reviewed and it is shown that Eq. (1.1) is the condition that a low-temperature phonon region will be well separated from the critical region, with free-particle behavior at intermediate temperatures. Section III presents Popov's theory[9] of the dilute gas in two dimensions, and a similar separation of temperature regions is shown to follow from condition (1.2). The theory is rederived in Sec. IV from a renormalization-group treatment consisting of two steps: first a quantum-mechanical renormalization which effectively replaces the two-body interaction by a t matrix, and then a classical renormalization to eliminate the divergence associated with the absence of Bose condensation in two dimensions.[5] Section V discusses the general scaling behavior and concludes with some remarks on universality beyond the extreme asymptotic domain, as well as on possible physical applications of the theory. The main results for the superfluid density are summarized in Fig. 1.

II. THE DILUTE GAS FOR $d > 2$

We begin with the second-quantized Hamiltonian in d dimensions[12]

$$H=-\int d^dx\,\psi^\dagger(\mathbf{x})\nabla^2\psi(\mathbf{x})+\frac{1}{2}\int d^dx\,d^dx'\psi^\dagger(\mathbf{x})\psi^\dagger(\mathbf{x}')\tilde{v}(|\mathbf{x}-\mathbf{x}'|)\psi(\mathbf{x})\psi(\mathbf{x}') , \tag{2.1}$$

where $\tilde{v}$ is a two-body interaction with strength v and range a. For a definite example we will sometimes take a repulsive potential of the form

$$\tilde{v}(r)=ve^{-r^2/a^2} , \tag{2.2}$$

but the results are valid more generally. For a system with number density n, the standard Bogoliubov quasiparticle theory,[2,3,11] generalized from $d=3$ to any $d>2$, consists of the following elements. (i) The quasiparticle excitation spectrum $\varepsilon(k)$ is given by[12]

$$\varepsilon^2(k)=\varepsilon_0^2+2\mu\varepsilon_0 , \tag{2.3a}$$

with

$$\varepsilon_0=k^2 , \tag{2.3b}$$

where μ is the chemical potential. (ii) For strong two-body scattering, e.g., hard spheres,[13] we replace the potential by a t matrix[14,15,9]

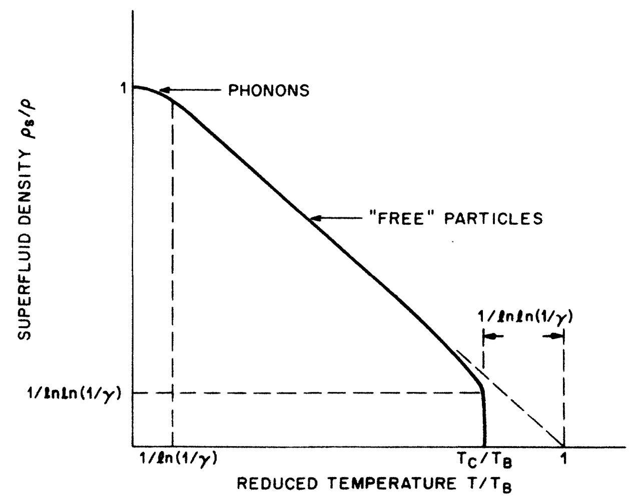

FIG. 1. Schematic representation of the temperature dependence of the superfluid density in the dilute Bose gas in two dimensions. There are three different regimes; a low-temperature phonon region for $T/T_B<1/\ln(1/na^2)\equiv 1/\ln(1/\gamma)$, an intermediate free-particle region for $1/\ln(1/\gamma)<T/T_B\lesssim 1$, and a critical region $|T-T_B|/T_B\lesssim 1/\ln\ln(1/\gamma)$, where ρ_s/ρ experiences a universal jump of order $1/\ln\ln(1/\gamma)$ at a critical temperature T_c which is displaced from T_B by a quantity of order $|T_c-T_B|/T_B\sim 1/\ln\ln(1/\gamma)$. The dilute limit is defined by the condition $\ln\ln(1/\gamma)\gg 1$, which guarantees that the three regions will be well separated.

$$t(\mathbf{k}_1,\mathbf{k}_2,z)=\tilde{v}(|\mathbf{k}_1-\mathbf{k}_2|)-\int\frac{d^dk_3}{(2\pi)^d}\tilde{v}(|\mathbf{k}_1-\mathbf{k}_3|)\times(2k_3^2-z)^{-1}t(\mathbf{k}_3,\mathbf{k}_2,z) , \tag{2.4}$$

where $\tilde{v}(k)$ is the Fourier transform of $\tilde{v}(r)$ and z is an internal frequency. In the limit $\mathbf{k}_1,\mathbf{k}_2\to 0$ we have, for the potential (2.2),

$$t(0,0,z)=\frac{c_1va^d}{1+c_2va^2A_d(za^2/8)} , \tag{2.5}$$

where c_1 and c_2 are nonsingular numerical constants and

$$A_d(\alpha)=\mathrm{P}\int_0^\infty\frac{x^{(d-2)/2}e^{-x}}{x-\alpha}dx . \tag{2.6}$$

For $d>2$, $A_d(0)\approx\Gamma(\frac{1}{2}(d-2))>0$, we can define

$$t(0,0,0)=t_0 \tag{2.7}$$

as the t matrix at zero k and ω, so for $va^2\gg 1$, $t_0\sim a^{d-2}$. For more general potentials than (2.2) the same relation holds with a equal to the scattering length. (iii) The number density can be written as

$$n=n_0+n' , \tag{2.8}$$

where the condensate part n_0 is to lowest order given by

$$\mu=n_0t_0 , \tag{2.9}$$

and the noncondensate part n' by

$$n'=\int\frac{d^dk}{(2\pi)^d}\left[\frac{\varepsilon_0+\mu-\varepsilon}{2\varepsilon}+\frac{\varepsilon_0+\mu}{\varepsilon}\frac{1}{e^{\beta\varepsilon}-1}\right] . \tag{2.10}$$

Equations (2.7)–(2.10) can be used to calculate $\mu(n,T)$ and $n_0(n,T)$ and thus to obtain the complete thermodynamics of the dilute Bose system. The superfluid density is given by a Landau quasiparticle formula[11] based on the Bogoliubov spectrum (2.3),

$$\frac{\rho_s}{\rho}=1-\frac{\beta}{\rho d}\int\frac{d^dk}{(2\pi)^d}k^2\frac{e^{\beta\varepsilon}}{(e^{\beta\varepsilon}-1)^2} , \tag{2.11}$$

where $\beta=(k_BT)^{-1}$ and $\rho=mn=n/2$ is the mass density.[16]

The Bogoliubov theory (2.7)–(2.11) forms the basis for a systematic expansion of the dilute Bose gas in the weak-coupling limit

$$g^2\equiv n^{(d-2)/d}t_0\ll 1 . \tag{2.12}$$

To see that the single coupling constant g is an appropriate expansion parameter at all temperatures we define the following temperature scales:

$$T_a\equiv\hbar^2/2ma^2 \tag{2.13}$$

is the microscopic scale associated with the potential;

$$T_n=4\pi(\hbar^2/2m)n^{2/d}=4\pi n^{2/d} \tag{2.14}$$

is the temperature scale associated with the density; and the Bose-Einstein transition temperature of the ideal gas is given by[1]

$$T_E = b_d T_n \;, \tag{2.15}$$

where $b_d = [\zeta(d/2)]^{-2/d}$, with $\zeta(x)$ the Riemann zeta function.

At $T=0$ we have[17]

$$n'(0)/n \sim t_0^{d/2} n^{(d-2)/2} \sim g^d \ll 1 \;, \tag{2.16}$$

$$\mu(0) \approx n t_0 \sim g^2 T_n \;. \tag{2.17}$$

The excitation spectrum is phononlike for $k < 2\mu^{1/2}$ and free-particle-like for $k > 2\mu^{1/2}$. At low temperatures, $T < \mu(0)$, the superfluid density (2.11) has the phonon form

$$\frac{\rho_s}{\rho} - 1 \sim -\left[\frac{T}{T_n}\right]^{d+1} \frac{1}{g^{(d+2)}} \;, \tag{2.18}$$

and the density of noncondensed particles behaves like

$$\frac{n'(T)}{n} - \frac{n'(0)}{n} \sim -\left[\frac{T}{T_n}\right]^{d-1} \frac{1}{g^{2-d}} \;. \tag{2.19}$$

The transition temperature T_B defined from the conditions $\rho_s = 0$ or $n' = n$ is in lowest order given by the Bose-Einstein temperature (2.15), with corrections coming from additional terms in (2.9) and (2.10), which vanish as powers of g. Over most of the temperature range $\mu(0) \lesssim T \lesssim T_B$, the densities ρ_s and n_0 are approximately equal to their free-particle values[1]

$$\frac{\rho_s}{\rho} - 1 = \frac{n_0}{n} - 1 = \left[\frac{T}{T_E}\right]^{d/2} . \tag{2.20}$$

Equations (2.10) and (2.11) are uniformly valid for all T up to $T_B \approx T_E$ as long as $g^2 \ll 1$. The crossover from (2.18) to (2.20) can be calculated numerically using the basic formulas (2.7)–(2.11).

Near T_B and for $d \le 4$ there are fluctuation corrections coming from critical phenomena associated with the two-component Bose order parameter.[10,18] These occur in a critical region of width[18,9]

$$\frac{T - T_B}{T_B} \lesssim g^{2(d-2)/(4-d)} \;, \tag{2.21}$$

and they cause a shift in transition temperature from T_B to T_c, as well as a change in the critical exponents from their mean-field values given in (2.18) and (2.19) to the usual scaling exponents[10,18] inside the region (2.21). For $d > 4$ the mean-field theory holds, with merely a shift in T_c proportional to g^2.

III. DILUTE GAS FOR $d=2$: QUASIPARTICLE THEORY

The foregoing derivation clearly breaks down in two dimensions for two different reasons.

(i) The quantum-mechanical t matrix (2.4) vanishes[7–9] at low k and ω, since according to (2.6),

$$A_2(\alpha) \sim -\ln\alpha, \quad \alpha \to 0 \;. \tag{3.1}$$

This result implies that the coupling constant g cannot be defined via Eqs. (2.7) and (2.12).

(ii) The Bose-Einstein condensation temperature (2.15) vanishes[1] (since $b_d = [\zeta(d/2)]^{-2/d}$ diverges logarithmically as $d \to 2$) due to classical phase fluctuations.[5] This effect can be seen in the expression for n', Eq. (2.10), which has a logarithmic divergence in its lower-momentum cutoff in both the interacting ($\mu \neq 0$) and noninteracting cases ($\mu = 0$).

A heuristic way to derive a consistent theory is to notice that the quasiparticle formula for ρ_s, Eq. (2.11), remains well behaved for $d=2$, so long as μ is positive. However, the integral diverges for small μ, since Eq. (2.3) implies that the integrand has the form $k^2/(k^4 + 2\mu k^2)$ in this limit. Of course there is no condensate so μ cannot be obtained from (2.9). Instead we will make the ansatz that μ is given by[7]

$$\mu = n \, | \, t(0,0,z=\mu) \, | \sim n / | \ln(a^2\mu) | \;, \tag{3.2}$$

thus providing a lower cutoff $k_0 \sim \mu^{1/2}$ which regularizes the integral expression for ρ_s in Eq. (2.11) at all temperatures. As in the previous case of dimension $d > 2$, k_0 is the momentum below which interaction effects are important and above which the system behaves like an ideal gas. In the present case, however, it is also clear that the logarithmic divergence in Eq. (2.11) will yield a transition temperature T_B, determined by setting $\rho_s = 0$ in Eqs. (2.11) and using (2.3) and (3.2), which will be of the form

$$T_B \sim n / | \ln(k_0 a) | \sim n / | \ln(\mu a^2) | \sim n / \ln\ln(1/na^2) \;.$$

It turns out that our conjectured Eq. (3.2) is essentially the result obtained earlier by Popov,[9] from a careful analysis of the diagrams associated with the Hamiltonian (2.1). Popov introduced a momentum-space sphere of radius k_0, and considered all $k < k_0$ as contributing to a quasicondensate $n_0(k_0)$ and all $k > k_0$ as contributing to $n'(k_0)$. In order to eliminate all divergences in $\ln k_0$, he found it necessary to introduce another cutoff $\tilde{k}_0 > k_0$, and to consider all diagrams which were second order in t as corrections to Eq. (3.2). In the end he obtained a formula for n from which the singular dependence on k_0 and $\tilde{k}_0$ was eliminated:

$$n = -\frac{\mu}{8\pi}[\ln(a^2\mu/2) + 1] - \int \frac{d^2k}{(2\pi)^2} \frac{\varepsilon_0}{\varepsilon} \frac{1}{e^{\beta\varepsilon} - 1} \;, \tag{3.3}$$

where the quasiparticle energy is still given by (2.3); the superfluid density is obtained from (2.11), for $d=2$. It can be shown moreover (see the Appendix), that to leading order Eqs. (3.2) and (3.3) are identical, so in a sense our heuristic guess in Eq. (3.2) represents a simple derivation of Popov's result.

Equations (2.3), (3.3), and (2.11) constitute a self-consistent theory of the Bose gas in two dimensions and it is natural to ask what the condition analogous to (1.1) is, which defines the range of validity of the theory. In the Appendix we rewrite the equations in a form convenient for asymptotic evaluation, and we obtain the following results.

In addition to the microscopic temperature scale T_a, Eq. (2.13), and the scale T_n, Eq. (2.14), which are related by

$$1/4\pi\gamma \equiv T_a/T_n \tag{3.4}$$

[cf. Eq. (1.2)], the critical temperature in the limit $\ln\ln(1/\gamma) \gg 1$ is

$$T_B \approx T_n/\ln\ln(1/\gamma) = T_n/\ln\ln(1/na^2) \ . \tag{3.5}$$

There is also a fourth temperature scale $\mu(0)$, separating the low-temperature phonon region from the "free-particle" region, and given by

$$\mu(0) \approx 2T_n/\ln(1/\gamma) \ . \tag{3.6}$$

The behavior of the superfluid density in the two regions $[T \lesssim \mu(0)$ and $\mu(0) \lesssim T \lesssim T_B]$ is given by Eqs. (1.5) and (1.4), respectively.

Finally, we may ask at what temperature fluctuation effects associated with the critical behavior will modify ρ_s. Since the critical point is expected to be of the Kosterlitz-Thouless[6] type for an interacting Bose system in $d=2$, we may estimate that ρ_s will be perturbed when it reaches a magnitude which is of the order of its universal value[19] just below the Kosterlitz-Thouless temperature T_c

$$\rho_s(T=T_c^-) = \frac{2m^2T_c}{\pi^2\hbar^2} = \frac{4\rho_s(0)}{\pi\ln\ln(1/\gamma)} \ . \tag{3.7}$$

Since, according to Eq. (1.4), ρ_s behaves linearly for $T \lesssim T_B$ we find

$$|T_c - T_B|/T_B \sim 1/\ln\ln(1/\gamma) \ . \tag{3.8}$$

Thus the dilute limit is indeed given by Eq. (1.2), which guarantees that there will be a well-defined free-particle domain (1.4) for $\mu(0) \lesssim T \lesssim T_B$ with a phonon region (1.5) at low temperatures $T \lesssim \mu(0)$, and a critical region close to T_c, which is itself close to the mean-field T_B according to Eq. (3.8). The superfluid density as a function of temperature is shown schematically in Fig. 1.

IV. RENORMALIZATION-GROUP TREATMENT

We now derive the foregoing results, particularly Eq. (3.5), using the renormalization group.[10] We begin by transforming the path-integral representation for the partition function appropriate to the Hamiltonian (2.1) to the form[4]

$$F = -\frac{T}{V}\ln[\mathrm{Tr}\exp(-L)] \ , \tag{4.1}$$

where L is an imaginary-time Lagrangian with spatial cutoff[12] $a=1$,

$$L = \int d^dx \int_0^\beta d\tau \left[\Gamma^{-1}\psi^* \frac{\partial\psi}{\partial\tau} + |\nabla\psi|^2 - \mu|\psi|^2 + \frac{v}{2}|\psi|^4 + f \right] , \tag{4.2}$$

V is the volume of the system, and Γ is an energy parameter which controls the strength of the quantum fluctuations and is initially equal to 1. The classical limit is $\Gamma=0$. The interaction $v \sim \tilde{v}(r=a)$ is the two-body potential at short distances, and f is the constant part of the free-energy density which absorbs effects due to the cutoff a. The other parameters have essentially the same meaning as above except that the field ψ is now a c number, since Eq. (4.1) involves a sum over all classical paths. [The chemical potential in Eq. (4.2) formally differs by a (v-dependent) constant from that in Eq. (2.1) due to the self-interaction implied by the form of Eq. (4.2). Since this term will just be canceled by the zero-temperature self-energy we can ignore its effects, which we have thus dropped out of Eq. (4.2).] The density is given by

$$n = -\frac{\partial}{\partial\mu}F(T,\mu,v,f,\Gamma) \ . \tag{4.3}$$

The fixed point which describes the ideal-gas transition from zero density to the superfluid at zero temperature is the noninteracting fixed point with $v=0$ and $\mu=0$ in Eq. (4.2). This is the controlling fixed point for the low-temperature almost-ideal behavior and we therefore expand around this point. The basic idea is to divide the renormalization into two stages: first a regime in which the quantum fluctuations dominate and then, when the renormalized temperature becomes large enough, a classical regime where the quantum fluctuations are negligible. We shall find that for $d=2$ the renormalized interaction will tend towards zero in the quantum regime, justifying the expansion in powers of v.

The renormalization transformation is carried out by integrating over a momentum shell[10] of width dl and summing over all frequencies. At wave vector k and Matsubara frequency $\omega_n = 2\pi nT$ the bare propagator is

$$\langle \psi^*(\mathbf{k},\omega_n)\psi(\mathbf{k}',\omega_{n'})\rangle = \beta\delta_{n+n'}\delta_{\mathbf{k}+\mathbf{k}'}(-i\Gamma^{-1}\omega_n + k^2 - \mu)^{-1} \ . \tag{4.4}$$

The momentum integrals yield a factor $K_d dl$ where $K_d = 2^{1-d}\pi^{-d/2}\Gamma(d/2)$, and the frequency sums are equivalent to the integral $\int d\omega/2\pi$ at zero temperature. After integrating out the degrees of freedom in the momentum shell we rescale according to

$$x \to xe^l \ , \tag{4.5a}$$

$$\psi \to \psi e^{\zeta l} \ , \tag{4.5b}$$

$$\tau \to \tau e^{zl} \ , \tag{4.5c}$$

and adjust a combination of ζ and z so that the coefficient of $|\nabla\psi|^2$ in (4.2) remains identically one, yielding

$$2\zeta + z = 2 - d + O(v_l^2) \ , \tag{4.6a}$$

where l subscripts denote renormalized quantities. The other parameters of the renormalized Lagrangian are given by

$$\frac{d\Gamma_l}{dl} = -(d+2\zeta)\Gamma_l + O(v_l^2) \ , \tag{4.6b}$$

$$\frac{dT_l}{dl} = zT_l \ , \tag{4.6c}$$

$$\frac{d\mu_l}{dl}=(d+z+2\zeta)\mu_l-C_\mu v_l+O(v_l^2)\ , \tag{4.6d}$$

$$\frac{dv_l}{dl}=(d+z+4\zeta)v_l-C_v v_l^2+O(v_l^3)\ , \tag{4.6e}$$

and

$$\frac{df_l}{dl}=(d+z)f_l+C_f\ , \tag{4.6f}$$

where the coefficients C_μ, C_v, and C_f, which depend on μ_l, Γ_l, and T_l, are calculated from the diagrams in Figs. 2(a), 2(b), and 2(c), respectively. The result is

$$C_\mu=\frac{2K_d\Gamma_l}{\exp[\Gamma_l(1-\mu_l)/T_l]-1}\ , \tag{4.7a}$$

$$C_v=K_d\Gamma_l^2\left[\frac{1}{2(1-\mu_l)}\coth[\Gamma_l(1-\mu_l)/2T_l]+\frac{1}{T_l}\operatorname{csch}^2[\Gamma_l(1-\mu_l)/2T_l]\right]\ , \tag{4.7b}$$

$$C_f=K_dT_l[\ln(1-e^{-\Gamma_l(1-\mu_l)/T_l})-\ln(1-e^{-\Gamma_l/T_l})]\ . \tag{4.7c}$$

We have dropped an additive constant from C_f which can be adjusted by changing the original definition of f in Eq. (4.2). We now have the freedom to choose the rescaling factor z, or equivalently by Eq. (4.6), ζ.

A. Quantum regime

In the low-temperature regime, $T \ll 1$, the spectrum of the dominant fluctuations is strongly quantum in character. It is thus convenient to keep the controlling parameter for these fluctuations fixed so we set

$$\Gamma_l=1\ , \tag{4.8a}$$

yielding

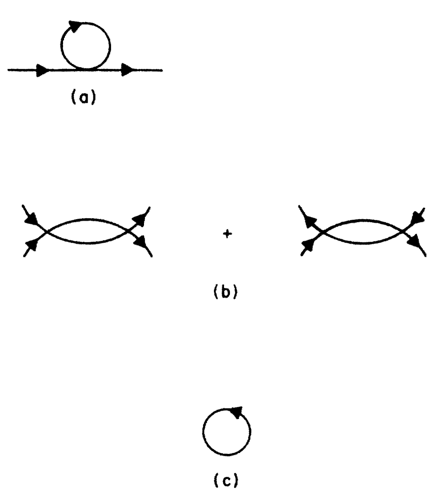

FIG. 2. The diagrams necessary for calculating the coefficients (a) C_μ, (b) C_v, and (c) C_f appearing in Eq. (4.6).

$$\zeta=\frac{-d}{2}+O(v_l^2)\ , \tag{4.8b}$$

and

$$z=2+O(v_l^2)\ . \tag{4.8c}$$

Since $T_l \ll 1$ we can use the approximation

$$\frac{dv_l}{dl}=(2-d)v_l-\frac{K_dv_l^2}{2(1-\mu_l)}\ , \tag{4.9}$$

and thus v flows to zero for $d \geq 2$, implying that after an initial transient perturbation theory in v_l is justified. In the regime of interest $\mu_l \ll 1$, so that we can ignore the μ_l in Eq. (4.9) and obtain in *two dimensions*

$$v_l \approx \frac{K_2}{2l}=\frac{1}{4\pi l}\ , \tag{4.10}$$

for large l. We are thus justified in neglecting everywhere terms of relative order v_l^2 since $\int v_l^2 < \infty$ for $d \geq 2$, and hence only finite renormalizations will arise from these factors. Equation (4.10) is just the statement that the scattering rate is renormalized by multiple scattering at low frequencies and long wavelengths, and is equivalent to (3.1).

From Eq. (4.7) it is apparent that nonzero temperature effects will start to appear when $T_l \sim 1$. We therefore stop the first stage of renormalization at a scale $\tilde{l}$ given by

$$T_{\tilde{l}}=e^{2\tilde{l}}T=1\ , \tag{4.11a}$$

where the absence of a subscript denotes bare parameters, i.e., $T_{l=0}\equiv T$. Note that until this scale is reached we are justified in using the $T=0$ limit of Eq. (4.7) with only overall constant multiplicative errors arising from the regime where $T_l=O(1)$. From Eq. (4.7a) we see that the corrections to the trivial renormalization of μ_l from C_μ are exponentially small for small T_l, so that to the desired order we simply have

$$\mu_{\tilde{l}}=\mu e^{2\tilde{l}}\ , \tag{4.11b}$$

where $\mu_{l=0}\equiv\mu$.

B. Classical regime

At scales of order $\tilde{l}$ and larger, the temperature dependence of the fluctuations becomes important. In particular, the fluctuations will be dominated by the lowest Matsubara frequency, $\omega_n=0$. In order to make the renormalization asymptotically classical it is convenient to make a different choice for the rescaling factor and let the coefficient Γ_l in Eq. (4.2) vary. We now fix T_l at unity for $l>\tilde{l}$, although this choice is certainly not unique. In this *classical regime* we then have

$$z=0+O(v_l^2)\ , \tag{4.12a}$$

$$\zeta=\tfrac{1}{2}(2-d)+O(v_l^2)\ , \tag{4.12b}$$

$$T_l=1\ , \tag{4.12c}$$

$$\frac{d\Gamma_l}{dl}=-2\Gamma_l+O(v_l^2)\ . \tag{4.12d}$$

With $T_l=1$ the important coefficient in Eq. (4.6f) is

$$C_f=K_d[\ln(1-e^{-\Gamma_l(1-\mu_l)})-\ln(1-e^{-\Gamma_l})] \quad (4.13a)$$

$$\approx K_d\ln(1-\mu_l)+O(\Gamma_l)\ . \quad (4.13b)$$

Using the recursion relations Eqs. (4.6) and (4.12) we renormalize until a scale l^* is reached, at which

$$v_{l^*}=1\ . \quad (4.14)$$

We then expect the renormalized critical chemical potential to be $\mu_{cl^*}\sim 1$ since there are no longer any small parameters in the effective Lagrangian (except the strongly irrelevant Γ). In *two dimensions* we have, from (4.6e), (4.10), and (4.12),

$$v_l=e^{2(l-\tilde{l})}v_{\tilde{l}}\approx e^{2(l-\tilde{l})}(4\pi\tilde{l})^{-1}\ , \quad (4.15)$$

where we have ignored the negligible effects of the $O(v_l^2)$ term in Eq. (4.6e) in the classical regime. From Eq. (4.14) we require

$$l^*-\tilde{l}\approx\tfrac{1}{2}\ln\tilde{l}\ . \quad (4.16)$$

We now focus on the renormalization-group equations for the chemical potential and free-energy density in $d=2$. To the required order these are obtained from Eqs. (4.6), (4.7), (4.12), and (4.13) in the limit of small Γ_l:

$$\frac{d\mu_l}{dl}\approx 2\mu_l-2K_2v_l\ , \quad (4.17a)$$

and

$$\frac{df_l}{dl}\approx 2f_l-2K_2\mu_l\ , \quad (4.17b)$$

where we have dropped terms of relative order μ_l since they will remain small during almost all the classical part of the renormalization. We note that μ, f, and v all have the same growth rate so there are "resonances" in the solutions to Eq. (4.17) which give rise to singular contributions to f_{l^*}. We integrate Eq. (4.17a) using Eq. (4.15) for v_l and then substitute into Eq. (4.17b), obtaining for f_{l^*} a part independent of $\mu_{\tilde{l}}$ and a singular $\mu_{\tilde{l}}$-dependent part,

$$f_{\rm sing}(l^*)\approx -K_2\int_{\tilde{l}}^{l^*}e^{2(l^*-l)}\mu_{\tilde{l}}e^{2(l-\tilde{l})}dl$$
$$=-e^{2(l^*-\tilde{l})}K_2\mu_{\tilde{l}}(l^*-\tilde{l})\ . \quad (4.18)$$

From Eq. (4.18), using Eqs. (4.16) and (4.11a) for l^* and $\tilde{l}$, $K_2=1/2\pi$, and Eq. (4.11b) for $\mu_{\tilde{l}}$, we find

$$f_{\rm sing}(l^*)\approx-(4\pi)^{-1}e^{2l^*}\mu\ln\ln(1/T)\ . \quad (4.19)$$

We can now obtain the original free energy from the invariance of the partition function via the relation

$$F(T,\mu,v,f=0,\Gamma=1)$$
$$=Te^{-dl^*}F[T_{l^*}=1,\mu_{l^*},v_{l^*}=1,f_{l^*},\Gamma_{l^*}=0]\ . \quad (4.20)$$

To find the critical density, we differentiate both sides with respect to μ. There are three terms involving, respectively, derivatives of e^{-2l^*}, μ_l^*, and f_l^*. However, since l^* is independent of μ, the first term vanishes and the second yields an uninteresting constant because $\partial F/\partial\mu_{l^*}=-n_{l^*}$ is of order unity at the critical point. We are thus left with only the singular contribution from $\partial f_{l^*}/\partial\mu$ which yields

$$n_c\approx(T/4\pi)\ln\ln(1/T)\ . \quad (4.21)$$

We can invert this to obtain, at fixed density (restoring the units[12]),

$$T_c\approx T_B\approx\frac{(\hbar^2/2m)4\pi n}{\ln\ln(1/na^2)}\ , \quad (4.22)$$

which agrees with Eq. (3.5). The behavior of the superfluid density in the different regimes discussed above can be derived by similar methods.

V. DISCUSSION AND CONCLUSION

The results we have derived in this paper for the transition temperature and superfluid density are only valid in the limit that $\ln\ln(1/na^2)$ is large, which is, of course, impossible to achieve in any physical system. We will argue, however, that there is a much larger regime in which the behavior is universal up to the single parameter,

$$\gamma\equiv n^{2/d}a^2\ , \quad (5.1)$$

where a is a length parametrizing the range of the interactions. Quite generally, the universality of scaling functions results from the slowness of a renormalization-group flow as it passes near a particular fixed point. A well-known example is the Wilson-Fisher fixed point near four dimensions, which describes the usual universality of critical phenomena.[10] Another example is the ideal Bose gas fixed point ($T=\mu=v=0$) discussed in the previous section, which controls the low-temperature thermodynamics of the dilute system. In the interacting case there is also a fixed point associated with the critical behavior (for $2<d<4$) or a fixed line describing the Kosterlitz-Thouless transition in two dimensions. (The renormalization-group flows associated with the above behavior are difficult to represent in a flow diagram because of the unphysical nature of the ideal Bose gas at positive chemical potential.)

The universality of the low-temperature thermodynamics is due to the irrelevance, near the zero-temperature noninteracting fixed point, of all other possible interactions and wave-vector dependences in the Hamiltonian relative to the slow transients associated with γ. Provided the renormalization-group flows pass near to this fixed point (which will be the case when the system is sufficiently dilute) the behavior will be controlled by the slow flows into and away from the fixed point calculated in the previous section. In the limit

$$\gamma\ll 1 \quad (5.2)$$

the other irrelevant operators will decay rapidly and will yield negligible contributions. We therefore expect that for $2\le d<4$ (i.e., down to and including $d=2$) the superfluid density will obey a universal scaling form

$$\rho_s(T)/\rho \approx P(T/T_n,\gamma) . \tag{5.3}$$

In dimensions $d>2$, this is equivalent to the expression derived by Weichman *et al.*[4] In this case, $P(\tau,0)$ is the ideal-gas result with $P(\tau,0)=0$ for $\tau \geq \tau_{c0}$ and

$$\tau_{c0} \sim (d-2) . \tag{5.4}$$

For small but nonzero γ the main changes are for $\tau \ll 1$ and $\tau_c - \tau \ll 1$. In these regimes, there will be crossovers controlled by γ, to a phonon regime and to an X-Y critical regime, respectively. Thus, for example, near to τ_c the superfluid density will have the scaling form

$$P(\tau,\gamma) \approx [(\tau_{c0}-\tau)/\tau_{c0}]\hat{P}_c((\tau_{c0}-\tau)/\gamma^{\psi}) , \tag{5.5}$$

where $\psi=(d-2)^2/2(4-d)=(d-2)/2\phi$, with $\phi=(4-d)/(d-2)$ the crossover exponent away from the ideal-gas behavior.[18] The function $\hat{P}_c(y)$ will vanish for $y \to y_c^+$ as

$$\hat{P}_c(y) \sim (y-y_c)^{(d-2)\nu_{xy}} , \tag{5.6}$$

where y_c yields a shift in $\tau_c(\gamma)$ and ν_{xy} is the d-dimensional X-Y correlation length exponent. The details of the crossover function $\hat{P}_c$ can only be calculated perturbatively in $d=4-\varepsilon$.[18]

In *two dimensions* the behavior is quite different although the form in Eq. (5.3) will still obtain. In this case there is no small parameter, and it is not known how to evaluate the flows analytically from the neighborhood of the zero-temperature fixed point to the Kosterlitz-Thouless fixed line which controls the ordered phase, though the behavior resulting from these flows should still be universal for sufficiently small γ. If we set $\gamma=0$ in Eq. (5.3), the superfluidity must disappear at any nonzero temperature so that $P(\tau>0,0)=0$. Thus, in contrast to $d>2$, the interactions parametrized by γ are needed to obtain any nontrivial result at all. For nonzero γ there will be a critical value $\tau_c(\gamma)$ above which P vanishes, a Kosterlitz-Thouless[6] critical region for $\tau \to \tau_c(\gamma)$ from below, with[19]

$$P(\tau_c(\gamma),\gamma)=(4/\pi)\tau_c(\gamma) , \tag{5.7}$$

and a phonon regime for $\tau \to 0$, as shown in Fig. 1.

The results we have obtained in this paper yield

$$\tau_c(\gamma) \approx 1/\ln\ln(1/\gamma) , \tag{5.8}$$

for γ extremely small such that

$$\tau_c(\gamma) \ll 1 . \tag{5.9}$$

Some form of logarithmic dependence on γ should have been anticipated since the exponent ψ in Eq. (5.5) vanishes as $d \to 2$. In this true asymptotic regime, with Eq. (5.9) satisfied, the relative width of the phonon region at low temperature is $1/\ln(1/\gamma)$ and the relative width of the Kosterlitz-Thouless critical region is $1/\ln\ln(1/\gamma)$. In this region we have

$$P(\tau,\gamma) \approx \left[\frac{\tau_c(\gamma)-\tau}{\tau_c(\gamma)}\right]\hat{P}_c\left[\frac{\tau_c(\gamma)-\tau}{[\ln\ln(1/\gamma)]^{-2}}\right] . \tag{5.10}$$

The crossover function $\hat{P}_c$ will, in $d=2$, contain both the universal jump[19] at $y=y_c$ and a square-root cusp as $y \to y_c^+$. We cannot, however, calculate its detailed form since this would require a solution of the two-dimensional X-Y model in a regime near T_c where the vortex fugacity is of order unity.

Away from the asymptotic region (5.9) which we have studied, but still with $\gamma \ll 1$, we expect P to be a very slow function of γ. Thus for a large range of densities, the form of $\rho_s(T)/\rho$ as a function of T will be almost universal, that is it will depend very weakly on γ. This *quasiuniversality* is associated with the existence of an almost marginal *line* in the hyperspace of Hamiltonians with $T=0$, $\mu=0$, but nonzero interaction coefficients v, w, u, etc., in Eq. (4.2). For $\gamma \ll 1$ the renormalization-group flows will rapidly approach this line and will vary slowly along it, thus providing the quasiuniversality defined by weak dependence of the superfluid density on the parameter γ, over a large range of temperatures and densities. On the other hand, in contrast to the behavior in the extreme asymptotic region (5.9), both the phonon and the critical regimes will occupy a significant fraction of the temperature range. Away from the dilute limit, i.e., for $\gamma=O(1)$, the behavior of ρ_s depends in general on the details of the interactions, and universality only appears in the critical behavior near T_c, and in the phonon region $T \ll \mu$. Even there, only the exponents and certain amplitude ratios are universal but the amplitudes themselves are not.

A potential application of the quasiuniversality we have found is to thin films of ^{4}He, studied by Reppy and co-workers.[20] In this system, the first few layers of helium appear to be inert, or at least not superfluid. A natural (although certainly questionable) assumption is that there is an active density $\rho-\rho_I$, with ρ_I the inert density, which behaves like a Bose gas with a mass m^* and an interaction range a^*. We can then use the results obtained above with $m \to m^*$, $a \to a^*$, $T_n \to T_n^*$ (determined by m^*), and $\rho \to \rho-\rho_I$. We thus expect that as long as $\rho-\rho_I$ is small so that, say, $T_c \ll 1$ K, the superfluid density normalized by $\rho_s(0)$, which by assumption is $\rho-\rho_I$, will be quasiuniversal over a wide range of T_c values. Eventually, if T_c could be reduced to the asymptotic regime with $\ln\ln(1/\gamma) \gg 1$, the form would change, but this is certainly not feasible experimentally. Both the effects of the underlying ^{4}He layers and potential effects of disorder[21] are an interesting and important area for future theoretical and experimental investigation.

We conclude by noting that another system which has been conjectured to exhibit two-dimensional dilute Bose gas behavior occurs in the resonating-valence-bond theory of high-T_c superconductors.[22]

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APPENDIX

The dilute gas in $d=2$ is described by Eqs. (2.3), (3.3), and (2.11) which we rewrite as

$$\frac{8\pi n}{\mu}=\ln\left[\frac{8\pi n}{\mu}\right]+\ln\left[\frac{1}{4\pi n a^2}\right]-1-\frac{2T}{\mu}I_1(\mu/T)\ , \tag{A1}$$

$$\frac{8\pi\rho_s}{\mu}=\frac{4\pi n}{\mu}-\frac{T}{\mu}I_2(\mu/T)\ , \tag{A2}$$

$$I_j(\alpha)=\int_0^\infty dx\left[\frac{(x^2+\alpha^2)^{1/2}-\alpha}{(x^2+\alpha^2)^{1/2}}\right]\frac{(xe^x)^{j-1}}{(e^x-1)^j}\ . \tag{A3}$$

We wish to evaluate $\rho_s(T)$ in the limit (1.2). The asymptotic forms of I_j are

$$I_j(\alpha)\approx\ln(1/\alpha)+\bar{c}_j,\quad \alpha\ll 1 \tag{A4a}$$

$$I_1(\alpha)\approx\tfrac{1}{3}I_2(\alpha)\approx\zeta(3)/\alpha^2+O(\alpha^{-3}),\quad \alpha\gg 1 \tag{A4b}$$

where the $\bar{c}_j$ are numerical constants. We first show that $\mu(T)$ changes only by a factor of order unity in the range $0\le T\le T_B$. At $T=0$ the second term on the right-hand side of (A1) dominates and we have[7]

$$\mu(0)\approx 8\pi n/\ln(1/\gamma)=2T_n/\ln(1/\gamma)\ . \tag{A5}$$

At T_B we use Eqs. (A1), (A2), and (A4a) to find

$$\mu(T_B)=4T_n/\ln(1/\gamma)\ . \tag{A6}$$

This result is consistent with our assumption that $\mu(T)\ll T_B$ for all T, so that Eq. (A2) implies $T_B=T_n/\ln(T_B/\mu_c)$ or

$$T_B\approx T_n/\ln\ln(1/\gamma)\ , \tag{A7}$$

the result quoted in Eq. (3.5). At low temperatures $T<\mu(0)$, Eq. (A2) yields (1.5), and in the intermediate temperature domain $\mu(0)\ll T<T_B$, Eq. (A2) yields

$$1-\rho_s/\rho\approx(T/T_n)\ln(T/\mu)\approx T/T_B\ , \tag{A8}$$

where we have used (A4a) and (A7).

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