Excitonic modes in a Bose-condensed electron-hole gas in the pairing approximation

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(Received 14 August 1987)

In studying the low-temperature state of an electron-hole gas in an optically pumped semiconductor there has been increasing realization that the pair approximation (similar to the Gor'kov approximation for BCS superfluids) can deal with both the low-density phase of excitons in a Bose-condensed state as well as the high-density excitonic insulator phase. Using functional differentiation techniques, we use this pair approximation to give a systematic derivation of the two-particle Green's function and the associated collective modes. Our equations of motion give what is often called the generalized random-phase approximation (GRPA). The collective modes are shown to correspond to the solution of the standard electron-hole ladder and bubble diagram sum, but with 2×2 matrix propagators. Our model calculations are for a simple two-band direct-band-gap semiconductor with parabolic bands and a positive band gap (the semiconductor limit as opposed to the semimetal limit) and thus might be appropriate for optically pumped Cu_2O. In the appropriate limits, our formalism leads to the phonon modes discussed (in the late 1960s) by Maksimov and Kozlov in the excitonic insulator and by Keldysh and Kozlov in the Bose-condensed state of excitons. In the low-density limit, the excitation of these collective modes is crucial in understanding how the Bose condensate is depleted at higher temperatures. Besides the excitonic modes, our equations of motion allow a systematic study of the complete spectrum of collective modes in the GRPA, including plasmons. Throughout, we emphasize the formal similarity with the theory of collective modes and excited Cooper-pair states in BCS superfluids.

I. INTRODUCTION

By optical pumping, a gas of electrons and holes can be created in a semiconductor. As is well known, the Coulomb interaction between electrons and holes may lead to the formation of excitons. If the density n of the electron-hole pairs is small ($na_0^3 \ll 1$, where $a_0 \equiv \hbar^2\epsilon_0/me^2$ is the Bohr radius of an exciton, m the reduced mass, and ϵ_0 the dielectric constant of the medium), these excitons behave almost like Bose particles[1] and a Bose-Einstein condensation might be expected to occur in the gas under the usual conditions. Properties of this condensed exciton phase have been the subject of much theoretical (for recent discussion, see Refs. 2 and 3) as well as experimental investigation (see, for example, Refs. 4 and 5). Due to the small mass of the exciton and the correspondingly high critical temperature for Bose condensation, this system has been of especial interest in the search for a Bose condensate in something else besides liquid ^{4}He or spin-polarized hydrogen.

In this theoretical study, we work with a simple direct-band-gap two-band semiconductor model[3] and attempt to give a unified discussion of both the single-particle and collective excitations arising from the condensate of electron-hole pairs at $T=0$ K. We limit ourselves to the semiconductor region (positive band gap) and assume the number of electrons and holes is fixed.

As the number of pairs increases, however, the excitons no longer behave like ideal Bosons. The correction due to their fermionic structure must then be taken into account. Moreover, as n reaches the Mott concentration ($na_0^3 \approx 1$) where the exciton wave functions start to overlap, the pairs start to dissociate. At still higher concentration of pairs ($na_0^3 \gg 1$), the system consists of a degenerate two-component plasma of electrons and holes. Under certain conditions, pairing of electrons and holes in momentum space can still occur in this degenerate gas. These new bound electron-hole pairs may then undergo a Bose condensation leading to a new state sometimes called the "excitonic" insulator.[6-9] These pairs are, formally, the complete analogue of Cooper pairs in superconductors. The properties of the excitonic insulator were extensively investigated in the later 1960s (for reviews, see Refs. 9 and 10) mainly in the semimetal limit (negative band gap $E_g < 0$ or band overlap) in which the conduction- and valence-band minima are separated by a finite wave vector $\mathbf{w}$. As demonstrated by Zittartz,[11] the excitonic insulator in this semimetal limit is not favored in the more realistic situation of anisotropic band structure or by the presence of impurities. These two effects put severe limitations on the observability of such a phase. However, in this paper our interest in the excitonic insulator condensed phase (in our direct-band-gap semiconductor model) is motivated by the reasons elegantly expressed by Comte and Nozières.[3] In particular, studying the properties of this simple model over a wide range of densities gives greater insight into the physics involved in the Bose condensate of excitons at low and intermediate densities.

In the simple model considered here, the electrons and holes are treated as two distinct species of particles. We neglect all *inter*band scattering processes (in particular, electron-hole exchange) apart from the direct Coulomb interaction that lends to the binding of electron-hole pairs into excitons. Although the interband transitions may be quite small, they have important physical conse-

quences. As discussed in Refs. 12 and 13, for example, they lead to the destruction of the gauge invariance of the Hamiltonian (2.2) and hence to the destruction of true superfluid flow characteristics.[14] Nevertheless, we will limit ourselves to our simple model and try to extract the essential physics in it. This will set the basis for future work which includes weak interband transitions. To some extent, our model is appropriate for Cu_2O, which seems to be the most promising candidate[5,15] to observe Bose condensation of the long-lived excitons (lifetime ~ 1 μs). However, in this paper, we do not consider the complications arising from the optical pumping procedure.

The description of the low-temperature properties of an electron-hole gas as a function of the density is a complex many-body problem involving the self-consistent formation of a bound state in a two-component gas of interacting fermions. Quite apart from its physical realization, the model many-body problem we discuss in this paper is of considerable methodological interest. Many attempts have already been made to describe this system, using a variety of techniques including canonical transformations, Green's functions, variational techniques, as well as mapping of the electron-hole Hamiltonian onto an effective exciton Hamiltonian. These different approaches are reviewed in Refs. 2 and 4.

In this paper we discuss the collective excitations of the electron-hole gas in a systematic manner using a Green's function formalism in conjunction with functional differential techniques. We recover (and generalize somewhat) the results already obtained by Keldysh and Kozlov[1] at low density and those of Kozlov and Maksimov[8] as well as Jerome, Rice, and Kohn[9] at high density. Our theory of collective modes corresponds to the generalized random-phase approximation (GRPA) and is formally quite similar to that first developed for BCS superfluids.[16–18] While most of our discussion in this paper is for zero temperature, we make some brief remarks about the finite temperature case.

In the pairing self-energy approximation we use, the single-particle Green's function is a 2×2 matrix, the off-diagonal terms describing the pairing of electrons and holes (i.e., the exciton condensate). These off-diagonal terms vanish above a certain critical temperature T_C, indicating that there is no longer a condensate of excitons in the system. In superconductivity the vanishing of the off-diagonal elements indicates the disappearance (or ionization) of Cooper pairs above T_C. It is important to remember, however, that whereas Cooper pairs only exist in a Bose-condensed state, in our electron-hole system, excitons may exist both in the condensate ($n=1$ and $\mathbf{q}=\mathbf{0}$) and outside the condensate (excitons in excited states $n=2,3\ldots$ and/or with finite center-of-mass momentum). Thus excitons exist above T_C. In the simple pairing approximation, on the other hand, only the Bose-condensed excitons in the zero momentum state are included. To adequately deal with the depletion of the condensate or with estimating T_C, one must generalize the pairing self-energy approximation to include the excitation of the collective modes which we discuss in this paper. We hope to discuss this extension in a future paper.

Section II of this paper is basically a review of the so-called pairing formalism for the single-particle Green's functions on which the rest of our analysis is based. Section III proceeds with the systematic derivation of the GRPA equations of motion for the various coupled two-particle Green's functions. In Sec. IV, for orientation, we discuss the structure of the GRPA equations in the absence of an exciton condensate. The zero-temperature normal mode solutions of these equations are discussed in Sec. V, with emphasis on the low-density case. In Sec. VI we briefly discuss some possible extensions of this work.

II. THE PAIRING APPROXIMATION

We consider a simple nondegenerate two-band semiconductor with a direct band gap E_g at low temperatures. The density n of electron-hole pairs is assumed to be controlled by optical pumping, the number of electrons being equal to the number of holes ($N_e=N_h$). The dispersion relation of the electrons and holes in each band is given by (we set $\hbar=1$ throughout this paper)

$$E_e(\mathbf{k})=E_g+\frac{k^2}{2m_e}$$
$$E_h(\mathbf{k})=\frac{k^2}{2m_h}\ , \tag{2.1}$$

and the effective Hamiltonian $K=H-\mu_e N_e-\mu_h N_h$ of the system is taken as

$$\begin{aligned}K=&\int d\mathbf{r}\,\psi_e^\dagger(\mathbf{r})\left[-\frac{\nabla^2}{2m_e}-\mu_e\right]\psi_e(\mathbf{r})+\int d\mathbf{r}\,\psi_h^\dagger(\mathbf{r})\left[-\frac{\nabla^2}{2m_h}-\mu_h\right]\psi_h(\mathbf{r})\\&+\tfrac{1}{2}\int d\mathbf{r}\int d\mathbf{r}'[\psi_e^\dagger(\mathbf{r})\psi_e^\dagger(\mathbf{r}')V(\mathbf{r}-\mathbf{r}')\psi_e(\mathbf{r}')\psi_e(\mathbf{r})+\psi_h^\dagger(\mathbf{r})\psi_h^\dagger(\mathbf{r}')V(\mathbf{r}-\mathbf{r}')\psi_h(\mathbf{r}')\psi_h(\mathbf{r})\\&-2\psi_e^\dagger(\mathbf{r})\psi_e(\mathbf{r})V(\mathbf{r}-\mathbf{r}')\psi_h^\dagger(\mathbf{r}')\psi_h(\mathbf{r}')]\\&+\int d\mathbf{r}[\psi_e^\dagger(\mathbf{r})W(\mathbf{r},\tau)\psi_e(\mathbf{r})-\psi_h^\dagger(\mathbf{r})W(\mathbf{r},\tau)\psi_h(\mathbf{r})+\psi_e^\dagger(\mathbf{r})\eta^*(\mathbf{r},\tau)\psi_h^\dagger(\mathbf{r})+\psi_h(\mathbf{r})\eta(\mathbf{r},\tau)\psi_e(\mathbf{r})]\ .\end{aligned} \tag{2.2}$$

Here μ_e and μ_h are the chemical potentials of the electrons and holes, respectively, and the chemical potential of electron-hole pairs is given by $\mu=\mu(n,T)=\mu_e+\mu_h$. $V(\mathbf{r})=e^2/\epsilon_0 r$ is the Coulomb interaction in the semiconductor medium. The external fields W, η and η^* are added for the purpose of generating the equations of motion of the two-particle Green's functions by functional differentiation. They are usually set to zero at the end of the derivation. The terms involving η and η^* are symmetry breaking terms. As a concrete example, one might consider [letting $\eta(\mathbf{r},t)\rightarrow A\nabla_r/2m$, where A is proportional to the strength of the field] the excitation of electron-hole pairs by an external electromagnetic field.[19]

In order to describe the condensed phase of this system, one must consider the 2×2 matrix single-particle Green's function

$$G(1,2)\equiv\begin{bmatrix} G_e(1,2) & F(1,2) \\ F^\dagger(1,2) & -G_h(2,1)\end{bmatrix} = \begin{bmatrix} -\langle T\psi_e(1)\psi_e^\dagger(2)\rangle & -\langle T\psi_e(1)\psi_h(2)\rangle \\ -\langle T\psi_h^\dagger(1)\psi_e^\dagger(2)\rangle & -\langle T\psi_h(2)\psi_h^\dagger(1)\rangle\end{bmatrix}, \quad (2.3)$$

where $\langle\cdots\rangle$ stands for thermal average with the Hamiltonian K given by (2.2), and the numbers 1 and 2 represent space-imaginary time coordinates. The off-diagonal terms of this matrix Green's function[7,20] take into account the pairing of electrons and holes (the Bose condensate of excitons) in much the same way as one does for Cooper pairs in superconductivity.

Defining

$$W(1)=\begin{bmatrix} W(1) & \eta^*(1) \\ \eta(1) & W(1)\end{bmatrix}, \quad (2.4)$$

and using (2.2), the equation of motion for $G(1,2)$ is found to be

$$G(1,2)=G_0(1,2)+\int d\bar{3}\,G_0(1,\bar{3})W(\bar{3})G(\bar{3},2) + \int d\bar{3}\int d\bar{4}\,G_0(1,\bar{3})\Sigma(\bar{3},\bar{4})G(\bar{4},2)\,, \quad (2.5)$$

where $G_0(1,3)$ is defined by

$$\left[-I\frac{\partial}{\partial\tau_1}+m^{-1}\frac{\nabla_1^2}{2}+\mu\right]G_0(1,3)=I\delta(1,3)\,, \quad (2.6)$$

with

$$I=\begin{bmatrix}1&0\\0&1\end{bmatrix};\quad m^{-1}=\begin{bmatrix}m_e^{-1}&0\\0&m_h^{-1}\end{bmatrix};\quad \mu=\begin{bmatrix}\mu_e&0\\0&-\mu_h\end{bmatrix}. \quad (2.7)$$

The self-energy matrix has the expansion

$$\Sigma(3,4)=I\delta(3,4)\int d\bar{5}\,V(3,\bar{5})[G_e(\bar{5},\bar{5}^+)-G_h(\bar{5},\bar{5}^+)] - V(3,4)G(3,4) - \int d\bar{5}\int d\bar{6}\,V(3,\bar{5})G(3,\bar{6})\frac{\delta\Sigma(\bar{6},4)}{\delta W(\bar{5})}\,, \quad (2.8)$$

where $V(3,5)\equiv V(\mathbf{r}_3-\mathbf{r}_5)\delta(\tau_3-\tau_5)$. In the pairing formalism developed for G_1, only the Hartree-Fock-Gor'kov terms of the matrix self-energy $\Sigma(3,4)$ are considered. [The Fock term, of course, can be easily generalized to the screened Coulomb potential $V_S(3,4)$ if we sum higher-order bubble diagrams.] Since the Hartree term vanishes because of the global neutrality of the electron-hole gas, we are left with

$$\Sigma(3,4)\approx -V(3,4)G(3,4)\,. \quad (2.9)$$

Using (2.9) with a static potential V_S (and with the external field set to zero), (2.5) is readily solved to give[20,2,3]

$$G(\mathbf{p},z)=\begin{bmatrix} G_e(z) & F(z) \\ F^+(z) & -G_h(-z)\end{bmatrix}, \quad (2.10a)$$

with

$$G_e(\mathbf{p},z)=\frac{u_\mathbf{p}^2}{z-\varepsilon_{1\mathbf{p}}}+\frac{v_\mathbf{p}^2}{z-\varepsilon_{2\mathbf{p}}}\,,$$
$$-G_h(-\mathbf{p},-z)=\frac{v_\mathbf{p}^2}{z-\varepsilon_{1\mathbf{p}}}+\frac{u_\mathbf{p}^2}{z-\varepsilon_{2\mathbf{p}}}\,, \quad (2.10b)$$
$$F(\mathbf{p},z)=F^\dagger(\mathbf{p},z)=u_\mathbf{p}v_\mathbf{p}\left[\frac{1}{z-\varepsilon_{1\mathbf{p}}}-\frac{1}{z-\varepsilon_{2\mathbf{p}}}\right],$$

where $(i=e,h)$

$$\varepsilon_{1,2\mathbf{p}}=\zeta_\mathbf{p}\pm\varepsilon_\mathbf{p}$$
$$\zeta_\mathbf{p}=\frac{\eta_{e\mathbf{p}}-\eta_{h-\mathbf{p}}}{2}\;;\quad \varepsilon_\mathbf{p}=(\eta_\mathbf{p}^2+\Delta_\mathbf{p}^2)^{1/2} \quad (2.11)$$
$$\eta_{i\mathbf{p}}=E_i(\mathbf{p})+\Sigma_i(\mathbf{p})-\mu_i\;;\quad \eta_\mathbf{p}=\frac{\eta_{e\mathbf{p}}+\eta_{h-\mathbf{p}}}{2}\,,$$

and the coherence factors $u_\mathbf{p}$ and $v_\mathbf{p}$ are defined by

$$u_\mathbf{p}^2=\frac{1}{2}\left[1+\frac{\eta_\mathbf{p}}{\varepsilon_\mathbf{p}}\right];\quad v_\mathbf{p}^2=\frac{1}{2}\left[1-\frac{\eta_\mathbf{p}}{\varepsilon_\mathbf{p}}\right];\quad u_\mathbf{p}v_\mathbf{p}=\frac{\Delta_\mathbf{p}}{2\varepsilon_\mathbf{p}}\,. \quad (2.12)$$

We note that in the degenerate case ($m_e=m_h$), $\zeta_\mathbf{p}=0$ and hence $\varepsilon_{1,2\mathbf{p}}=\pm\varepsilon_\mathbf{p}$. This simple case leads to results which are formally identical to the single-particle Green's function for a BCS superfluid.

The diagonal part of $\Sigma(\mathbf{p})$ is the exchange self-energy

$$\Sigma_{ex}(\mathbf{p})=-\sum_\mathbf{q}V(\mathbf{p}-\mathbf{q})\langle c_\mathbf{q}^\dagger c_\mathbf{q}\rangle\,. \quad (2.13)$$

The gap function $\Delta_\mathbf{p}$ (which is the nondiagonal part of Σ) is the order parameter for the condensed phase and is given by

$$\Delta_\mathbf{p}=\sum_\mathbf{q}V(\mathbf{p}-\mathbf{q})\langle c_\mathbf{q}d_{-\mathbf{q}}\rangle\,, \quad (2.14)$$

where c and d are second-quantized destruction operators for electrons and holes, respectively. Working at $T=0$ K, one has $\varepsilon_{1\mathbf{p}}>0$ and $\varepsilon_{2\mathbf{p}}<0$, and the distribution functions of electrons and holes are given by

$$n_{e\mathbf{p}}=n_{h\mathbf{p}}=v_\mathbf{p}^2\,. \quad (2.15)$$

Using (2.12)–(2.15) and taking account of the spin degeneracy, we have to solve the self-consistent system of equations

$$N=2\sum_{\mathbf{p}} v_{\mathbf{p}}^2 , \tag{2.16a}$$

$$\Delta_{\mathbf{p}}=\sum_{\mathbf{q}} V(\mathbf{p}-\mathbf{q})\frac{\Delta_{\mathbf{q}}}{2(\eta_{\mathbf{q}}^2+\Delta_{\mathbf{q}}^2)^{1/2}} , \tag{2.16b}$$

where N is the number of electron-hole pairs.

Defining $\psi_{\mathbf{p}}\equiv\Delta_{\mathbf{p}}/2\varepsilon_{\mathbf{p}}$, Eq. (2.16b) can be rewritten in the form[2,7,20]

$$\left[E_g+\frac{p^2}{2m}-\mu-\sum_{\mathbf{q}} V(\mathbf{p}-\mathbf{q})[1-\mathrm{sgn}(\eta_{\mathbf{q}})(1-4\psi_{\mathbf{q}}^2)^{1/2}]\right]\psi_{\mathbf{p}} =\mathrm{sgn}(\eta_{\mathbf{p}})(1-4\psi_{\mathbf{p}}^2)^{1/2}\sum_{\mathbf{q}} V(\mathbf{p}-\mathbf{q})\psi_{\mathbf{q}} , \tag{2.17}$$

To lowest order in the density n, (2.17) and (2.16a) become, respectively,

$$\left[E_g+\frac{p^2}{2m}-\mu_0\right]\psi_{\mathbf{p}}=\sum_{\mathbf{q}} V(\mathbf{p}-\mathbf{q})\psi_{\mathbf{q}} , \tag{2.18a}$$

$$N=2\sum_{\mathbf{p}}\psi_{\mathbf{p}}^2 . \tag{2.18b}$$

Equation (2.18a) is the familiar Wannier equation for an exciton in momentum space (with zero center-of-mass momentum). Its eigenvalues are

$$-E_g+\mu_0=-\frac{E_0}{n^2} , \quad n=1,2,\ldots \tag{2.19}$$

where $E_0=e^2/2\epsilon_0 a_0$ is the exciton Rydberg. Equation (2.18) describes a gas of noninteracting excitons at $T=0$ K. We expect to find all these excitons condensed in the lowest excitonic state ($n=1$) and so

$$\mu_0=E_g-E_0 , \tag{2.20}$$

and

$$\psi_{\mathbf{p}}=\left[\frac{n}{2}\right]^{1/2}\phi_1(\mathbf{p}) . \tag{2.21}$$

Here $\phi_1(\mathbf{p})$ is the Fourier transform of the $1s$-hydrogen wave function describing the relative motion of the $1s$ exciton and is given by

$$\phi_1(\mathbf{p})=\frac{(64\pi a_0^3)^{1/2}}{(1+a_0^2 p^2)^2} . \tag{2.22}$$

In the next order in density (order n), the correction to the chemical potential is found to be[1,20,2]

$$\mu=\mu_0+\tfrac{13}{3}\pi(na_0^3)E_0 , \tag{2.23}$$

which arises from the exchange repulsion between the electrons and holes that are involved in two excitons. As noted in Ref. 2, to this order in density, there should also be a correction arising from the van der Waals attraction between two excitons. This correction was evaluated in Ref. 20. It comes from screening effects which are not taken into account in the pairing formalism but which are easily included.

As the density of pairs increases, the occupancy of the states also increases. In the high-density limit ($na_0^3\gg 1$), no exciton bound state can exist in r space. As with any degenerate Fermi gas, the electrons and holes fill their respective bands up to some Fermi level k_F (note that $N_e=N_h$). The normal electron-hole plasma phase is obtained by setting $\Delta_{\mathbf{p}}=0$. $u_{\mathbf{p}}^2$ and $v_{\mathbf{p}}^2$ are then step functions as in a normal metal, with $v_{\mathbf{p}}^2=\Theta(k_F-p)$. As in the case of overlapping bands,[6–10] this ground state is unstable with respect to the electron-hole interaction.[3] Pairing of electrons and holes will appear at the Fermi surface in a manner similar to Cooper pairs in a superconductor. There is then a solution with a small gap $\Delta_{\mathbf{p}}$ in the single-particle spectrum arising from the field produced by the Bose-Einstein condensate "Cooper" pairs. In this case, the chemical potential of the pairs is found to be[2]

$$\mu=\frac{k_F^2}{2m}-\frac{2e^2}{\pi\epsilon_0}k_F . \tag{2.24}$$

The single-particle excitations in the pairing approximation are given by the poles of the matrix Green's function in Eq. (2.10), i.e., $\varepsilon_{1\mathbf{p}}$ and $\epsilon_{2\mathbf{p}}$. To zeroth order in the density we obtain

$$\varepsilon_{1\mathbf{p}}\approx E_g+\frac{p^2}{2m_e}-\mu_e , \tag{2.25a}$$

$$\varepsilon_{2\mathbf{p}}\approx -\frac{p^2}{2m_h}+\mu_h . \tag{2.25b}$$

Using (2.20), we can write approximately $\mu_e\approx E_g-\frac{1}{2}E_0$, $\mu_h\approx-\frac{1}{2}E_0$. The dispersion relations (2.25) are sketched in Fig. 1. One pole is electronlike and the other one holelike, as in a semiconductor. The only difference here, which arises from the condensate, is the position of the chemical potential at E_g-E_0. The quantity

$$2\varepsilon_{\mathbf{p}}=\varepsilon_{1\mathbf{p}}-\varepsilon_{2\mathbf{p}}\approx E_0+\frac{p^2}{2m} \tag{2.26}$$

at low density is clearly the energy needed to excite a

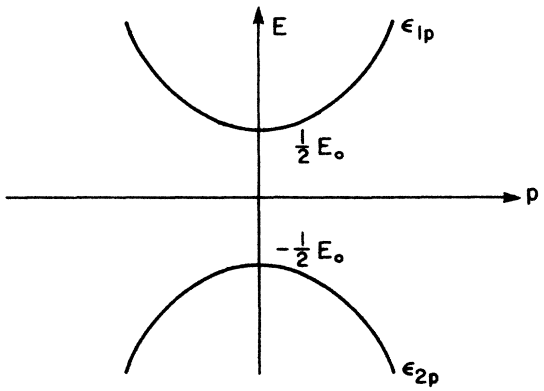

FIG. 1. Schematic plot of the single-particle excitations in the pairing approximation in the low-density limit. $\varepsilon_{1\mathbf{p}}$ and $\varepsilon_{2\mathbf{p}}$ are defined via (2.10) and (2.11).

condensed $1s$ exciton ($\mathbf{q}=\mathbf{0}$) to some scattering state, the minimum such energy being the binding energy E_0 of the exciton. As we will see in Sec. IV, the lowest value of $2\varepsilon_p$ is the minimum energy of a two-particle excitation process at $T=0$ K in the pairing approximation. Thus (2.26) means that the only allowed processes involve creation or destruction of Bose-condensed excitons. Clearly this does not include excitation of such excitons into states of finite center-of-mass momentum or into states ($n=2,3,\ldots$). In order to allow for such excitonic states, one has to consider repeated scattering (T matrix) of electrons and holes in the single-particle self-energy diagrams. On the other, as we show in later sections, such excitations will arise in the two-particle Green's functions G_2 which are generated by functional differentiation of the pairing self-energy.

In the high-density electron-hole plasma phase ($\Delta_p=0$), we have

$$\varepsilon_{1p}\approx\begin{cases}\dfrac{(p^2-k_F^2)}{2m_e} & \text{if } p>k_F\ ,\\ \dfrac{(k_F^2-p^2)}{2m_h} & \text{if } p<k_F\ ,\end{cases}\tag{2.27a}$$

$$\varepsilon_{2p}\approx\begin{cases}\dfrac{(k_F^2-p^2)}{2m_h} & \text{if } p>k_F\ ,\\ \dfrac{(p^2-k_F^2)}{2m_e} & \text{if } p<k_F\ .\end{cases}\tag{2.27b}$$

In the excitonic insulator phase, these dispersion relations are modified as indicated in Fig. 2. In this high-density limit, using

$$2\varepsilon_p=\varepsilon_{1p}-\varepsilon_{2p}\approx 2\left[\frac{(p^2-k_F^2)^2}{4m}+\Delta_p^2\right]^{1/2},\tag{2.28}$$

we see that minimum energy needed to break a condensed bound pair is equal to $2\Delta_{k_F}$. Taking $m_e=m_h$ in Fig. 2, we obtain the same dispersion relations as in a superconductor. The excitonic insulator phase is thus formally very similar to superconductivity as regards the single-particle excitations. As we shall show in the next section, there is a similar formal correspondence between the collective modes.

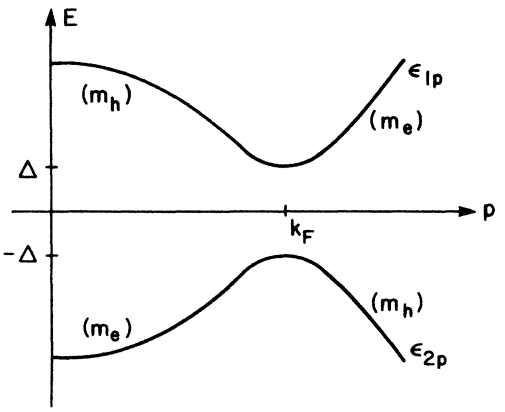

FIG. 2. Schematic plot of the single-particle excitations in the pairing approximation in the high-density limit. In this case, there is a gap of 2Δ at the Fermi momentum k_F.

III. EQUATIONS OF MOTION FOR THE TWO-PARTICLE GREEN'S FUNCTIONS

In order to derive the collective modes of the electron-hole gas, we use the functional differentiation approach described by Kadanoff and Baym.[21] This was originally used in the theory of the collective modes for superconductors by Ambegaokar and Kadanoff.[16] Since then it has been used to discuss collective modes in a wide variety of condensed phases.[22] This approach has several advantages. (a) If the approximation used for G_1 is conserving, then G_2 derived by functional differentiation of G_1 is also automatically conserving. (b) It is easily generalized to deal with finite temperatures as well as impurity scattering. (c) It exhibits the structure of the theory in a very transparent way, especially with regard to symmetry properties.

Based on the pairing self-energy approximation, the theory of collective modes which we will obtain corresponds to what is called the generalized RPA. In the case of superconductors, this approximation was originally worked out using decoupling methods on equations of motion.[23,24] Some of this early literature gives useful insights into the physics involved in the formally similar problem we are dealing with. In turn, our formulation of the generalized RPA may allow a more complete analysis of the collective modes and excited Cooper pair states (excitons) in BCS superfluids.

Consider a coupling to some external scalar field as in (2.2),

$$W(1,2)=\begin{bmatrix}W_e(1,2) & \eta^*(1,2)\\ \eta(1,2) & W_h(2,1)\end{bmatrix}.\tag{3.1}$$

We can generate the 16 possible two-particle Green's functions of the condensed phase by functional differentiation of the matrix Green's function G_1

$$L_{abcd}(1,2,1',2')=-\frac{\delta G_{ab}(1,1')}{\delta W_{cd}(2',2)}\ ,\tag{3.2}$$

starting from the identity

$$\frac{\delta G_{ab}(1,1')}{\delta W_{cd}(2',2)}=-\sum_{i,j}\int d\bar{3}\int d\bar{4}\,G_{ai}(1,\bar{3})\frac{\delta G_{ij}^{-1}(\bar{3},\bar{4})}{\delta W_{cd}(2',2)}\times G_{jb}(\bar{4},1')\ ,\tag{3.3}$$

and the matrix equation of motion for the inverse Green's function

$$G^{-1}(1,2)=G_0^{-1}(1,2)-\Sigma(1,2)-W(1,2)\ .\tag{3.4}$$

Defining an effective interaction $\Xi_{jkl}(3,6,4,5)$ by

$$\Xi_{ijkl}(3,6,4,5)\equiv\frac{\delta\Sigma_{ij}(3,4)}{\delta G_{kl}(5,6)}\ , \tag{3.5}$$

one finds the equations of motion for the coupled L's by functional differentiation of (3.4). The final result is (setting the external fields to zero)

$$L_{abcd}(1,2,1',2')=-G_{ac}(1,2')G_{db}(2,1')+\int d\bar{3}\int d\bar{4}\int d\bar{5}\int d\bar{6}\,G_{ai}(1,2)\Xi_{ijkl}(\bar{3},\bar{6},\bar{4},\bar{5})L_{klcd}(\bar{5},2,\bar{6},2')G_{jb}(\bar{4},1')\ , \tag{3.6}$$

where summation over repeated indices is implied. The dispersion relations of the collective modes are given by the condition that the homogeneous part of the system of equations (3.6) has solutions. Clearly, the frequencies of the collective modes cannot depend on the specific external fields used. Since these frequencies are what we are interested in, we shall leave the indices c and d implicit.

We now apply this general formalism to the single-particle self-energy used in the pairing approximation discussed in Sec. II. This self-energy is given by (we have not set $W=0$ at this stage)

$$\Sigma(3,4)=I\delta(3,4)\int d\bar{5}\,V(3,\bar{5})[G_e(\bar{5},\bar{5}^+)-G_h(\bar{5},\bar{5}^+)]$$
$$-V_S(3,4)G(3,4)\ , \tag{3.7}$$

where the exchange term involves a statically screened Coulomb potential V_S. The resulting theory will give us a theory of the collective modes for both high and low densities, just as the pairing formalism gives single-particle excitations appropriate to both high and low densities.

When extracting the effective interaction from (3.7), however, we will not assume that V_S is a functional of G_1. Thus (3.7) may not generate a conserving approximation in the Baym-Kadanoff sense.[25] At low density however, where we can use $V_S\approx V$, the approximation is conserving. At high density, the introduction of V_S allows an easy way of including the screening of the bare Coulomb interaction. In the extreme limit of strong screening, V_S is often replaced by a zero-range potential.[9]

The system of equations obtained in this way is what is often called the generalized random phase approximation for a Bose-condensed system with a matrix single-particle propagator,

$$L_{ab}(1,2,1',2')=-G_{ac}(12')G_{db}(2,1')+\int d\bar{3}\int d\bar{4}\,G_{ai}(1,\bar{3})V(\bar{3},\bar{4})L_{jj}(\bar{4},2,\bar{4}^+,2')G_{ib}(\bar{3},1')$$
$$-\int d\bar{3}\int d\bar{4}\,G_{ai}(1,\bar{3})V_S(\bar{3},\bar{4})L_{ij}(\bar{3},2,\bar{4},2')G_{jb}(\bar{4},1')\ . \tag{3.8}$$

The second term on the right-hand side (rhs) of (3.8) comes from the Hartree part of the self-energy and generates the bubble diagrams in the diagrammatic expansion of L_{ab}. The third term comes from the Fock and Gor'kov parts of the self-energy and generates the ladder diagrams. Our equation for L_{ab} includes the usual Bethe-Salpeter ladder diagrams that are responsible for the formation of excitons, which appear here as collective modes of the electron-hole gas. Because of the presence of the condensate, however, our ladder diagrams are expressed in terms of 2×2 matrix propagators. This will cause a modification of the exciton dispersion relations, as discussed in Sec. V.

Defining the Fourier transformed $L_{ab}(K_i,K_k,K_j,K_l)$ by

$$L_{ab}(1,2,1',2')\equiv\frac{1}{\beta^4}\sum_{\substack{K_i,K_j,\\K_k,K_l}}e^{iK_iX_1+iK_kX_2-iK_jX_{1'}-iK_lX_{2'}}L_{ab}(K_i,K_k,K_j,K_l)\ , \tag{3.9}$$

where $K\equiv(\mathbf{k},i\omega_k)$ and $X\equiv(\mathbf{r},\tau)$. Here, the $i\omega$ associated with K_i, K_k, K_j, and K_l are Fermi-Matsubara frequencies. Denoting $K_i\to P+Q$, $K_j\to P$, $K_l\to K+Q$, $K_k\to K$, we obtain in momentum space

$$L_{ab}(\mathbf{p},\mathbf{k};Q)=-\frac{1}{\beta}\sum_{i\omega_p}G_{ac}(P+Q)G_{db}(P)\delta_{\mathbf{p},\mathbf{k}}$$
$$+\frac{1}{\beta}\sum_{i\omega_p}\sum_{\mathbf{p}''}G_{ai}(P+Q)G_{ib}(p)V(\mathbf{q})L_{kk}(\mathbf{p}'',\mathbf{k};Q)$$
$$-\frac{1}{\beta}\sum_{i\omega_p}\sum_{\mathbf{p}''}G_{ai}(P+Q)G_{jb}(P)V_S(\mathbf{p}-\mathbf{p}'')L_{ij}(\mathbf{p}'',\mathbf{k};Q)\ , \tag{3.10}$$

where we have used the contracted response function defined by

$$\frac{1}{\beta^2}\sum_{i\omega_p,i\omega_k}L_{ab}(P+Q,K,P,K+Q)\equiv\beta L_{ab}(\mathbf{p},\mathbf{k};Q)\ . \tag{3.11}$$

We note that Q is associated with a Bose-Matsubara frequency since it describes a collective mode (a pole of a two-particle Green's function). The choice of K_i, K_k, K_j, K_l made above corresponds to choosing $\tau_1 = \tau_{1'}$, $\tau_2 = \tau_{2'}$ so that Q is the momentum and frequency of the center of mass of the pairs at $(1,1')$ and $(2,2')$. The explicit matrix form of the system of equations is ($L_{ij} \equiv L_{ijcd}$)

$$\begin{bmatrix} L_{11} \\ L_{12} \\ L_{21} \\ L_{22} \end{bmatrix}(\mathbf{p},\mathbf{k};Q) = -\frac{1}{\beta}\sum_{i\omega_p} \begin{bmatrix} G_{1c}G_{d1} \\ G_{1c}G_{d2} \\ G_{2c}G_{d1} \\ G_{2c}G_{d2} \end{bmatrix} \delta_{\mathbf{p},\mathbf{k}}$$
$$+\frac{1}{\beta}\sum_{i\omega_p}\sum_{\mathbf{p}''} \begin{bmatrix} G_{11}G_{11}+G_{12}G_{21} & 0 & 0 & G_{11}G_{11}+G_{12}G_{21} \\ G_{11}G_{12}+G_{12}G_{22} & 0 & 0 & G_{11}G_{12}+G_{12}G_{22} \\ G_{21}G_{11}+G_{22}G_{21} & 0 & 0 & G_{21}G_{11}+G_{22}G_{21} \\ G_{21}G_{12}+G_{22}G_{22} & 0 & 0 & G_{21}G_{12}+G_{22}G_{22} \end{bmatrix} V(\mathbf{q}) \begin{bmatrix} L_{11} \\ L_{12} \\ L_{21} \\ L_{22} \end{bmatrix}(\mathbf{p}'',\mathbf{k};Q)$$
$$-\frac{1}{\beta}\sum_{i\omega_p}\sum_{\mathbf{p}''} \begin{bmatrix} G_{11}G_{11} & G_{11}G_{21} & G_{12}G_{11} & G_{12}G_{21} \\ G_{11}G_{12} & G_{11}G_{22} & G_{12}G_{12} & G_{12}G_{22} \\ G_{21}G_{11} & G_{21}G_{21} & G_{22}G_{11} & G_{22}G_{21} \\ G_{21}G_{12} & G_{21}G_{22} & G_{22}G_{12} & G_{22}G_{22} \end{bmatrix} V_S(\mathbf{p}-\mathbf{p}'') \begin{bmatrix} L_{11} \\ L_{12} \\ L_{21} \\ L_{22} \end{bmatrix}(\mathbf{p}'',\mathbf{k};Q) , \tag{3.12}$$

where the first G in each product has argument $P+Q$ and the second G has argument P.

Evaluating the frequency sums with the aid of (2.10)–(2.12) for $T=0$ K, we obtain

$$\begin{bmatrix} L_{11}+L_{22} \\ L_{12}+L_{21} \\ L_{12}-L_{21} \\ L_{11}-L_{22} \end{bmatrix}(\mathbf{p},\mathbf{k};Q) \equiv \begin{bmatrix} B_{11} \\ B_{12} \\ B_{21} \\ B_{22} \end{bmatrix}(\mathbf{p},\mathbf{k};Q)$$
$$= \frac{4}{\Lambda_{pq}}\sum_{\mathbf{p}''} \begin{bmatrix} \bar{\gamma}^2\bar{\varepsilon} & 0 & 0 & 0 \\ l\bar{\gamma}\bar{\omega} & 0 & 0 & 0 \\ \gamma\bar{\gamma}\bar{\varepsilon} & 0 & 0 & 0 \\ -m\bar{\gamma}\bar{\omega} & 0 & 0 & 0 \end{bmatrix} V(\mathbf{q}) \begin{bmatrix} B_{11} \\ B_{12} \\ B_{21} \\ B_{22} \end{bmatrix}(\mathbf{p}'',\mathbf{k};Q)$$
$$-\frac{1}{\Lambda_{pq}}\sum_{\mathbf{p}''} \begin{bmatrix} \bar{\gamma}^2\bar{\varepsilon} & l\bar{\gamma}\bar{\omega} & \gamma\bar{\gamma}\bar{\varepsilon} & -m\bar{\gamma}\bar{\omega} \\ l\bar{\gamma}\bar{\omega} & l^2\bar{\varepsilon} & \gamma l\bar{\omega} & -lm\bar{\varepsilon} \\ \gamma\bar{\gamma}\bar{\varepsilon} & \gamma l\bar{\omega} & \gamma^2\bar{\varepsilon} & -\gamma m\bar{\omega} \\ -m\bar{\gamma}\bar{\omega} & -lm\varepsilon & -\gamma m\bar{\omega} & m^2\bar{\varepsilon} \end{bmatrix} V_S(\mathbf{p}-\mathbf{p}'') \begin{bmatrix} B_{11} \\ B_{12} \\ B_{21} \\ B_{22} \end{bmatrix}(\mathbf{p}'',\mathbf{k};Q) . \tag{3.13}$$

Since we are interested in finding the collective modes, we have left out the inhomogeneous terms on the rhs of (3.13) and integrate over the spin variables. The coherence factors are defined by (we follow Ref. 1 in introducing γ and $\bar{\gamma}$)

$$\begin{aligned} \gamma &\equiv \gamma_{\mathbf{pp}'} = u_{\mathbf{p}}u_{\mathbf{p}'} + v_{\mathbf{p}}v_{\mathbf{p}'} , \\ l &\equiv l_{\mathbf{pp}'} = u_{\mathbf{p}}u_{\mathbf{p}'} - v_{\mathbf{p}}v_{\mathbf{p}'} , \\ m &\equiv m_{\mathbf{pp}'} = u_{\mathbf{p}}v_{\mathbf{p}'} + v_{\mathbf{p}}u_{\mathbf{p}'} , \\ \bar{\gamma} &\equiv \bar{\gamma}_{\mathbf{pp}'} = u_{\mathbf{p}}v_{\mathbf{p}'} - v_{\mathbf{p}}u_{\mathbf{p}'} , \end{aligned} \tag{3.14}$$

with the further abbreviations $\mathbf{p}' \equiv \mathbf{p}+\mathbf{q}$ ($\mathbf{p}'$ will have this meaning throughout the rest of this paper),

$$\Lambda_{pq} \equiv \bar{\omega}^2 - \bar{\varepsilon}^2, \quad \bar{\omega} \equiv \omega + \zeta_{\mathbf{p}} - \zeta_{\mathbf{p}'}, \quad \bar{\varepsilon} \equiv \varepsilon_{\mathbf{p}} + \varepsilon_{\mathbf{p}'} . \tag{3.15}$$

A system of equations similar to (3.13) has already been considered by Kozlov and Maksimov[8] in their treatment of the collective excitations limit in the high-density limit in semimetals. We shall use (3.13) to discuss the collective modes in both the low- and high- density limits in the case of a positive band gap.

We can easily make contact with the equations of Jérome, Rice, and Kohn (JRK),[9] who used the same method to generate the two-particle Green's functions in the high-density limit. In their analysis, JRK neglected the Fock terms in the self-energy in Eq. (3.7). That means that the terms $-V_S(3,4)G_{11}(3,4)$ and $-V_S(3,4)G_{22}(3,4)$ are absent in their equations. [They assumed that these terms were already included in the dispersion relations (2.1).] As a result, terms with $i=j$ are also absent in the Fock term in (3.10) and in the following equations. As easily seen, it is then sufficient to consider only the three quantities B_{11}, B_{12}, B_{21}. Upon defining

$$L(Q)\equiv\sum_{\mathbf{p}}V^{1/2}(\mathbf{q})B_{11}(\mathbf{p},\mathbf{k};Q)\ ,$$
$$B_{\mathbf{p}_0}(Q)\equiv\sum_{\mathbf{p}}V_S(\mathbf{p}_0-\mathbf{p})B_{12}(\mathbf{p},\mathbf{k};Q)\ , \tag{3.16}$$
$$A_{\mathbf{p}_0}(Q)\equiv\sum_{\mathbf{p}}V_S(\mathbf{p}_0-\mathbf{p})B_{21}(\mathbf{p},\mathbf{k};Q)\ ,$$

and redefining the coherence factors as $\gamma\to l(\mathbf{q})$, $l\to\eta(\mathbf{q})$, $m\to m(\mathbf{q})$, $\bar{\gamma}\to p(\mathbf{q})$, we recover the system of equations (4.9)–(4.11) of JRK.

The coupled equations (3.13) can be further simplified by defining the new functions $C_{11}=B_{11}/\bar{\gamma}$, $C_{12}=B_{12}/l$, $C_{21}=B_{21}/\gamma$, $C_{22}=-B_{22}/m$ (assuming $\mathbf{q}\neq\mathbf{0}$). It is then seen explicitly that $C_{11}=C_{21}$ and $C_{12}=C_{22}$, so that we are left with a system of only two equations involving $G_2^{\pm}\equiv C_{11}\pm C_{12}$:

$$(\bar{\omega}-\bar{\varepsilon})G_2^{+}(\mathbf{p},\mathbf{k};Q)=\sum_{\mathbf{p}_0}[2\bar{\gamma}_{\mathbf{p}\mathbf{p}'}\bar{\gamma}_{\mathbf{p}_0\mathbf{p}_0'}V(\mathbf{q})-\gamma_{\mathbf{p}\mathbf{p}_0}\gamma_{\mathbf{p}'\mathbf{p}_0'}V_S(\mathbf{p}-\mathbf{p}_0)]G_2^{+}(\mathbf{p}_0,\mathbf{k};Q)$$
$$+\sum_{\mathbf{p}_0}[2\bar{\gamma}_{\mathbf{p}\mathbf{p}'}\bar{\gamma}_{\mathbf{p}_0\mathbf{p}_0'}V(\mathbf{q})-\bar{\gamma}_{\mathbf{p}\mathbf{p}_0}\bar{\gamma}_{\mathbf{p}'\mathbf{p}_0'}V_S(\mathbf{p}-\mathbf{p}_0)]G_2^{-}(\mathbf{p}_0,\mathbf{k};Q)\ , \tag{3.17a}$$
$$-(\bar{\omega}+\bar{\varepsilon})G_2^{-}(\mathbf{p},\bar{k};Q)=\sum_{\mathbf{p}_0}[2\bar{\gamma}_{\mathbf{p}\mathbf{p}'}\bar{\gamma}_{\mathbf{p}_0\mathbf{p}_0'}V(\mathbf{q})-\bar{\gamma}_{\mathbf{p}\mathbf{p}_0}\bar{\gamma}_{\mathbf{p}'\mathbf{p}_0'}V_S(\mathbf{p}-\mathbf{p}_0)]G_2^{+}(\mathbf{p}_0,\mathbf{k};Q)$$
$$+\sum_{\mathbf{p}_0}[2\bar{\gamma}_{\mathbf{p}\mathbf{p}'}\bar{\gamma}_{\mathbf{p}_0\mathbf{p}_0'}V(\mathbf{q})-\gamma_{\mathbf{p}\mathbf{p}_0}\gamma_{\mathbf{p}'\mathbf{p}_0'}V_S(\mathbf{p}-\mathbf{p})]G_2^{-}(\mathbf{p}_0,\mathbf{k};Q)\ , \tag{3.17b}$$

where, as before, $\mathbf{p}_0'\equiv\mathbf{p}_0+\mathbf{q}$. Our approach to the collective modes of the electron-hole gas is somewhat different from that originally used by Keldysh and Kozlov[1] at low densities. In particular, our work is strictly within the GRPA while they include a subset of higher-order diagrams. As a result, our system of equations (3.17) is not easily compared to the equivalent set given by Eqs. (40) and (41) of Ref. 1, although the two theories bear some obvious similarities. As we will show in the next section, our system of equations reproduces the long-wavelength acoustic mode obtained by Keldysh and Kozlov[1] at low density. More generally, however, all the collective modes (within the GRPA) can be extracted from (3.17).

The generalized RPA treatment of collective modes in BCS superconductors leads to the same kind of matrix equations[26] given by (3.12) and (3.13), with V_S now being the short-range attractive interaction responsible for superconductivity. In this case also, the system of four coupled equations can be reduced to a set of two coupled equations, as we have shown above. This fact can be useful in simplifying the discussion of collective modes in BCS superconductors as well as in superfluid ^{3}He.

IV. GRPA FOR THE NORMAL PHASE

In the concluding part of the last section, we limited our attention to the zero-temperature case. Partly this was for simplicity of analysis but, more importantly, it was done because the pairing approximation is not adequate to discuss the depletion of the exciton condensate at finite temperature. Since it does not include the possibility of excitons outside the condensate, the temperature at which Δ vanishes is related to the ionization of the Bose-condensed excitons (with binding energy E_0).

We recall that the matrix self-energy given by (2.8) can also be written in the form

$$\Sigma(3,4)=\Sigma_H(3,4)+\int d\bar{5}\int d\bar{6}\,V(3,\bar{5})L(3,\bar{5},\bar{6},\bar{5}^{+})G^{-1}(\bar{6},4)\ , \tag{4.1}$$

where Σ_H is the Hartree self-energy matrix. The behavior of the function Δ (the nondiagonal part of the self-energy Σ) is thus clearly linked to the two-particle Green's function L. In particular, if we want to calculate the Bose-Einstein condensation temperature as given by $\Delta(T,n)=0$, we have to include the right physics in the two-particle Green's functions L in (4.1). The pairing approximation corresponds to only keeping the first line on the rhs of (3.10). Thus one is ignoring the effect of excitons with finite center-of-mass momentum, whose excitation is expected to be the main cause of the depletion of the condensate of excitons in the low-density case.[3] As a specific case, let us consider Cu_20. In this material,[5,15] $\epsilon_0\approx 10$ and $E_g=2.2$ eV so that one finds $a_0\approx 10$ Å and $E_0\approx 0.15$ eV. Thus the temperature associated with the ionization of the $1s$ exciton is $T\approx 2000$ K. In contrast, the ordinary Bose-Einstein condensation temperature for a gas of excitons is of the order of $T_{\rm BE}\approx 50$ K, assuming a concentration of $n=1.0\times 10^{19}$ cm^{-3} and taking $m_e=m_h\approx 1.5m_0$ (where m_0 is the mass of a bare electron).

On the other hand, the GRPA equations of motion *generated* from the pairing approximation *do* include the excitons outside the condensate. Setting Δ [and the associated anomalous Green's function $F(\mathbf{p},\omega)$] equal to zero in the GRPA equations in (3.10) will describe the "normal phase" of our system, including excitons. We gain considerable insight into the complicated structure of the GRPA equations of motion by first considering this finite temperature "normal" phase.

Before doing this, it is useful to explicitly exhibit the structure of the two-particle electron-hole Green's func-

tion $L_{1212}(1,2,1',2')$ within the Hartree-Fock-Gor'kov (HFG) approximation. This is given by [see (3.6)]

$$L^0_{1212}(1,2,1',2')=G_e(1,2')G_h(1',2)\ . \tag{4.2}$$

We assume $m_e=m_h$, in which case $\varepsilon_{1,2\mathbf{p}}=\pm\varepsilon_{\mathbf{p}}$. Fourier transforming using the conventions of Sec. III and carrying out the frequency sums, we obtain

$$\begin{aligned}L^0_{1212}(\mathbf{p},\mathbf{k};Q)&=\frac{1}{\beta}\sum_{\omega_p}G_e(P+Q)G_h(-P)\delta_{\mathbf{p},\mathbf{k}}\\&=\left[-\frac{u^2_{p+q}u^2_p}{\omega-(\varepsilon_p+\varepsilon_{p+q})}+\frac{v^2_{p+q}v^2_p}{\omega+(\varepsilon_p+\varepsilon_{p+q})}\right][1-f(\varepsilon_p)-f(\varepsilon_{p+q})]\delta_{\mathbf{p},\mathbf{k}}\\&\quad+\left[-\frac{u^2_{p+q}v^2_p}{\omega+(\varepsilon_p-\varepsilon_{p+q})}+\frac{v^2_{p+q}u^2_p}{\omega-(\varepsilon_p-\varepsilon_{p+q})}\right][f(\varepsilon_p)-f(\varepsilon_{p+q})]\delta_{\mathbf{p},\mathbf{k}}\ .\end{aligned} \tag{4.3}$$

Since $\varepsilon_p,\varepsilon_{p+q}>0$, the Fermi factors $f(\epsilon)$ all vanish at $T=0$ K. This example shows a general feature, namely that the independent-particle approximation for the response functions L^0_{ij} have poles at $\pm(\varepsilon_p-\varepsilon_{p+q})$ at $T\neq 0$ K, in addition to the ones at $\pm(\varepsilon_p+\varepsilon_{p+q})$ which were in evidence at zero temperature in the equations of motion at the end of Sec. III.

Finally, we consider the GRPA equations in Sec. III with $\Delta=0$. This allows us to make contact with the description of an *e-h* system viewed as a normal two-component plasma. To be specific, we take $c=1$ and $d=2$ in (3.12). Physically this corresponds to coupling into an off-diagonal scalar field denoted by $\eta^*(1,2)$ [see (3.1) and (3.2)]. In this case, (3.12) reduces to

$$\begin{aligned}L_{11}(\mathbf{p},\mathbf{k};Q)&=\frac{2}{\beta}\sum_{i\omega_p}\sum_{\mathbf{k}''}G_e(P+Q)G_e(P)V(\mathbf{q})[L_{11}(\mathbf{k}'',\mathbf{k};Q)+L_{22}(\mathbf{k}'',\mathbf{k};Q)]\\&\quad-\frac{1}{\beta}\sum_{i\omega_p}\sum_{\mathbf{k}''}G_e(P+Q)G_e(P)V_S(\mathbf{p}-\mathbf{k}'')L_{11}(\mathbf{k}'',\mathbf{k};Q)\ ,\end{aligned} \tag{4.4a}$$

$$L_{12}(\mathbf{p},\mathbf{k};Q)=\frac{1}{\beta}\sum_{i\omega_p}\sum_{\mathbf{k}''}G_e(P+Q)G_h(-P)V_S(\mathbf{p}-\mathbf{k}'')L_{12}(\mathbf{k}'',\mathbf{k};Q)\ , \tag{4.4b}$$

$$L_{21}(\mathbf{p},\mathbf{k};Q)=\frac{1}{\beta}\sum_{i\omega_p}\sum_{\mathbf{k}''}G_h(-P-Q)G_e(P)V_S(\mathbf{p}-\mathbf{k}'')L_{21}(\mathbf{k}'',\mathbf{k};Q)\ , \tag{4.4c}$$

$$\begin{aligned}L_{22}(\mathbf{p},\mathbf{k};Q)&=\frac{2}{\beta}\sum_{i\omega_p}\sum_{\mathbf{k}''}G_h(-P-Q)G_h(-P)V(\mathbf{q})[L_{11}(\mathbf{k}'',\mathbf{k};Q)+L_{22}(\mathbf{k}'',\mathbf{k};Q)]\\&\quad-\frac{1}{\beta}\sum_{i\omega_p}\sum_{\mathbf{k}''}G_h(-P-Q)G_h(-P)V_S(\mathbf{p}-\mathbf{k}'')L_{22}(\mathbf{k}'',\mathbf{k};Q)\ .\end{aligned} \tag{4.4d}$$

In the normal phase ($\Delta=0$), the pairing approximation for G_1 corresponds to the usual Hartree-Fock approximation. Since the Hartree term vanishes in the expression for the self-energy, (2.5) reduces to a set of two uncoupled equations for $G_e(1,2)$ and $G_h(1,2)$ with the solutions

$$G_e(\mathbf{p},z)=\frac{1}{z-\eta_{e\mathbf{p}}}\ , \tag{4.5a}$$

$$G_h(\mathbf{p},z)=\frac{1}{z-\eta_{h-\mathbf{p}}}\ . \tag{4.5b}$$

It is clear that the equations for L_{12} and L_{21} are similar and, moreover, both are uncoupled from the response functions L_{11} and L_{22}. The eigenvalue equation for L_{12} can be written more explicitly as

$$\begin{aligned}L_{12}(\mathbf{p},\mathbf{k};Q)=-\sum_{\mathbf{k}''}&\frac{[1-f(\eta_{e\mathbf{p}'})-f(\eta_{h-\mathbf{p}})]}{\omega-(\eta_{h-\mathbf{p}}+\eta_{e\mathbf{p}'})}\\&\times V_S(\mathbf{p}-\mathbf{k}'')L_{12}(\mathbf{k}'',\mathbf{p};Q)\ .\end{aligned} \tag{4.6}$$

This is immediately recognized as the usual equation for an exciton, generalized to finite temperature. (See, for example, Ref. 2.) Since we fix the number of particles in the system, the chemical potentials for a normal phase at $T=0$ K would be given (neglecting the exchange self-energy) by $\mu_e\approx E_g+k_F^2/2m_e$, $\mu_h\approx k_F^2/m_h$, correspond-

ing to a degenerate gas at all densities. Since the Bose degeneracy temperature T_{BE} of the excitons is roughly the same as the Fermi degeneracy temperature T_F, one expects that in the normal phase ($T \geq T_{BE}$), the Fermi factors in (4.6) reduce to Boltzmann distributions with $f(\eta) \ll 1$.

If we can neglect the exchange term V_S in (4.4a) and (4.4d), we can combine them to find a closed equation for the quantity

$$\sum_{\mathbf{p}} [L_{11}(\mathbf{p},\mathbf{k};Q)+L_{22}(\mathbf{p},\mathbf{k};Q)] \equiv A(\mathbf{k},Q) \ . \tag{4.7}$$

This has self-sustaining solutions given by the standard RPA dispersion relation for a two-component plasma,

$$1=V(\mathbf{q})\frac{2}{\beta}\sum_{P}[G_e(P+Q)G_e(P) + G_h(-P-Q)G_h(-Q)] \ . \tag{4.8}$$

As discussed in the literature,[27] the solutions of this equation include the usual high-frequency out-of-phase plasmon as well as a low-frequency in-phase acoustic plasmon (albeit strongly damped). We note that for $q=0$, we can solve (4.4a) and (4.4d) exactly and we find that the exchange term has no effect on the plasmon frequency, which justifies our neglect of it for small q.

V. COLLECTIVE MODES IN THE LOW-DENSITY LIMIT AT $T=0$

In this section we discuss the collective modes associated with the GRPA equations given in (3.17) in the presence of a Bose condensate.[28] Owing to the complexity of these equations, we will consider only the long-wavelength limit $q \to 0$. The rhs of these equations may be viewed as corrections to the noninteracting particle-hole excitation spectrum given by the poles of the L^0 functions. At $T=0$ K and to first order in the density ($na_0^3 \ll 1$), these are given by [using (2.11)]

$$\omega_+ = \varepsilon_{1p'} - \varepsilon_{2p} \simeq E_g - \mu + \frac{(\mathbf{p}+\alpha\mathbf{q})^2}{2m} + \frac{q^2}{2M} + S(\mathbf{p}) \ , \tag{5.1a}$$

where $\alpha = m_h/M$ and

$$S(\mathbf{p}) = n\sum_{\mathbf{p}_1} V(\mathbf{p}-\mathbf{p}_1)\phi_1(\mathbf{p}_1)[\phi_1(\mathbf{p})-\phi_1(\mathbf{p}_1)] \ . \tag{5.1b}$$

One finds $\omega_- = \varepsilon_{2p'} - \varepsilon_{1p}$ is given by $-\omega_+$ with α replaced by $\beta = m_e/M$. We consider the terms on the rhs of (3.17) as a small perturbation and set $q=0$ in them. We can now use the approximation $\gamma^2_{\mathbf{p}'\mathbf{p}_0'} \approx \gamma^2_{\mathbf{p}\mathbf{p}_0} = 1-\tilde{\gamma}^{\,2}_{\mathbf{p}\mathbf{p}_0}$. The limit $q \to 0$ in the Hartree term involving $V(\mathbf{q})$ will only be taken at the end since $\tilde{\gamma}_{\mathbf{p}\mathbf{p}}=0$ (we will approximate $\tilde{\gamma}_{\mathbf{p}\mathbf{p}'}$ to first order in $\mathbf{q}$). Defining

$$A_{\mathbf{p}\mathbf{p}_0} \equiv 2\tilde{\gamma}_{\mathbf{p}\mathbf{p}'}\tilde{\gamma}_{\mathbf{p}_0\mathbf{p}_0'}V(\mathbf{q}) \ , \tag{5.2}$$

with these approximations, (3.17) becomes

$$(\omega-\omega_+)G_2^+(\mathbf{p},\mathbf{k};Q)+\sum_{\mathbf{p}_0} V_S(\mathbf{p}-\mathbf{p}_0)G_2^+(\mathbf{p}_0,\mathbf{k};Q) = \sum_{\mathbf{p}_0} A_{\mathbf{p}\mathbf{p}_0}[G_2^+(\mathbf{p}_0,\mathbf{k};Q)+G_2^-(\mathbf{p}_0,\mathbf{k};Q)] + \sum_{\mathbf{p}_0}\tilde{\gamma}^{\,2}_{\mathbf{p}\mathbf{p}_0}V_S(\mathbf{p}-\mathbf{p}_0)[G_2^+(\mathbf{p}_0,\mathbf{k};Q)-G_2^-(\mathbf{p}_0,\mathbf{k};Q)] \ , \tag{5.3a}$$

$$(\omega-\omega_-)G_2^-(\mathbf{p},\mathbf{k};Q)-\sum_{\mathbf{p}_0} V_S(\mathbf{p}-\mathbf{p}_0)G_2^-(\mathbf{p}_0,\mathbf{k};Q) = -\sum_{\mathbf{p}_0} A_{\mathbf{p}\mathbf{p}_0}[G_2^+(\mathbf{p}_0,\mathbf{k};Q)+G_2^-(\mathbf{p}_0,\mathbf{k};Q)] + \sum_{\mathbf{p}_0}\tilde{\gamma}^{\,2}_{\mathbf{p}\mathbf{p}_0}V_S(\mathbf{p}-\mathbf{p}_0)[G_2^+(\mathbf{p}_0,\mathbf{k};Q)-G_2^-(\mathbf{p}_0,\mathbf{k};Q)] \ . \tag{5.3b}$$

Accurate to terms of order n, we can write

$$\tilde{\gamma}^2_{\mathbf{p}\mathbf{p}_0} \approx \frac{n}{2}[\phi_1(\mathbf{p})-\phi_1(\mathbf{p}_0)]^2 \ . \tag{5.4}$$

The left-hand side (lhs) of (5.3) gives us the exciton energies for $V_S \to V$ (i.e., bare Coulomb interaction) renormalized by exchange and band-filling corrections. The rhs are corrections of order n to the energy of these "bare" excitons.

One can solve (5.3) formally by introducing the auxiliary functions $G_2^{0\pm}$ defined as the solutions of

$$(\omega-\omega_\pm)G_2^{0\pm}(\mathbf{p},\mathbf{k};Q)\pm\sum_{\mathbf{p}_0} V(\mathbf{p}-\mathbf{p}_0)G_2^{0\pm}(\mathbf{p}_0,\mathbf{k};Q) = \pm\delta_{\mathbf{p},\mathbf{k}} \ . \tag{5.5}$$

These solutions can be shown to be given by

$$G_2^{0+}(\mathbf{p},\mathbf{k};Q) = \sum_r \frac{\phi_r(\mathbf{p}+\alpha\mathbf{q})\phi_r^*(\mathbf{k}+\alpha\mathbf{q})}{\omega+i\delta-(\omega_{r\mathbf{q}}-\mu)} \ , \tag{5.6a}$$

$$G_2^{0-}(\mathbf{p},\mathbf{k};Q) = -\sum_r \frac{\phi_r(\mathbf{p}+\beta\mathbf{q})\phi_r^*(\mathbf{k}+\beta\mathbf{q})}{\omega+i\delta+(\omega_{r\dot{\mathbf{q}}}-\mu)} \ , \tag{5.6b}$$

where $\phi_r(\mathbf{p})$ are the eigenfunctions of the Schrödinger equation for the Wannier exciton with $\alpha=m_h/M$, $\beta=m_e/M$, and $M=m_e+m_h$. $r \equiv \{n,l,m\}$ is the set of quantum number for the exciton states whose renormalized energies, in complete analogy with the hydrogen atom, are given by [$S(\mathbf{p})$ is defined in (5.1)]

$$\omega_{r\mathbf{q}}=E_g-\frac{E_0}{n^2}+\frac{q^2}{2M}+\sum_{\mathbf{p}}|\phi_r(\mathbf{p})|^2 S(\mathbf{p}) . \tag{5.7}$$

While we shall speak in term of these "bound" excitonic states, formally the sum over r in (5.6) also includes the scattering e-h states as well.

With the aid of (5.5) and (5.6), (5.3) can now be written as

$$B^{\pm}(\mathbf{p},\mathbf{k};Q)=\sum_{\mathbf{p}_0\mathbf{k}_0}\bar{\gamma}^2_{\mathbf{k}_0\mathbf{p}_0}V(\mathbf{k}_0-\mathbf{p}_0)\times B^{0\mp}(\mathbf{p},\mathbf{k}_0;Q)B^{-}(\mathbf{p}_0,\mathbf{k};Q)+\sum_{\mathbf{p}_0\mathbf{k}_0}A_{\mathbf{k}_0\mathbf{p}_0}B^{0\pm}(\mathbf{p},\mathbf{k}_0;Q)B^{+}(\mathbf{p}_0,\mathbf{k};Q) , \tag{5.8}$$

where we have defined

$$B^{\pm}(\mathbf{p},\mathbf{k};Q)\equiv G_2^{+}(\mathbf{p},\mathbf{k};Q)\pm G_2^{-}(\mathbf{p},\mathbf{k};Q) . \tag{5.9}$$

Neglecting the $\mathbf{q}$ dependence of the wave functions in (5.6), (5.8) can be rewritten, after some work, as a system of coupled linear equations (both s and r represent the complete set of quantum numbers $\{n,l,m\}$)

$$A_s(Q)=\sum_r R_{sr}(Q)X_r^{-}(Q)A_r(Q)+\sum_r R_{sr}(Q)X_r^{+}(Q)f_r^{*}(\mathbf{q})B(Q) , \tag{5.10a}$$

$$B(Q)=2\sum_r V(\mathbf{q})f_r(\mathbf{q})X_r^{+}(Q)A_r(Q)+2\sum_r V(\mathbf{q})f_r^{*}(\mathbf{q})f_r(\mathbf{q})X_r^{-}(Q)B(Q) , \tag{5.10b}$$

where we have defined

$$A_r(Q)\equiv\sum_{\mathbf{k}\mathbf{p}\mathbf{p}_0}\phi_r^{*}(\mathbf{p}_0)\bar{\gamma}^2_{\mathbf{p}\mathbf{p}_0}V(\mathbf{p}_0-\mathbf{p})B^{-}(\mathbf{p},\mathbf{k};Q) , \tag{5.11}$$

$$B(Q)\equiv\sum_{\mathbf{p}\mathbf{k}}\bar{\gamma}_{\mathbf{p}\mathbf{p}'}V(\mathbf{q})B^{+}(\mathbf{p},\mathbf{k};Q) , \tag{5.12}$$

$$R_{sr}\equiv\sum_{\mathbf{p}\mathbf{p}_0}\phi_s^{*}(\mathbf{p}_0)\phi_r(\mathbf{p})\bar{\gamma}^2_{\mathbf{p}\mathbf{p}_0}V(\mathbf{p}-\mathbf{p}_0) , \tag{5.13}$$

$$X_r^{\pm}(Q)\equiv\frac{1}{\omega+i\delta-(\omega_{r\mathbf{q}}-\mu)}\pm\frac{1}{\omega+i\delta+(\omega_{r\mathbf{q}}-\mu)} , \tag{5.14}$$

$$f_r(\mathbf{q})\equiv\sum_{\mathbf{p}}\phi_r(\mathbf{p})\bar{\gamma}_{\mathbf{p}\mathbf{p}'} . \tag{5.15}$$

Clearly, one can immediately solve (5.10b) to give

$$B(Q)=\frac{2\Sigma_r V(\mathbf{q})f_r(\mathbf{q})X_r^{+}(Q)A_r(Q)}{[1-2\Sigma_r V(\mathbf{q})f_r^{*}(\mathbf{q})f_r(\mathbf{q})X_r^{-}(Q)]} , \tag{5.16}$$

and so (5.10a) reduces to

$$A_s(Q)=\sum_r[R_{sr}X_r^{-}(Q)+2C_s(Q)f_r(\mathbf{q})V(\mathbf{q})X_r^{+}(Q)]A_r(Q) , \tag{5.17}$$

where

$$C_s(Q)\equiv\frac{\Sigma_r R_{sr}X_r^{+}(Q)f_r^{*}(\mathbf{q})}{[1-2\Sigma_r V(\mathbf{q})f_r^{*}(\mathbf{q})f_r(\mathbf{q})X_r^{-}(Q)]} . \tag{5.18}$$

From the definitions (5.11) and (5.12), we see that $B(Q)=0$ if we ignore the Hartree term $V(\mathbf{q})\to 0$. On the other hand, $A_r(Q)=0$ if we ignore the exchange term, $V(\mathbf{p}_0-\mathbf{p})\to 0$.

Equation (5.17) is of the form

$$A_s(Q)=\sum_r D_{sr}A_r(Q) , \tag{5.19}$$

and thus the collective modes are given by the solutions of

$$\det(I-D)=0 . \tag{5.20}$$

It is of course impossible to solve exactly (5.20) and we must introduce some approximation. We suppose that, when the density is low, the original exciton modes will not be affected too much by the presence of the condensate. We thus look for solutions of (5.20) with $\omega\approx\omega_{r'\mathbf{q}}-\mu$, where $r'=(n',l',m')$. This is equivalent to keeping in the summation over $r=(n,l,m)$ in (5.17) only the terms with a denominator having $r=r'$, i.e., the dominant terms in the sum. It is easily seen that (5.20) then reduces to

$$\det(I'-D')=0 , \tag{5.21}$$

or

$$1-D_{r'r'}=0 . \tag{5.22}$$

I' is the unit matrix of the same dimension. The condition in (5.20) can be interpreted as a coupling between the different excitonic modes in the system due to the presence of the condensate [compare with (4.6) in the normal phase where excitons are well defined]. In our dominant-term approximation, the coupling between energy levels is ignored. For $n'=1$, the solution of (5.22) is easy since one may show that $f_{n=1}(\mathbf{q})=0$ to first order in q. In this case we obtain the phononlike dispersion relation

$$\omega=\left[\left[\frac{q^2}{2M}\right]^2+R_{11}\left[\frac{q^2}{M}\right]\right]^{1/2} , \tag{5.23}$$

with R_{11} given by[20,2]

$$R_{11}=\sum_{\mathbf{p}}|\phi_1(\mathbf{p})|^2 S(\mathbf{p}) . \tag{5.24}$$

This is identical to the lowest-order correction to the chemical potential found in the pairing approximation,[2] i.e., $R_{11}=\mu-\mu_0$. Using this fact, one sees that (5.23) is identical to the result first derived by Keldysh and Kozlov[1] using a somewhat different formulation. In the limit $q\to 0$, we note that the frequency given by (5.23) satisfies our assumption that $\omega\approx(\omega_{1q}-\mu)$.

The dispersion relation for the state r' is more generally given by

$$1-R_{r'r'}X_{r'}^{-}(Q)-2C_{r'}(Q)f_{r'}(\mathbf{q})V(\mathbf{q})X_{r'}^{+}(Q)=0 . \tag{5.25}$$

Approximating $C_{r'}$ by keeping only the dominant term as explained above, we find

$$C_{r'}=\frac{R_{r'r'}X_r^+ f_{r'}^*(\mathbf{q})}{1-2V(\mathbf{q})|f_{r'}(\mathbf{q})|^2 X_{r'}^-}\ , \tag{5.26}$$

and hence (5.25) reduces to

$$1-2[V(\mathbf{q})|f_{r'}(\mathbf{q})|^2+R_{r'r'}]X_{r'}^- +2R_{r'r'}V(\mathbf{q})|f_{r'}(\mathbf{q})|^2[(X_{r'}^-)^2-(X_{r'}^+)^2]=0\ , \tag{5.27}$$

if the denominator of (5.26) does not vanish. All terms proportional to $V(\mathbf{q})$ are associated with the Hartree term.

To order na_0^3, (5.27) leads to the dispersion relation (at this point, we relabel r' by r)

$$\omega^2=(\omega_{r\mathbf{q}}-\mu)^2+2(\omega_{r\mathbf{q}}-\mu)[R_{rr}+2V(\mathbf{q})f_r^*(\mathbf{q})f_r(\mathbf{q})]\cdot +8R_{rr}V(\mathbf{q})f_r^*(\mathbf{q})f_r(\mathbf{q})\ . \tag{5.28}$$

For r corresponding to an s state ($l=0$) one may show that the Hartree term involving $V(\mathbf{q})$ vanishes since $f_r(\mathbf{q})=0$ in the limit $q\to 0$. Using (5.7), we can write (5.28) in the form

$$\omega^2=\left[\frac{q^2}{2M}\right]^2+(W_r+R_{rr}+\beta_r)\frac{q^2}{M} +W_r^2+2W_rR_{rr}+4W_r\beta_r+4R_{rr}\beta_r\ , \tag{5.29}$$

where [ΔE_r is the last term in (5.7)]

$$W_r=E_0\left[1-\frac{1}{n_r^2}\right]+\Delta E_r-\Delta\mu\ , \qquad \omega_{r\mathbf{q}}-\mu=W_r+\frac{q^2}{2M},\quad \beta_r=2V(\mathbf{q})f_r^*(\mathbf{q})f_r(\mathbf{q})\ . \tag{5.30}$$

For excited states ($n=2,3,\ldots$), the dominant-term approximation gives

$$\omega\simeq W_r+R_{rr}+\beta_r+\frac{q^2}{2M}\ . \tag{5.31}$$

Thus the excited-state exciton kinetic energy is not modified, in contrast to the $1-s$ exciton given by (5.23). Clearly, further studies are needed taking account of the off-diagonal terms of $R_{rr'}$, but this will probably require numerical work.

In order to discuss high-frequency collective modes, we neglect the exchange term in (5.10) (including both Hartree and Fock terms seems very difficult). In this approximation, both $A_r(Q)$ and R_{sr} vanish. Equation (5.10b) then leads to the condition

$$1-\sum_r V(\mathbf{q})f_r^*(\mathbf{q})f_r(\mathbf{q})X_r^-(Q)=0\ . \tag{5.32}$$

We use the $q\to 0$ approximation

$$f_r(\mathbf{q})\simeq\sqrt{n/2}\,\mathbf{q}\cdot\nabla_p\phi_1(\mathbf{p})\delta_{r,\mathbf{p}}\ . \tag{5.33}$$

In the absence of bound states, (5.33) can be obtained from (5.15) by the substitutions $\mathbf{r}\to\mathbf{p}$, $\phi_r(\mathbf{p}_0)\to\phi_{\mathbf{p}}(\mathbf{p}_0)=\delta_{\mathbf{p}_0\mathbf{p}}$, and

$$\omega_{n\mathbf{q}}-\mu\to\omega_{\mathbf{p},\mathbf{q}}-\mu=E_0+\frac{p^2}{2m}+\frac{q^2}{2M}\ . \tag{5.34}$$

Equation (5.32) thus becomes

$$1=2V(\mathbf{q})\sum_{\mathbf{p}}\frac{\tilde{\gamma}_{\mathbf{p}\mathbf{p}'}^2\left[E_0+\frac{p^2}{2m}\right]}{\omega^2-\left[E_0+\frac{p^2}{2m}\right]^2} \tag{5.35}$$

in the $q\to 0$ limit and ignoring the exchange term.

This dispersion relation can be also obtained directly from (3.13). Neglecting $V_S(\mathbf{p}-\mathbf{p}'')$ in (3.13), we find

$$1=2V(\mathbf{q})\sum_{\mathbf{p}}\frac{\tilde{\gamma}_{\mathbf{p}\mathbf{p}'}^2(\varepsilon_{\mathbf{p}}+\varepsilon_{\mathbf{p}'})}{\bar{\omega}^2-(\varepsilon_{\mathbf{p}}+\varepsilon_{\mathbf{p}'})^2}\ , \tag{5.36}$$

a result which in fact is valid in both the high- and low-density limits. Taking the low-density limit of (5.36) gives (5.35) if we include the bound states.

Approximating the denominator in (5.35) by ω^2-G^2, where G is an effective gap, we find after some algebraic manipulation that[2]

$$\omega^2=E_0^2+\omega_{\mathrm{pl}}^2\ , \tag{5.37}$$

where $G\simeq E_0$ and

$$\omega_{\mathrm{pl}}^2=\frac{4\pi ne^2}{m}\ . \tag{5.38}$$

This shows clearly that the excitons must first be broken before the electrons and holes can give rise to this high-frequency collective excitation. Similarly, in the high-density limit, (5.36) gives[8]

$$\omega^2=(2\Delta)^2+\omega_{\mathrm{pl}}^2\ , \tag{5.39}$$

in complete formal analogy with (5.37). However, in this limit, one has $\omega_{\mathrm{pl}}\gg 2\Delta$ while in the low-density limit, $\omega_{\mathrm{pl}}\simeq E_0$.

The plasmon damping is quite different in the presence of an exciton condensate. In the normal phase discussed in Sec. IV, the excitons are associated with the poles at $\pm(\varepsilon_{\mathbf{p}}+\varepsilon_{\mathbf{p}+\mathbf{q}})$ while the plasmon modes are associated with the poles at $\pm(\varepsilon_{\mathbf{p}}-\varepsilon_{\mathbf{p}+\mathbf{q}})$. In the Bose-condensed phase, however, these collective modes are more complicated because now the noninteracting spectrum involves poles at both $\pm(\varepsilon_{\mathbf{p}}+\varepsilon_{\mathbf{p}+\mathbf{q}})$ and $\pm(\varepsilon_{\mathbf{p}}-\varepsilon_{\mathbf{p}+\mathbf{q}})$. In particular, at $T=0$ K, the single-particle spectrum into which the plasmon can decay is given by $(\varepsilon_{\mathbf{p}}+\varepsilon_{\mathbf{p}+\mathbf{q}})$ instead of the normal $(\varepsilon_{\mathbf{p}}-\varepsilon_{\mathbf{p}+\mathbf{q}})$. As a result, the plasmon mode can be damped even in the long-wavelenth limit $q\to 0$. Physically this arises because the charge fluctuations are coupled into the order parameter fluctuations and this opens up a new decay mechanism. We note that the same kind of plasmon damping occurs in BCS superconductors,[23,24] although it does not seem to have been ever worked out in detail.

VI. CONCLUDING REMARKS

The pairing approximation for the single-particle self-energies has been extensively studied in recent years as a simple way to discuss the exciton condensate in a two-band direct-band-gap semiconductor (in both the low-

and high-density limits). Using functional derivative techniques, we have used this simple approximation for the single-particle matrix Green's function to generate the various two-particle Green's functions G_2. We have used the resulting equations of motion (generalized RPA) to give a systematic analysis of the collective modes as function of the density. Our results make contact with and generalize earlier theoretical studies.[1,6–9] In the low-density limit, we gave a detailed discussion of the fluctuations of the Bose condensate of 1s excitons, i.e., excitons in higher-energy states ($n=2,3,\ldots$) as well as those with finite center-of-mass momentum.

At finite temperatures, the 1s-exciton condensate will not be depleted by ionization but rather by excitation of excitons (e-h collective modes) with finite center-of-mass momentum. A major extension of the present paper would be to generalize the BCS-like self-consistent equation for the order parameter Δ [see (2.14)] to include the t-matrix self-energy arising from such excitons. If one is mainly interested in the temperature region close to the Bose-condensation phase boundary, it is probably sufficient to base this extension on the t matrix for free excitons [see (5.5)]. This will be discussed in more detail elsewhere (see also Ref. 28).

In the normal phase where there is no exciton condensate, the GRPA equations of motion for G_2 describe a phase in which the excitons are completely uncoupled from the density fluctuations (see Sec. IV). We call attention to the extensive work[29] on the free energy of this phase which includes both the contributions of electron and hole Fermi quasiparticles as well as the Bose collective modes (the excitons) in a t-matrix approximation.

The simple direct-gap semiconductor model we have discussed in this paper may be appropriate for optically pumped Cu_2O. Further work is needed, however, before one can make contact with experiment.[15] In particular, in evaluating the decay luminescense of a system with an exciton condensate, we would need to calculate the temperature-dependent weight of the excitonic modes we discussed in Sec. V. The only study of the decay emission spectrum in the literature is based on a weakly interacting Bose gas model.[30] One can show[28] that the GRPA equations of motion leads to results consistent with this work. Ultimately one would also have to take into account the external pump mechanism as well as the various relaxation and decay processes before addressing the question of to what extent such a Bose condensate of excitons would exhibit superfluid flow characteristics.[12–14]

Note added in proof. (1) After this work had been completed, we became aware of similar work using a generalized kinetic equation approach, including the effect of the laser pumping field, by S. Schmitt-Rink, D. S. Chemla, and H. Haug (Phys. Rev., in press).

(2) Related work on the collective modes of e-h condensates in a high magnetic field is discussed by Y. Kuramoto, J. Phys. Soc. Jpn. **42**, 1143 (1976).

ACKNOWLEDGMENT

This research was supported by a grant from the Natural Sciences and Engineering Research Council (NSERC) of Canada.

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