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Fluctuation and bifurcation of the path described by generalized random walks

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With the aid of the recursion relation for generalized random walks (GRW's), a model is proposed which shows fluctuation and bifurcation of a path. The jumping probabilities of the GRW's are specified by $P^\alpha(m+\alpha\cdot 1,N;\Theta_N)=P_0^\alpha+(\alpha\cdot 1)\frac{1}{2}\Lambda_N\exp(\xi_m/M)$ $[\Theta_N=(\Lambda_N,\xi)]$ with $\alpha\cdot 1=1,0,-1$ as $\alpha=+,0,-$, respectively. The P_0^α's are the usual jumping probabilities between sites m and $m+\alpha\cdot 1$. The function $\Lambda_N\exp(\xi_m/M)$ represents deviations from the usual processes. Λ_N is a global coupling function of N, while ξ is a parameter between the walker and the environment. M is a range of sites which the walker can visit. For a weakly coupled case, $\Lambda_N\exp(\xi_m/M)\simeq\Lambda_N(1+\xi_m/M)$, a linearization yields fluctuation of the path, whereas the form $\Lambda_N\exp(\xi_m/M)$ yields a case showing bifurcation of the path. The analysis is performed by using the path-integral representation of the formal solution to a Fokker-Planck equation derived from the GRW's. A generalization of the present model to a two-dimensional case is given.

I. INTRODUCTION

As a probabilitic description of dynamical processes, random walks are extensively used in various fields.[1,2] One of the features common to nonlinear, nonequilibrium phenomena is fluctuation[3-6] or bifurcation of the path.[7]

In the previous papers,[8-11] generalized random walks GRW's were proposed to study nonlinear, nonequilibrium processes. For the GRW's, specifications of the jumping probabilities are made such that they represent the processes under consideration. The continuum limit of the recursion relation for the GRW's results in a Fokker-Planck (FP) equation. The FP equation can be solved as an initial-value problem and its fundamental Green's function can be expressed by a path integral. The path-integral representation yields "deterministic paths"; that is, extremum paths obtained by

$$\frac{d}{dt}\left[\frac{\partial L}{\partial \dot{x}}\right]-\frac{\partial L}{\partial x}=0 \quad \left[\dot{x}=\frac{dx}{dt}\right], \tag{1.1}$$

where $L(x,\dot{x})$ is a Lagrangian (or Onsager-Machlup function[12]), and L determines the path-integral representation of the fundamental Green's function,

$$G(x\mid x_0;t)=\lim_{n\to\infty}\left[\frac{1}{\tilde{A}}\right]\times\int\frac{dx_{n-1}}{\tilde{A}}\cdots\times\int\frac{dx_1}{\tilde{A}}\exp\left[-\int_0^t L(x,\dot{x})dt\right], \tag{1.2}$$

where $\tilde{A}$ is a normalization factor. The potential part of L is expressed by the first and the second moments determined by the jumping probabilities of the GRW's.

It is important to note that the walker plays the role of an element of the dynamical system, or a test particle in the (dynamical) system. The jumping probability denotes a transition between the states of the element or particle.

In this paper, we study a dynamical model system which shows fluctuation and path bifurcation. The processes are described by the GRW's in which the jumping probabilities are specified by an exponential form with respect to site m,

$$P^\alpha(m\mid m-\alpha\cdot 1,N;\Theta_N)=P_0^\alpha+\frac{\alpha\cdot 1}{2}\Lambda_N e^{\xi_m/M} \quad [\Theta_N=(\Lambda_N,\xi)], \tag{1.3}$$

with $\alpha\cdot 1=1$, 0, -1 as $\alpha=+$, 0, $-$, respectively. The P_0^α's are the jumping probabilities in the usual random walks, and the second term represents the deviations from the usual processes. The first term P_0^α denotes the difference in the homogeneous field due to the presence of an external force, and the second term represents the contribution due to the nonsteady, nonhomogeneous part of the environment characterized by the function Λ_N and parameter ξ. We study two cases: a weak-coupling case in which ξ is small enough, $\exp(\xi_m/M)\cong 1+\xi_m/M$, and a strong-coupling case in which ξ is not so small. Physically, this means the random walks are influenced weakly or strongly by a contribution expressed by $\Lambda_N\exp(\xi_m/M)$.

II. BASIC EQUATIONS

To study the behavior of "elements" which constitute the dynamical system, we start with a recursion relation of generalized random walks. The recursion relation of the GRW's reads

$$W(m,N)=\sum_{\alpha=\pm,0} P^{\alpha}(m \mid m-\alpha\cdot 1,N-1;\Theta_{N-1}) \times W(m-\alpha\cdot 1,N-1) \quad [\Theta_{N-1}=(\Lambda_{N-1},\xi)]\ , \tag{2.1}$$

$$\sum_{\alpha=\pm,0} P^{\alpha}(m+\alpha\cdot 1 \mid m,N;\Theta_N)=1\ , \tag{2.2}$$

with $\alpha\cdot 1=1,0,-1$ as $\alpha=+,0,-$, respectively. $P^{\alpha}(m \mid m-\alpha\cdot 1,N-1;\Theta_{N-1})$'s are functions of site $m\in[-L,L]$, step $N\in[0,\infty)$, and Θ_{N-1} is a symbol showing the coupling between the walker (element) and the environment. The important thing is a specification of the jumping probabilities P^{α}'s such that the specification represents the processes under consideration.

We consider processes in which the jumping probabilities are expressed by[13]

$$P^{\alpha}(m \mid m+\alpha\cdot 1,N;\Theta_N)=P_0^{\alpha}+\frac{\alpha\cdot 1}{2}\Lambda_N e^{\xi_m/M} \quad [\Theta_N=(\Lambda_N,\xi),\ \ M=2\mid L\mid] \tag{2.3}$$

where the second term is a function taking a small value and representing deviations from the usual processes denoted by P_0^{α}. Λ_N is a global coupling function of N, and ξ is a parameter between the walker and the environment, and M denotes a range of sites which the walker can visit.

Replacing m,N by x $(=ma)$, t $(=Nt_0)$, and taking the continuum limit ($a\to 0$, $t_0\to 0$, while a^2/t_0 remains fixed) in the recursion relation (2.1) results in a Fokker-Planck equation expressed by

$$\frac{\partial w}{\partial t}=-\frac{\partial}{\partial x}K^{(1)}(x,t;\beta)w+\tfrac{1}{2}\frac{\partial^2}{\partial x^2}K_0^{(2)}w \quad [\beta=(\lambda(t),\xi)]\ , \tag{2.4}$$

where

$$\begin{bmatrix} K^{(1)}(x,t;\beta) \\ K_0^{(2)} \end{bmatrix}=\begin{bmatrix} \frac{a}{t_0}[p^{+}(x,t;\beta)-p^{-}(x,t;\beta)] \\ \frac{a^2}{t_0}[p^{+}(x,t;\beta)+p^{-}(x,t;\beta)] \end{bmatrix}, \tag{2.5}$$

and

$$p^{\alpha}(x,t;\beta)=p_0^{\alpha}+\frac{\alpha\cdot 1}{2}\lambda(t)e^{(\xi x/Ma)} \quad \left[\sum_{\alpha=\pm,0} p^{\alpha}(x,t;\beta)=1,\ \ \beta=(\lambda(t),\xi)\right]. \tag{2.6}$$

Here note that $K^{(1)}$ contains a symbol β $[=(\lambda(t),\xi)]$ showing the coupling between the walker and the environment, while $K_0^{(2)}$ is constant as seen in (2.6), where $\alpha\cdot 1$ takes 1 and -1 as $\alpha=+$ and $-$, respectively. The quantities w, p^{α}, p_0^{α}, and $\lambda(t)$ are continuous functions corresponding to W, P^{α}, P_0^{α}, and Λ_N, respectively.

To get a fundamental Green's function for (2.4), we express w and t as follows:

$$\tilde{w}(x,\tilde{t})=K_0^{(2)}w(x,t)\ , \quad \tilde{t}=K_0^{(2)}t \quad [K_0^{(2)}=(a^2/t_0)(p_0^{+}+p_0^{-})]\ . \tag{2.7}$$

The FP equation (2.4) then becomes

$$\frac{\partial\tilde{w}(x,\tilde{t})}{\partial\tilde{t}}=-\frac{\partial}{\partial x}c^{(1)}(x,\tilde{t};\beta)\tilde{w}+\frac{1}{2}\frac{\partial^2}{\partial x^2}\tilde{w}\ , \quad c^{(1)}(x,\tilde{t};\beta)=K^{(1)}(x,\tilde{t}/K_0^{(2)};\beta)/K_0^{(2)}\ . \tag{2.8}$$

We consider following conditions:

$$\tilde{w}(x,0)=K_0^{(2)}\delta(x-x_0)\ , \quad \tilde{w}(\pm\infty,\tilde{t})=0\ . \tag{2.9}$$

Specifically, for the Uhlenbeck-Ornstein (UO) processes in which $c^{(1)}$ is linearized as $c^{(1)}(x,\tilde{t};\beta)=-c_0x$ ($c_0=$const), the Green's function G_0 is exactly expressed in the form

$$G_0(x \mid x_0;\Delta\tilde{t})=\left[\frac{c_0}{\pi(1-e^{-2c_0\Delta\tilde{t}})}\right]^{1/2} \times\exp\left[-\frac{c_0(x-x_0e^{-c_0\Delta\tilde{t}})^2}{1-e^{-2c_0\Delta\tilde{t}}}\right]. \tag{2.10}$$

When $\Delta\tilde{t}$ is very small, we can rewrite G_0 as follows:

$$G_0(x \mid x_{n-1};\Delta\tilde{t})=\left[\frac{1}{2\pi\Delta\tilde{t}}\right]^{1/2}\exp-\left[\frac{\Delta\tilde{t}}{2}\left[\frac{x-x_{n-1}}{\Delta\tilde{t}}-c^{(1)}(\bar{x},\tilde{t}_{n-1};\beta)\right]^2\right] \times\exp\left[-\frac{\Delta\tilde{t}}{2}\frac{\partial}{\partial x}c^{(1)}(x,\tilde{t}_{n-1};\beta)\right]\equiv\frac{1}{\tilde{A}}g(x \mid x_{n-1};\Delta\tilde{t})\ , \tag{2.11}$$

where

$$\tilde{t}_{n-1}=\tilde{t}-\Delta\tilde{t},\quad \tilde{A}=(2\pi\Delta\tilde{t})^{1/2},\quad \beta=(\lambda(\tilde{t}),\xi),\quad \tilde{x}=\tfrac{1}{2}(x+x_{n-1}) ,$$

and here we have adopted the midpoint rule in the evaluation of the function $c^{(1)}(x,\tilde{t};\beta)$. Finally, the path-integral representation for a solution of the FP equation is given by

$$w(x,t)=\frac{1}{K_0^{(2)}}\int dx_0 G(x\mid x_0;t)K_0^{(2)}w(x_0,0) \tag{2.12}$$

and

$$\begin{aligned} G(x\mid x_0;t)&=\lim_{n\to\infty}\frac{1}{\tilde{A}}\int\frac{dx_{n-1}}{\tilde{A}}\cdots\int\frac{dx_1}{\tilde{A}}g(x\mid x_{n-1};\Delta\tilde{t})\cdots g(x_1\mid x_0;\Delta\tilde{t}) ,\\ &=\lim_{n\to\infty}\frac{1}{\tilde{A}}\int\frac{dx_{n-1}}{\tilde{A}}\cdots\int\frac{dx_1}{\tilde{A}}e^{-S[x(t)\mid x_0(0)]}\\ &\equiv\int_{x_0,0}^{x,t}dx(t)e^{-S[x(t)\mid x_0(0)]} , \end{aligned} \tag{2.13}$$

where

$$S[x(t)\mid x_0(0)]=\int_0^t L(x,\dot{x})dt \quad \left[\dot{x}=\frac{dx}{dt}\right] , \tag{2.14}$$

$$L(x,\dot{x})=\frac{\dot{x}^2}{2K_0^{(2)}}-\dot{x}\frac{K^{(1)}}{K_0^{(2)}}-V(x,t;\beta) , \tag{2.15}$$

$$U(x,t;\beta)(=-V)=\frac{K_0^{(2)}}{2}\left[\frac{\partial}{\partial x}\left[\frac{K^{(1)}}{K_0^{(2)}}\right]+\left[\frac{K^{(1)}}{K_0^{(2)}}\right]^2\right] . \tag{2.16}$$

V is a potential for the trajectory. We call function U $(=-V)$ a "field" for G, since U characterizes behaviors of G. For a general case that $K_0^{(2)}$ becomes $K^{(2)}(x)$, we get an expression corresponding to (2.13); see the Appendix and Ref. 10.

III. SPECIALIZED CASES

As mentioned in Sec. II, we consider the processes in which the functional form for p^α is an exponential form with respect to site x $(=ma)$; see (2.6). The function $K^{(1)}$ and the constant $K_0^{(2)}$ then become

$$K^{(1)}(x,t;\beta)=K_0^{(1)}+J(x,t;\beta)\quad(\beta=(\lambda(t),\xi)) , \tag{3.1}$$

$$K_0^{(1)}=\frac{a}{t_0}(p_0^+-p_0^-),\quad J(x,t;\beta)=\frac{a}{t_0}\lambda(t)e^{\xi x/(Ma)} , \tag{3.2}$$

$$K_0^{(2)}=\frac{a^2}{t_0}(p_0^++p_0^-) . \tag{3.3}$$

The Lagrangian $L(x,\dot{x})$ reads

$$L(x,\dot{x})=\frac{\dot{x}^2}{2K_0^{(2)}}-\dot{x}\frac{K_0^{(1)}+J}{K_0^{(2)}}+\frac{1}{2}\left[\frac{\partial}{\partial x}J+\frac{(K_0^{(1)}+J)^2}{K_0^{(2)}}\right] \tag{3.4}$$

from (2.15) and (2.16). The Euler-Lagrangian (EL) equation results in

$$\frac{d}{dt}\left[\frac{\partial L}{\partial\dot{x}}\right]-\frac{\partial L}{\partial x}=0 , \tag{3.5}$$

and it gives deterministic paths; that is, extremum paths. Substitution of (3.4) into (3.5) leads to

$$\frac{\ddot{x}}{K_0^{(2)}}-\frac{1}{K_0^{(2)}}\left[\frac{\partial J}{\partial t}\right]-\frac{1}{K_0^{(2)}}(K_0^{(1)}+J)\frac{\partial J}{\partial x}-\frac{1}{2}\frac{\partial^2 J}{\partial x^2}=0 . \tag{3.6}$$

Here note that

$$K_0^{(1)}=\text{const} . \tag{3.7}$$

A special case that $K_0^{(1)}=0$ means that an end-to-end distance for the walker's path is estimated by simple random walks having equal jumping probabilities $p_0^+=p_0^-$ in both directions. The difference between p^α and p_0^α [Eq. (2.6)] [or p^α and p_0^α in (2.3)] disappears when $\lambda(t)=0$.

In what follows, we consider two specialized cases for $J(x,t;\beta)[\beta=(\lambda(t),\xi)]$,

$$J(x,t;\beta)=\frac{a}{t_0}\lambda(t)\left[1+\frac{\xi x}{Ma}\right] \tag{3.8}$$

and

$$J(x,t;\beta)=\frac{a}{t_0}\lambda(t)e^{\xi x/(Ma)} . \tag{3.9}$$

The $\lambda(t)$ is a very small function of t. A positivity of $1/Ma$ in the definition of $p^\alpha(x,t;\beta)$ [see (2.6)], that is, $0\le p^\alpha(x,t;\beta)\le 1$, restricts a range of the function $\lambda(t)$,

$$|\lambda(t)|\,e^{\xi x/(Ma)}<2(1-p_0^\alpha) . \tag{3.10}$$

For the first case, (3.8), the local coupling parameter ξ is so small that $p^\alpha(x,t;\beta)$ in (2.6) is approximated by the linearization

$$\exp(\xi x/Ma) \cong 1+\xi x/Ma \ .$$

The walker's motions are expressed by a "harmonic field," $U(x,t;\beta)$ $[=-V(x,t;\beta)]$,

$$U(x,t;\beta)=\frac{\lambda(t)}{2}\left[\left[\frac{\xi}{Mt_0}\right]+\frac{\lambda(t)}{K_0^{(2)}}\left[\frac{a}{t_0}\right]^2\left[1+\frac{\xi x}{Ma}\right]^2\right] . \tag{3.11}$$

For the second case, (3.9), the local coupling parameter ξ is not small, that is, a strong coupling. The walker's motions are expressed by an "exponential field," $U(x,t;\beta)$ $[=-V(x,t;\beta)]$

$$U(x,t;\beta)=\frac{\lambda(t)}{2}e^{\xi x/Ma}\left[\left[\frac{\xi}{Mt_0}\right]+\frac{\lambda(t)}{K_0^{(2)}}\left[\frac{a}{t_0}\right]^2 e^{\xi x/Ma}\right] . \tag{3.12}$$

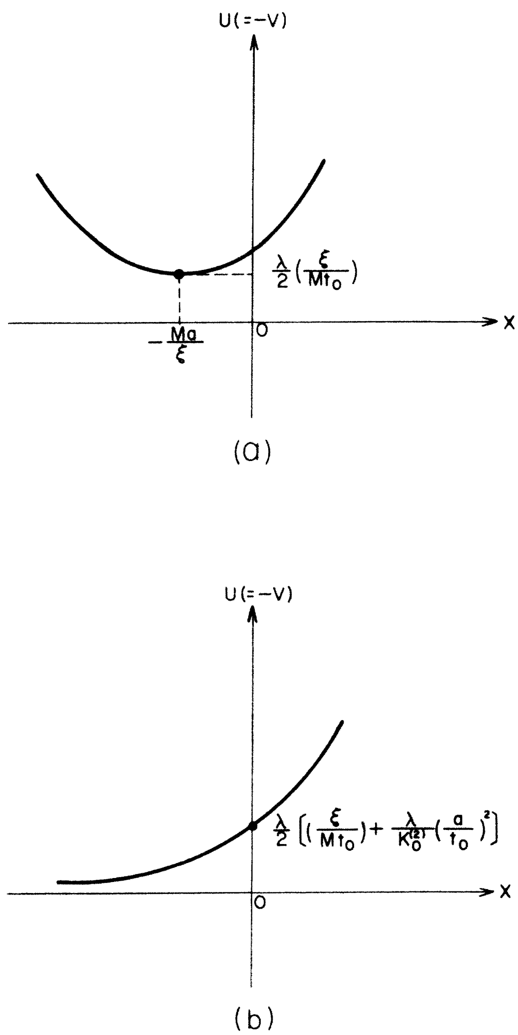

FIG. 1. Fields $U(x,t;\beta)$ $[=-V(x,t;\beta)]$ described by (3.11) and (3.12). The field specifies a behavior of G, while V is a potential for a trajectory determined by the extremum path.

Figure 1 shows behaviors of a field for G; that is, a potential for trajectory, $V(x,t;\beta)$ $[=-U(x,t;\beta)]$, given by (3.11) and (3.12).

IV. FLUCTUATION OF PATH

In the first case specialized by (3.11), we expect fluctuations of path in the field (or potential). The Lagrangian has a form

$$L(x,\dot{x})=\frac{\dot{x}^2}{2K_0^{(2)}}-\frac{\dot{x}}{K_0^{(2)}}[e^{(0)}(t)+e^{(1)}(t)x]+f^{(0)}(t)+f^{(1)}(t)x+f^{(2)}(t)x^2 , \tag{4.1}$$

where $e^{(n)}(t)$ $(n=0,1)$ and $f^{(n)}(t)$ $(n=0,1,2)$, are

$$\begin{aligned} &e^{(0)}(t)=K_0^{(1)}+\lambda(t)\frac{a}{t_0}, \quad e^{(1)}(t)=\lambda(t)\left[\frac{\xi}{Ma}\right]\left[\frac{a}{t_0}\right] ,\\ &f^{(0)}(t)=\tfrac{1}{2}e^{(1)}(t)+\frac{1}{2K_0^{(2)}}[e^{(0)}(t)]^2 ,\\ &f^{(1)}(t)=\frac{1}{K_0^{(2)}}e^{(0)}(t)e^{(1)}(t) ,\\ &f^{(2)}(t)=\frac{1}{2K_0^{(2)}}[e^{(1)}(t)]^2 . \end{aligned} \tag{4.2}$$

To study the fluctuations, we introduce a new variable,

$$x(t)=\bar{x}(t)+\eta(t) , \tag{4.3}$$

instead of x, where $\eta(0)=\eta(n\Delta t)=0$. The extremum path denoted by $\bar{x}$ is a special solution of the EL equation given by

$$\frac{\ddot{x}}{K_0^{(2)}}-\left[2f^{(2)}(t)+\frac{\dot{e}^{(1)}(t)}{K_0^{(2)}}\right]x-\frac{\dot{e}^{(0)}(t)}{K_0^{(2)}}-f^{(1)}(t)=0 . \tag{4.4}$$

Here we consider a case that boundary conditions $x(0)=0$ and $x(t)=x$ are set. We denote the $S[x(t)\,|\,x(0)]$ by $S[x(t)]$. The Taylor expansion of $S[\bar{x}+\eta]$ in (2.13) around $\bar{x}$ yields $S[\bar{x}]+(1/2!)\delta^2 S$, where

$$\delta^2 S=\int_0^t\left[\frac{\partial^2 L}{\partial \dot{x}^2}\dot{\eta}^2+2\frac{\partial^2 L}{\partial x\,\partial \dot{x}}\eta\dot{\eta}+\frac{\partial^2 L}{\partial x^2}\eta^2\right]dt . \tag{4.5}$$

The definition of the extremum path makes the first derivative of S zero. After substituting (4.1) into (4.5), the t integration for $\eta(t)\dot{\eta}(t)$ $(=(d/dt)[\eta^2(t)/2])$ becomes zero from the boundary condition; $\eta(0)=\eta(n\Delta t)=0$. Finally, we have

$$G(x\,|\,0;t)=e^{-S[\bar{x}(t)]}\tilde{G}(0\,|\,0;t) , \tag{4.6}$$

where

$$\tilde{G}(0\mid 0;t)=\lim_{n\to\infty}\left[\frac{1}{2\pi K_0^{(2)}\Delta t}\right]^{n/2}\int d\eta^{n-1}\exp\left[-\sum_{i=1}^{n-1}\left[\frac{1}{2K_0^{(2)}\Delta t}(\eta_i-\eta_{i-1})^2+\Delta t f_{i-1}^{(2)}\eta_{i-1}^2\right]\right]$$
$$=\lim_{n\to\infty}\left[\frac{1}{2\pi K_0^{(2)}\Delta t}\right]^{n/2}\int d\eta^{n-1}e^{-\eta^T Q\eta}\quad (d\eta^{n-1}=d\eta_1 d\eta_2\cdots d\eta_{n-1})\ . \tag{4.7}$$

A superscript "T" on $\eta^T[=(\eta_1,\eta_2,\ldots,\eta_{n-1})]$ signifies a transposed form of a column vector η. A symbol Q is an $(n-1)\times(n-1)$ matrix of elements given by

$$Q=\frac{1}{2K_0^{(2)}\Delta t}\begin{bmatrix}2 & -1 & & & 0\\ -1 & 2 & -1 & & \\ & -1 & 2 & \ddots & \\ & & \ddots & \ddots & -1\\ 0 & & & -1 & 2\end{bmatrix}+\Delta t\begin{bmatrix}f_1^{(2)} & & & & 0\\ & f_2^{(2)} & & & \\ & & f_3^{(2)} & & \\ & & & \ddots & \\ 0 & & & & f_{n-1}^{(2)}\end{bmatrix}\ . \tag{4.8}$$

In getting the matrix form of (4.7), we have used the boundary condition for $\eta_i=\eta(i\Delta t)$, $\eta_0=\eta_n=0$. With the aid of an unitary matrix reducing Q to a diagonal form, we can express the $\tilde{G}$ as follows:

$$\tilde{G}(0\mid 0;t)=\left[\frac{1}{2\pi K_0^{(2)}}\right]^{1/2}\lim_{n\to\infty}\left[\frac{1}{\Delta t(2K_0^{(2)}\Delta t)^{n-1}\mid Q\mid}\right]^{1/2},$$
$$(\mid Q\mid=\det Q)\ . \tag{4.9}$$

Setting $(2K_0^{(2)}\Delta t)^{n-1}\mid Q\mid=F_{n-1}$ and decomposing F_{n-1} by its co-factor, we have

$$F_k=2(1+K_0^{(2)}\Delta t^2 f_k^{(2)})F_{k-1}-F_{k-2}\quad (k=2,3,\ldots,n-1), \tag{4.10}$$

$$F_1=2(1+K_0^{(2)}\Delta t^2 f_1^{(2)})\ , \tag{4.11}$$

$$F_0=1\ . \tag{4.12}$$

The continuum limit $\Delta t\to 0$ in (4.10) yields a corresponding differential equation for $F(t)$ $(=\lim_{\Delta t\to 0}\Delta t F_k)$,

$$\frac{d^2F(t)}{dt^2}=2K_0^{(2)}f^{(2)}(t)F(t)\quad (=[e^{(1)}(t)]^2F(t)) \tag{4.13}$$

and the conditions (4.11) and (4.12) are expressed as its initial conditions for $F(t)$,

$$F(0)\quad (=\lim_{\Delta t\to 0}\Delta t F_0)=0\ , \tag{4.14}$$

$$\left.\frac{dF(t)}{dt}\right|_{t=0}=1\ . \tag{4.15}$$

With the aid of the solution $F(t)$ and (4.9), we rewrite the G in (4.6) as follows:

$$G(x\mid 0;t)=[2\pi K_0^{(2)}F(t)]^{-1/2}\exp\{-S[\bar{x}(t)]\}\ , \tag{4.16}$$

where $F(t)$ represents fluctuations of the extremum path. A constant λ_0 for $\lambda(t)$ in (4.2) yields

$$f_0^{(2)}=\frac{1}{2K_0^{(2)}}(e_0^{(1)})^2\quad \left[e_0^{(1)}=\lambda_0\left[\frac{a}{t_0}\right]\left[\frac{\xi}{Ma}\right]\right]\ . \tag{4.17}$$

In this case, a solution of (4.13) satisfying the conditions (4.14) and (4.15) becomes

$$F(t)=\frac{\sinh(e_0^{(1)}t)}{e_0^{(1)}} \tag{4.18}$$

and the expression (4.16) results,

$$G(x\mid 0;t)=\left[\frac{e_0^{(1)}}{2\pi K_0^{(2)}\sinh(e_0^{(1)}t)}\right]^{1/2}e^{-S[\bar{x}(t)]}\ . \tag{4.19}$$

Specifically, a case that

$$K_0^{(1)}+\left[\frac{a}{t_0}\right]\lambda_0=0 \tag{4.20}$$

results in

$$p_0^+-p_0^-=\lambda_0\ . \tag{4.21}$$

In this case $e^{(0)}(t)$ and $f^{(n)}(t)$ $(n=0,1)$ vanish in (4.2). Since the difference between p_0^+ and p_0^- arises from an inhomogeneity of the environment, we may contact the difference with the constant λ_0 representing the global coupling with the environment.

An action $S[\bar{x}(t)]$ for the case specified by (4.20) or (4.21) is easily evaluated by

$$G(x\mid 0;t)=\left[\frac{e_0^{(1)}}{2\pi K_0^{(2)}\sinh(e_0^{(1)}t)}\right]^{1/2}$$
$$\times\exp\left[-\frac{x^2e_0^{(1)}}{2K_0^{(2)}}[\coth(e_0^{(1)}t)-1]\right]\ . \tag{4.22}$$

This form has a behavior similar to the density matrix for a oscillator in the harmonic potential.[14] Figure 2 shows the behaviors of (4.20).

An initial distribution concentrated around $\bar{x}(0)$ loses its original shape due to the diffusion effect in a "harmonic" field U $(=-V)$, Figs. 2(a)–2(c): The behaviors of G are characterized by the field U. The path fluctuation around the extremum path $\bar{x}(t)$ is specified by $F(t)$, see (4.16). The present method is applicable to diffusion processes in a bistable potential.[15]

The second case specialized by (3.9) or (3.12) denotes processes influenced strongly by the field U: The coupling parameter ξ is not so small. The EL equation giving an extremum path becomes

$$\frac{\ddot{x}}{K_0^{(2)}} - \frac{K^{(1)}}{K_0^{(2)}}\left[\frac{dK^{(1)}}{dx}\right] - \frac{1}{2}\frac{d^2K^{(1)}}{dx^2} = 0 , \quad (4.23)$$

that is,

$$\ddot{x} - \left[\lambda_0 \frac{a}{t_0}\right]^2 \gamma e^{2\gamma x} - \frac{K_0^{(2)}}{2}\left[\lambda_0 \frac{a}{t_0}\right]^2 \gamma^2 e^{\gamma x} = 0 , \quad (4.24)$$

where $\gamma = \xi / Ma$.

In this case we have obtained path bifurcations.[10]

V. FLUCTUATIONS IN A TWO-DIMENSIONAL CASE

For the dynamical processes in higher dimensions, we have to restart with a recursion relation for the GRW's in a d-dimensional space. For simplicity, we consider the processes in a two-dimensional (2D) case and study fluctuations of path corresponding to that in Sec. III.

The recursion relation for the 2D space $[-L_1, L_1] \times [-L_2, L_2]$ reads

$$W(m_1,m_2,N) = \sum_{\alpha=\pm}{}' P_1^\alpha(m_1,m_2 \mid m_1-\alpha\cdot 1, m_2, N-1;\Theta_{N-1}) W(m_1-\alpha\cdot 1, m_2, N-1)$$
$$+ \sum_{\alpha=\pm}{}' P_2^\alpha(m_1,m_2 \mid m_1, m_2-\alpha\cdot 1, N-1;\Theta_{N-1}) W(m_1, m_2-\alpha\cdot 1, N-1)$$
$$(m_1 \in [-L_1, L_1], m_2 \in [-L_2, L_2]) . \quad (5.1)$$

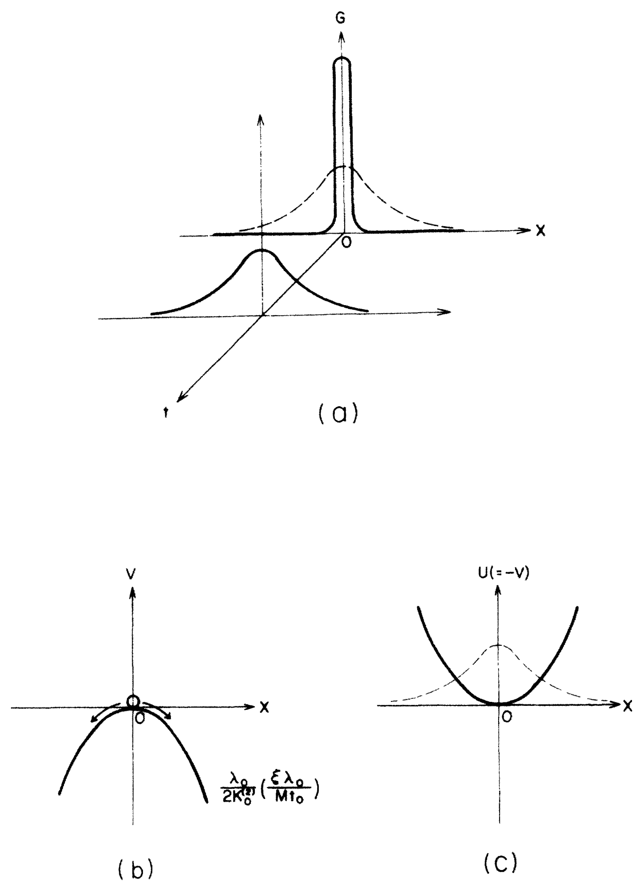

FIG. 2. (a) Behaviors of a propagator G given by (4.20). (b) Behaviors of a particle in an inverse "harmonic" potential. (c) A harmonic field for G.

Note that a prime in summation indicates the case $\alpha = 0$ is excluded; the symbol $\alpha\cdot 1$ takes 1 and -1 as $\alpha = +$ and $-$, respectively (Fig. 3).

Jumping probabilities between sites, P_1^α and P_2^α, denote transition rates satisfying the normalization condition;

$$\sum_{\alpha=\pm}{}' P_1^\alpha(m_1+\alpha\cdot 1, m_2 \mid m_1, m_2, N;\Theta_N)$$
$$+ \sum_{\alpha=\pm}{}' P_2^\alpha(m_1, m_2+\alpha\cdot 1 \mid m_1, m_2, N;\Theta_N) = 1 ,$$
$$\Theta_N = (\Lambda_N, \xi_1, \xi_2) . \quad (5.2)$$

The symbols ξ_i and Λ_N, etc. have the same meanings as those in the 1D case. Suppose that the jumping probabilities are expressed by

$$\begin{bmatrix} P_1^\alpha(m_1,m_2 \mid m_1-\alpha\cdot 1, m_2, N;\Theta_N) \\ P_2^\alpha(m_1,m_2 \mid m_1, m_2-\alpha\cdot 1, N, \Theta_N) \end{bmatrix} = \begin{bmatrix} P_{1,0}^\alpha \\ P_{2,0}^\alpha \end{bmatrix} + \frac{\alpha\cdot 1}{2}\Lambda_N \begin{bmatrix} e^{\xi_1 m_1/M_1} \\ e^{\xi_2 m_2/M_2} \end{bmatrix} . \quad (5.3)$$

Here we approximate $(\Lambda_N, \xi_1, \xi_2)$ in P_i^α by (Λ_N, ξ_i).

With the aid of continuum variables defined by

$$x^i = m_i a_i \quad \text{and} \quad t = N t_0 , \quad (5.4)$$

we get a FP equation for the 2D case,

$$\frac{\partial w(x', x^2;t)}{\partial t} = -\sum_{t=1}^{2} \frac{\partial}{\partial x^i} K_i^{(1)}(x^1, x^2, t;\beta_i) w + \frac{1}{2}\sum_{i=1}^{2} K_{i,0}^{(2)} \frac{\partial^2}{\partial x^i \partial x^i} w , \quad (5.5)$$

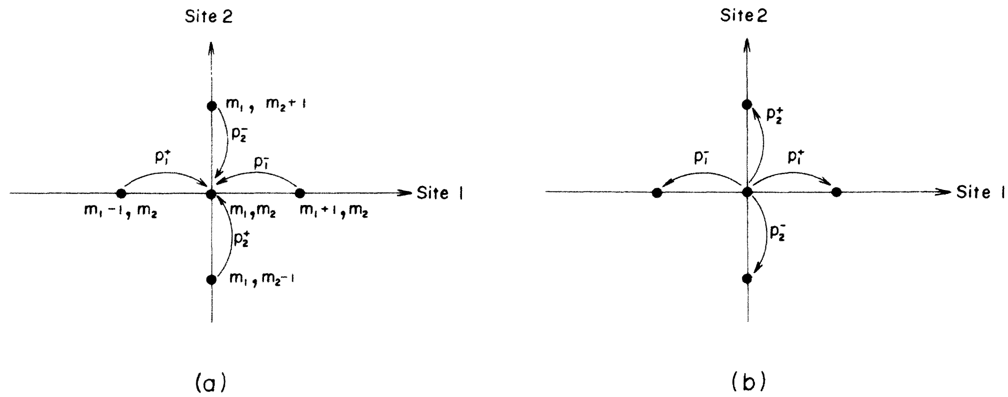

FIG. 3. Jumping probabilities between sites in two dimensions; the probabilities shown in (a) satisfy the normalization condition of (5.2), see (b).

after taking the continuum limit ($a_i \to 0$, $t_0 \to 0$, while a_i^2/t_0 remains fixed) of the recursion relation (5.1), where

$$K_i^{(1)}(x^1,x^2,t;\beta_i)=\frac{a_i}{t_0}[p_i^{+}(x^1,x^2,t;\beta_i)-p_i^{-}(x^1,x^2,t;\beta_i)]\ , \quad (5.6)$$

$$K_{i,0}^{(2)}=\frac{a_i^2}{t_0}(p_{i,0}^{+}+p_{i,0}^{-})\ \ [\beta_i=(\lambda(t),\xi_i)]\ , \quad (5.7)$$

and

$$p_i^{\alpha}(x^1,x^2,t;\beta_i)=p_{i,0}^{\alpha}+\frac{\alpha\cdot 1}{2}\lambda(t)e^{\xi_i x^i/M_i a_i}\ . \quad (5.8)$$

The small letters w,p^{α} represent continuous functions corresponding to discrete ones, W,P^0, respectively. Note that the $K_i^{(1)}$'s contain the components of the coupling denoted by β_i between the walker and the environment, while $K_{i,0}^{(2)}$'s are constant. The FP equation (5.5) does not have cross terms $K_{i,0}^{(2)}(\partial^2/\partial x^i \partial x^j)$ in the present specification for the jumping probabilities.

We express the $K_i^{(1)}$ functions by

$$K_i^{(1)}(x^1,x^2,t;\beta_i)=K_{i,0}^{(1)}+J_i(x^1,x^2,t;\beta_i)\ , \quad (5.9)$$

where

$$K_{i,0}^{(1)}=\frac{a_i}{t_0}(p_{i,0}^{+}-p_{i,0}^{-})\ , \quad (5.10)$$

$$J_i(x^1,x^2,t;\beta_i)=\frac{a_i}{t_0}e^{\xi_i x^i/M_i a_i}\ . \quad (5.11)$$

Following a procedure given in the Appendix, we get a Lagrangian for 2D,

$$L(x^1,x^2,\dot{x}^1,\dot{x}^2)=\sum_{i=1}^{2}\frac{\dot{x}_i^2}{2K_{i,0}^{(2)}}-\sum_{i=1}^{2}\dot{x}^i\frac{K_{i,0}^{(1)}+J_i}{K_{i,0}^{(2)}}+\frac{1}{2}\sum_{i=1}^{2}\left[\frac{\partial}{\partial x^i}J_i+\frac{(K_{i,0}^{(1)}J_i)^2}{K_{i,0}^{(2)}}\right]\ ,\quad (\dot{x}^i=dx^i/dt)\ . \quad (5.12)$$

With the aid of (5.12), we study diffusive motions of a polymer chain trapped in a fictitious tube, see Fig. 4, in which we apply conditions specified by

$$K_{i,0}^{(1)}=\text{const}\ \ [\text{cf. } (4.20) \text{ or } (4.21)]\ , \quad (5.13)$$

$$J_i(x^1,x^2,t;\beta_i)=\frac{a_i}{t_0}\lambda(t)\left[1+\frac{\xi_i x^i}{M_i a_i}\right]\ . \quad (5.14)$$

This represents a situation similar to the one-dimensional case. Namely, the first condition (5.13) states that an end-to-end distance for a polymer chain is estimated by simple random walks having unequal jumping probabilities in the x and y axes, respectively.[16] The second condition—that is, a linearization of field—specifies diffusive motions of a single polymer chain trapped in the tube.[17]

After some calculations similar to the 1D case, we get an expression showing fluctuations around an extremum path described by $\exp\{-S[\bar{x}^1(t),\bar{x}^2(t)]\}$,

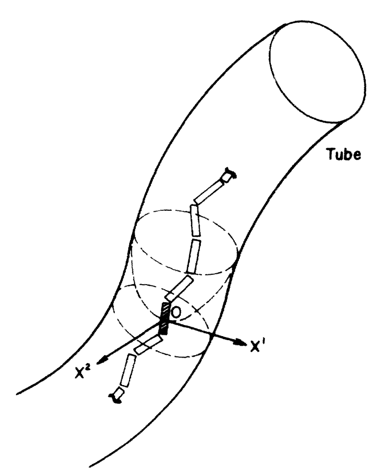

FIG. 4. A polymer chain trapped in a fictitious tube. A dotted concave curve denotes a field for a hatched monomer.

$$G(x^1,x^2\,|\,0,0,t)=e^{-S[\bar{x}^1(t),\bar{x}^2(t)]}\tilde{G}(0,0\,|\,0,0,t)$$
$$=\prod_{i=1}^{2}\{[2\pi K_{i,0}^{(2)}F_i(t)]^{-1/2}e^{-S[\bar{x}^i(t)]}\}\ , \tag{5.15}$$

where the $F_i(t)$'s characterize the fluctuations.

Specifically, for a case that $\lambda(t)=\lambda_0$ constant, we can get

$$F_i(t)=\frac{1}{\lambda_0\xi_i/M_it_0}\sinh\left[\left(\frac{\lambda_0\xi_i}{M_it_0}\right)t\right]\ ; \tag{5.16}$$

$$G(x^1,x^2\,|\,0,0,t)=\prod_{i=1,2}\left[\frac{\lambda_0\xi_i/M_it_0}{2\pi K_{i,0}^{(2)}\sinh[(\lambda_0\xi_i/M_it_0)t]}\right]^{1/2}e^{-S[\bar{x}^i(t)]}\ . \tag{5.17}$$

For a simpler case, similar to the 1D case, the factor $\exp\{-S[x^i(t)]\}$ can be evaluated explicitly, cf. (4.20).

VI. CONCLUDING REMARKS AND DISCUSSION

From a quite general point of view, we studied behaviors of a walker ("element") which is influenced by a "field" (or potential) in dynamical systems. Simple model processes showing fluctuation and bifurcation of the path were proposed. In one dimension, stochastic motions of particle in a "linearized field" yield a path fluctuation. In a strong field representing a sink or source, random motions became path bifurcations as studied in Ref. 10. For two dimensions the former model processes were applied to specify diffusive motions of a single polymeric chain dissolved in a solvent and trapped in a fictitious tube. Trajectories of the diffusive motion were expressed by the path-integral representation to a Fokker-Planck equation derived from generalized random walks: The continuum limit of the recursion relation of the GRW's gave us the corresponding FP equation, to which a formal solution is expressed by the path-integral representation.

Actual calculations were performed by taking the simplified cases, where we set

$$K_0^{(1)}=\frac{a}{t_0}(p_0^+-p_0^-)=\text{const}\quad\left[=\lambda_0\frac{a}{t_0}\right] \tag{6.1}$$

for a 1D case and

$$K_{i,0}^{(1)}=\frac{a_i}{t_0}(p_{i,0}^+-p_{i,0}^-)=\text{const} \tag{6.2}$$

for a 2D case, respectively.

When λ's $=0$ in (3.7) and (3.8) for one dimension and in (5.14) for two dimensions, these conditions mean that an end-to-end distance for the walker's path is estimated by simple usual random walks having equal jumping probabilities when $\lambda_0=0$.

For a weak field (potential) in one or two dimensions, in which ξ or ξ_i is very small, the diffusive processes are specified by fluctuation around the extremum path. For a strong field in which ξ is not so small, the exponential form of $p^\alpha(x,t;\xi)$ is specified by (2.6). A function $\lambda(t)$ (<0) denotes a field with a sink and $\lambda(t)$ (>0) denotes a field with a source. The diffusing particles are strongly influenced by the field. The processes yield bifurcations of path, as we have studied in Ref. 10.

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APPENDIX: A LAGRANGIAN IN d-DIMENSIONAL SPACE

To get a Lagrangian in d-dimensional space, we start with a Fokker-Planck equation, derived from the GRW's analogously to (5.5), for the d-dimensional (dD) space,

$$\frac{\partial w(\mathbf{x},t)}{\partial t}=-\sum_{i=1}^{d}\frac{\partial}{\partial x^i}K_i^{(1)}(\mathbf{x},t)w+\frac{1}{2}\sum_{i=1}^{d}\frac{\partial^2}{\partial x^i\partial x^i}K_{ii}^{(2)}(\mathbf{x},t)w\qquad[\mathbf{x}=(x^1,x^2,\ldots,x^d)]\ . \tag{A1}$$

where

$$K_i^{(1)}(\mathbf{x},t)=\frac{a_i}{t_0}[p_i^+(\mathbf{x},t)-p_i^-(\mathbf{x},t)]\ ,$$
$$K_{ii}^{(2)}(\mathbf{x},t)=\frac{a_ia_i}{t_0}[p_i^+(\mathbf{x},t)+p_i^-(\mathbf{x},t)]\ ,$$
$$\left[\sum_{i=1}^{d}\sum_{\alpha=\pm}{}'\,p_i^\alpha(\mathbf{x},t)=1\right]\ , \tag{A2}$$

and we have dropped a symbol β_i $[=(\lambda,\xi_i)]$ in the above expressions for simplicity of notation. By choosing a set of suitable forms for $p_i^\alpha(\mathbf{x},t)$, we can set $K_{ii}^{(2)}(\mathbf{x},t)=D_{ii}K^{(2)}(\mathbf{x})$, where D_{ii} is constant. The FP equation (A1) is then rewritten into

$$\frac{\partial\tilde{w}}{\partial\tilde{t}}=-\sum_{i=1}^{d}\frac{\partial}{\partial x^i}c_i^{(1)}(\mathbf{x},\tilde{t})\tilde{w}+\frac{1}{2}\sum_{i=1}^{d}D_{ii}\frac{\partial^2}{\partial x^i\partial x^i}\tilde{w}\ , \tag{A3}$$

$$\tilde{w}(\mathbf{x},\tilde{t})=K^{(2)}(\mathbf{x})w(\mathbf{x},t),\quad \tilde{t}=K^2(\mathbf{x})t\ , \tag{A4}$$

$$c_i^{(1)}(\mathbf{x},\tilde{t})=\frac{K_i^{(1)}(\mathbf{x},t)}{K_{ii}^{(2)}(\mathbf{x})}\ , \tag{A5}$$

cf. (2.8).

Specifically, when we take "Uhlenbeck-Ornstein (UO)" processes expressed by $c_i^{(1)}(\mathbf{x},t)=-c_ix^i$ ($c_i=$const), and consider the boundary conditions

$$\tilde{w}(\mathbf{x},0)=K^{(2)}(\mathbf{x})\delta(\mathbf{x}-\mathbf{x}_0)$$

and $\tilde{w}(\mathbf{x},t)\to 0$ as $\mathbf{x}\to\pm\infty$, a fundamental Green's function [cf. (2.10)] to (A2) is exactly expressed by

$$G_0(\mathbf{x}\mid\mathbf{x}_0;\Delta\tilde{t})=\frac{1}{(2\pi)^{d/2}\mid\sigma\mid^{1/2}}\times\exp\left[-\tfrac{1}{2}\sum_i\sigma_{ii}^{-1}(x^i-x_0^i e^{-c_i\Delta\tilde{t}})^2\right], \quad (A6)$$

where $\mid\sigma\mid$ is the determinant of the diagonal matrix,

$$\sigma_{ii}=\frac{D_{ii}}{2c_i}(1-e^{-2c_i\Delta\tilde{t}})\ . \quad (A7)$$

When $\Delta\tilde{t}$ is very small so that σ_{ii} is approximated by $D_{ii}\Delta\tilde{t}$ and $-c_i x^i$ is replaced by $c_i^{(1)}(\mathbf{x},t)$ again, the Green's function G_0 becomes

$$G_0(\mathbf{x}\mid\mathbf{x}_0;\Delta\tilde{t})=\frac{1}{(2\pi\Delta\tilde{t})^{d/2}\mid D\mid^{1/2}}\exp\left[-\tfrac{1}{2}\sum_i\frac{\Delta_{ii}}{\mid D\mid\Delta\tilde{t}}[x^i-x_0^i-\Delta\tilde{t}c_i^{(1)}(x,0)]^2\right]$$

$$\equiv\frac{1}{(2\pi\Delta\tilde{t})^{d/2}\mid D\mid^{1/2}}g(\mathbf{x}\mid\mathbf{x}_0;\Delta\tilde{t})\ , \quad (A8)$$

where Δ_{ii} is a cofactor of D_{ii} [cf. $D_{ii}=1$ in (2.11)]. The solution of (A2) is expressed by

$$w(\mathbf{x},t)=\frac{1}{K^{(2)}(\mathbf{x})}\int d\mathbf{x}_0 G(\mathbf{x}\mid\mathbf{x}_0;t)K^{(2)}(\mathbf{x}_0)w(x_0,0) \quad (A9)$$

and

$$G(\mathbf{x}\mid\mathbf{x}_0;t)=\lim_{n\to\infty}\frac{1}{\tilde{A}_d}\int\frac{d\mathbf{x}_{n-1}}{\tilde{A}_d}\int\frac{d\mathbf{x}_{n-2}}{\tilde{A}_d}\cdots\int\frac{d\mathbf{x}_1}{\tilde{A}_d}g(\mathbf{x}\mid\mathbf{x}_{n-1};\Delta\tilde{t})g(\mathbf{x}_{n-1}\mid\mathbf{x}_{n-2};\Delta\tilde{t})\cdots g(\mathbf{x}_1\mid\mathbf{x}_0;\Delta\tilde{t})\ , \quad (A10)$$

where

$$\tilde{A}_d=(2\pi\Delta\tilde{t})^{d/2}\mid D\mid^{1/2},\quad \Delta\tilde{t}=K^{(2)}(\mathbf{x})\Delta t\ . \quad (A11)$$

Here we adopt the "midpoint rule," $\tilde{x}^i=(x^i+x_{n-1}^i)/2$, and expand the $c^{(1)}$ around $\tilde{x}^i$. Note that $(x^1-x_{n-1}^1)(x^2-x_{n-1}^2)$ vanishes and the $(x^i-x_{n-1}^i)^2$ $(i=1,2)$ become $D\Delta\tilde{t}/\Delta_{ii}$, in the integrations of $g(\mathbf{x}\mid\mathbf{x}_{n-1};\Delta\tilde{t})$. Considerations of these facts lead us to an expression for (A8) in two dimensions,

$$G(x^1,x^2\mid x_{n-1}^1,x_{n-2}^2;\Delta\tilde{t})=\exp\left[-\frac{\Delta\tilde{t}\Delta_{11}}{2\mid D\mid}\left[\frac{x^1-x_{n-1}^1}{\Delta\tilde{t}}-c_1^{(1)}(\tilde{x}^1,\tilde{x}^2,t_{n-1})\right]^2\right]\times\exp\left[-\frac{\Delta\tilde{t}\Delta_{22}}{2\mid D\mid}\left[\frac{x^2-x_{n-1}^2}{\Delta\tilde{t}}-c_2^{(1)}(\tilde{x}^1,\tilde{x}^2,t_{n-1})\right]^2\right]\times\exp\left[-\frac{\Delta\tilde{t}}{2}\left[\frac{\partial c_1^{(1)}}{\partial x^1}+\frac{\partial c_2^{(1)}}{\partial x^2}\right]\right]\ . \quad (A12)$$

To see that the expression (A9) with (A10) and (A11) satisfies the FP equation (A1) for (A3), we consider an expression given by

$$\tilde{w}(\mathbf{x},\tilde{t}+\Delta\tilde{t})=\int\frac{d\mathbf{x}_{n-1}}{\tilde{A}_d}g(\mathbf{x}\mid\mathbf{x}_{n-1};\Delta\tilde{t})\tilde{w}(\mathbf{x}_{n-1},\tilde{t}) \quad (A13)$$

corresponding to (A9).

By introducing a new vector $\mathbf{y}$ $[=\mathbf{x}-\mathbf{x}_{n-1}-\Delta\tilde{t}\mathbf{c}^{(1)}(\mathbf{x},t)]$, and expanding $\tilde{w}(\mathbf{x}_{n-1},\tilde{t})$ around $\tilde{x}$, we have

$$\tilde{w}(\mathbf{x}_{n-1},\tilde{t})[=K^{(2)}(\mathbf{x}_{n-1})w(\mathbf{x}_{n-1},t)]=\tilde{w}(\mathbf{x},\tilde{t})-\mathbf{y}\cdot\frac{\partial}{\partial\mathbf{x}}\tilde{w}(\mathbf{x},\tilde{t})-\Delta\tilde{t}\mathbf{c}^{(1)}\cdot\frac{\partial}{\partial\mathbf{x}}\tilde{w}(\mathbf{x},\tilde{t})+\tfrac{1}{2}\mathbf{y}\mathbf{y}:\frac{\partial^2}{\partial\mathbf{x}\partial\mathbf{x}}\tilde{w}(\mathbf{x},\tilde{t})+\Delta\tilde{t}\mathbf{y}\mathbf{c}^{(1)}:\frac{\partial^2}{\partial\mathbf{x}\partial\mathbf{x}}\tilde{w}(\mathbf{x},\tilde{t})+O(\Delta\tilde{t}^2) \quad (A14)$$

and the Jacobian of a transformation becomes

$$\frac{dx_{n-1}^i}{dy^i}=-1+\frac{\Delta\tilde{t}}{2}\frac{\partial c_i^{(1)}}{\partial x^i}+O(\Delta\tilde{t}^2) . \tag{A15}$$

In (A14), . and : are inner and tensor products, $A\cdot B=\sum_i A_iB_i$ and $A:B=\sum_i\sum_j A_{ij}B_{ij}$.

By substituting (A12), (A14), and (A15) into (A13), we get

$$\tilde{w}(\mathbf{x},\tilde{t})=\exp\left[-\frac{\Delta\tilde{t}}{2}\left[\frac{\partial c_1^{(1)}}{\partial x^1}+\frac{\partial c_2^{(1)}}{\partial x^2}\right]\right]\left[1-\frac{\Delta\tilde{t}}{2}\left[\frac{\partial c_1^{(1)}}{\partial x^1}+\frac{\partial c_2^{(1)}}{\partial x^2}\right]\right]$$
$$\times\left[1-\Delta\tilde{t}\mathbf{c}^{(1)}\cdot\frac{\partial}{\partial\mathbf{x}}\right]\tilde{w}(\mathbf{x},\tilde{t}-\Delta\tilde{t})+\frac{\Delta\tilde{t}}{2}\mathbb{D}:\frac{\partial^2}{\partial\mathbf{x}\partial\mathbf{x}}\tilde{w}(\mathbf{x},\tilde{t}-\Delta\tilde{t}) , \tag{A16}$$

where $\mathbb{D}$ is a 2×2 matrix; $|\mathbb{D}|=D_{11}D_{12}$, and we have used the integrals

$$\int\int dy^1dy^2\exp[-\tfrac{1}{2}(A_{11}y^1y^1+A_{22}y^2y^2)]=\frac{2\pi}{\sqrt{A_{11}A_{22}}} , \tag{A17}$$

$$\int\int dy^1dy^2y^iy^i\exp[-\tfrac{1}{2}(A_{11}y^1y^1+A_{22}y^2y^2)]=\frac{2\pi}{(A_{11}A_{22})^{1/2}}A_{ii} . \tag{A18}$$

From (A16), we can derive the FP equation (A2) in the limit $\Delta\tilde{t}\to0$.

From (A10), we can define an action $S[\mathbf{x}(t)\,|\,\mathbf{x}_0(t)]$ by

$$G(\mathbf{x}\,|\,\mathbf{x}_0;t)=\lim_{n\to\infty}\frac{1}{\tilde{A}_d}\int\frac{d\mathbf{x}_{n-1}}{\tilde{A}_d}\int\frac{d\mathbf{x}_{n-2}}{\tilde{A}_d}\cdots\int\frac{d\mathbf{x}_1}{\tilde{A}_d}e^{-S[\mathbf{x}(t)\,|\,\mathbf{x}_0(0)]}$$
$$\equiv\int_{\mathbf{x}_0}^{\mathbf{x},t}dx\,(t)e^{-S[\mathbf{x}(t)\,|\,\mathbf{x}(0)]} , \tag{A19}$$

where

$$S[\mathbf{x}(t)\,|\,\mathbf{x}_0(0)]=\int_0^t L(x^1,x^2,\dot{x}^1,\dot{x}^2)dt . \tag{A20}$$

After replacing $\Delta\tilde{t}$ in $g(\mathbf{x}\,|\,\mathbf{x}_{n-1};t)$ by $\Delta tK^{(2)}(\mathbf{x})$, we obtain the Lagrangian expressed by

$$L(\mathbf{x},\dot{\mathbf{x}})=\frac{1}{2K^{(2)}|\mathbb{D}|}\left[\sum_{i=1}^{2}\Delta_{ii}\dot{x}^i\dot{x}^i-\sum_{i=1}^{2}\Delta_{i2}\dot{x}^i\left[\frac{K_i^{(1)}}{K^{(2)}|\mathbb{D}|}\right]\right.$$
$$\left.+\frac{K^{(2)}}{2}\sum_{i=1}^{2}\left[\frac{\partial}{\partial x^i}\left[\frac{K_i^{(1)}}{K^{(2)}}\right]+\Delta_{ii}\left[\frac{K_i^{(1)}}{K^{(2)}|\mathbb{D}|}\right]^2\right]\right] . \tag{A21}$$

This expression is reduced to the expression (2.15) with (2.16) for the 1D case and $K^{(2)}(\mathbf{x})=K_0^{(2)}$. For general cases, we can check the above expressions.[13,15]

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