Finite-temperature contributions to the specific heat of the electron-phonon system

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We calculate the $T^3\ln T$ contribution to the specific heat of metals due to the electron-phonon interaction and show that it may be regarded either as a Bose contribution or as a Fermi one. We find that the magnitude of the coefficient of the $T^3\ln T$ term is $\frac{12}{7}$ times larger than that calculated by previous authors and demonstrate that the difference between the results is due to our inclusion of the temperature dependence of the quasiparticle spectrum. We discuss renormalization effects due to the electron-phonon interaction and, by using experimentally determined values of the relevant parameters in our expressions, show that for ordinary metals one would not expect to be able to isolate this $T^3\ln T$ contribution in measurements of the specific heat.

I. INTRODUCTION

The low-temperature specific heat of normal metals has the form

$$C=\gamma T+\Gamma T^3\ln T+\beta^* T^3 \ . \tag{1}$$

The leading contribution at low temperature is the linear term, which is enhanced by the electron-phonon interaction, as was shown by Buckingham[1] and Buckingham and Schafroth.[2] It may be written as

$$\gamma=\gamma_0(1+\tilde{\lambda}) \ , \tag{2}$$

where $\tilde{\lambda}$ is a measure of the strength of the electron-phonon coupling, and γ_0 is the value of γ in the absence of the electron-phonon interactions. The $T^3\ln T$ contributions come from electron-electron interactions, both those induced by phonons and those due to the Coulomb interaction between electrons. Buckingham and Schafroth first showed, using perturbation theory to calculate the free energy, that there is a $T^3\ln T$ contribution to the specific heat due to phonon induced electron-electron interactions. The presence of this $T^3\ln T$ contribution was confirmed by Eliashberg[3] and by Nakajima and Watabe[4] who started from an expression for the thermodynamic potential in terms of Green's functions. The coefficient of this $T^3\ln T$ contribution is negative and it therefore gives a positive contribution to the specific heat at low temperatures.[5] The cubic term in Eq. (1) contains the leading-order contribution to the specific heat due to thermal excitation of phonons, as well as contributions from the electron-phonon interaction.

The $T^3\ln T$ contribution from the Coulomb interaction is analogous to that found in normal liquid ^{3}He. It is generally small, and it can be large only when the metal is almost unstable to small deformations of the Fermi surface. Under such circumstances, Γ is positive, and thus the $T^3\ln T$ term reduces the specific heat at low temperatures. It is particularly large if the system is almost ferromagnetic, since one then finds large contributions to Γ from spin fluctuations.[6] The large temperature-dependent contributions to the specific heat observed in heavy-fermion compounds[7–10] have been interpreted in terms of such a model.[11,12]

In the case of normal Fermi liquids it has been shown that the $T^3\ln T$ contribution to the specific heat may be regarded as a consequence of the interaction between quasiparticles being a nonanalytic function of the momenta of the two quasiparticles for small momentum differences.[13] In this paper we show that the $T^3\ln T$ term due to the electron-phonon interaction may be regarded as having a similar origin, and that the results for a Fermi liquid and for metals may be cast in similar forms. We also show that this $T^3\ln T$ term may be rewritten in a form similar to that suggested by Danino and Overhauser,[14] which has a natural interpretation as a contribution due to the damping of phonons. This establishes that these two contributions are one and the same.

Another topic we consider is the magnitude of Γ. We find this to have a larger numerical coefficient than that given in most previous calculations. This difference may be traced to the fact that earlier the temperature dependence of the quasiparticle energy was generally neglected. We also explore the effects of vertex renormalization, and show that these are not significant for the long-wavelength phonons that give rise to the $T^3\ln T$ terms.

This paper is organized as follows. In Sec. II we introduce a simple model which describes the $T^3\ln T$ term in the specific heat and we show that the $T^3\ln T$ term may be considered to be either a Fermi or a Bose contribution to the specific heat. In Sec. III we discuss renormalization of the $T^3\ln T$ terms by Coulomb interactions. Renormalization effects due to vertex corrections to the electron-phonon coupling are investigated in Sec. IV where we also make numerical estimates of the size of the $T^3 \ln T$ term. In Sec. V we briefly discuss our conclusions.

II. SIMPLE MODEL

In order to avoid inessential complications, we first describe a simple illustrative example. Consider electrons interacting with acoustic phonons in a model containing a single species of ion: the contribution to the thermodynamic potential Ω, is given to second order in the electron-phonon matrix element, $W(\mathbf{q})$, by

$$\Delta\Omega = \int_{-\infty}^{\infty} \frac{d\omega}{2\pi} n_B(\omega) \sum_{\mathbf{q}} |W(\mathbf{q})|^2 \times \{ \mathrm{Im}\chi(\mathbf{q},\omega)\mathrm{Re}D(\mathbf{q},\omega) + \mathrm{Re}\chi(\mathbf{q},\omega)\mathrm{Im}D(\mathbf{q},\omega) \} , \quad (3)$$

where

$$\chi(\mathbf{q},\omega) = 2\sum_{\mathbf{p}} \frac{n_{\mathbf{p}} - n_{\mathbf{p}+\mathbf{q}}}{\omega - (e_{\mathbf{p}} - e_{\mathbf{p}+\mathbf{q}})} ,$$

and

$$D(\mathbf{q},\omega) = \frac{n_i q^2}{M_i} \frac{[\hat{\mathbf{q}}\cdot\zeta(\mathbf{q},\alpha)]^2}{\omega^2 - \omega(\mathbf{q},\alpha)^2} ,$$

is the longitudinal part of the bare phonon propagator, $n_B(\omega)$ is the Bose-Einstein distribution function, and $n_{\mathbf{p}}$ is the Fermi-Dirac distribution function. $e_{\mathbf{p}}$ is the electron energy, $\zeta(\mathbf{q},\alpha)$ and $\omega(\mathbf{q},\alpha)$ are the polarization vector and angular frequency of the phonon with wave vector $\mathbf{q}$ and polarization α, M_i is the mass of an ion, and n_i is the number of ions per unit volume. Equation (3) may be written as

$$\Delta\Omega = \frac{n_i}{2M_i} \int_{-\infty}^{\infty} \frac{d\omega}{\pi} n_B(\omega) \sum_{\mathbf{q}} [\mathbf{q}\cdot\zeta(\mathbf{q},\alpha)]^2 |W(\mathbf{q})|^2 \times \left[P\frac{q^2 \mathrm{Im}\chi(q,\omega)}{\omega^2 - \omega^2(\mathbf{q},\alpha)} + \frac{q^2\pi}{2\omega(\mathbf{q},\alpha)} \mathrm{Re}\chi(q,\omega)[\delta(\omega-\omega(\mathbf{q},\alpha)) + \delta(\omega+\omega(\mathbf{q},\alpha))] \right] , \quad (4)$$

where P denotes a principal value integral. The second term in the large parentheses gives T^3 terms and since here we consider only terms up to $T^3 \ln T$ we shall ignore it. For later purposes we note that the remaining term in Eq. (4) may be rewritten as

$$\Delta\Omega = \tfrac{1}{2} \sum_{\mathbf{pq}} n_{\mathbf{p}}(1 - n_{\mathbf{p}+\mathbf{q}}) |W(\mathbf{q})|^2 \mathrm{Re}D(q, e_{\mathbf{p}} - e_{\mathbf{p}+\mathbf{q}}) . \quad (5)$$

We shall use the Debye model, in which the solid is treated as isotropic, and shall sum over all wave numbers within the Debye sphere, whose radius is q_D. Only low frequencies are important at the temperatures which are of interest here so that we may use the long-wavelength form of $\chi(q,\omega)$,

$$\mathrm{Im}(q,\omega) = \mathrm{Im}\chi(s) = \frac{\pi}{2} N^0 s \Theta(1 - |s|) , \quad (6)$$

where $N^0 = m p_F / \pi^2 \hbar^3$ is the bare density of electron states at the Fermi surface, $s = \omega q v_F^0$, and Θ is the unit step function. Here p_F is the Fermi momentum, $v_F^0 = p_F/m$ is the Fermi velocity, and m is the electron mass. We shall also replace the matrix element, $W(\mathbf{q})$, by a constant W. With these approximations Eq. (5) becomes

$$\Delta\Omega = \frac{n_i |W|^2}{2M_i} P \int_{-\infty}^{\infty} \frac{d\omega}{2\pi} n_B(\omega) \sum_{\mathbf{q}} \mathrm{Im}\chi(s) \frac{q^2}{\omega^2 - \omega(q)^2} , \quad (7)$$

where $\omega(q) = c_L q$ and c_L is the speed of longitudinal sound, and the contribution to the entropy is

$$\Delta S = -\frac{n_i |W|^2}{2M_i} \times \int_{-\infty}^{\infty} \frac{d\omega}{\pi} \frac{\partial n_B(\omega)}{\partial T} \sum_{\mathbf{q}} \mathrm{Im}\chi(s) \frac{q^2}{\omega^2 - \omega(q)^2} , \quad (8)$$

$\chi(q,\omega)$ is a slowly varying function of temperature, and therefore in Eq. (8) we have ignored its derivative with respect to temperature. The integrand in Eq. (8) can be expanded in powers of (ω/q), and the contribution from the leading power, (ω/q), is

$$\Delta S^{(1)} = \sum_{\mathbf{q}} \int_0^{\infty} d\omega \frac{\partial n_B(\omega)}{\partial T} \lambda^0 \frac{\omega}{q v_F^0} \Theta\left[1 - \frac{\omega}{q v_F^0}\right] = \lambda^0 n_e k_B^2 \frac{\pi^2}{2} \left[\frac{T}{T_F^0}\right] \left[\left[\frac{q_D}{2p_F}\right]^2 - \frac{\pi^2}{5}\left[\frac{T}{T_F^0}\right]^2 \right] , \quad (9)$$

where $\lambda^0 = N^0 |W|^2 n_i / 2M_i c_L^2$ is the electron-phonon coupling constant, $n_e = p_F^3 / 3\pi^2\hbar^3$ is the number density of electrons, and $T_F^0 = p_F^2/2m$ is the bare Fermi energy. From Eq. (9) one sees that the enhancement of the effective mass, $\tilde{\lambda}$, Eq. (2), due to the electron-phonon interaction is $\lambda^0 (q_D/2p_F)^2$. The rest of the contribution to order $T^3 \ln T$ is

$$\Delta S^{(2)} = -\int_0^{\infty} d\omega \frac{\partial n_B(\omega)}{\partial T} \lambda^0 \sum_{\mathbf{q}} \frac{\omega}{q v_F} \frac{\omega^2}{\omega^2 - (c_L q)^2} = -\frac{\pi^4}{20} n_e k_B^2 \lambda^0 \left[\frac{v_F^0}{c_L}\right]^2 \left[\frac{T}{T_F^0}\right]^3 \ln T . \quad (10)$$

The $T^3\ln T$ contribution to the specific heat is found by differentiating $\Delta S^{(2)}$, and is equal to

$$\Delta C = -\frac{3\pi^4}{20} n_e k_B^2 \lambda^0 \left[\frac{v_F^0}{c_L}\right]^2 \left[\frac{T}{T_F^0}\right]^3 \ln T \ . \quad (11)$$

This result may be obtained from the quasiparticle spectrum, as we now show. Equation (7) may be rewritten as

$$\Delta\Omega = \frac{1}{2}\sum_{\mathbf{pq}} n_{\mathbf{p}}(1-n_{\mathbf{p+q}})\frac{n_i |W|^2}{M_i}\frac{q^2}{(e_{\mathbf{p+q}}-e_{\mathbf{p}})^2-(c_L q)^2} \ . \quad (12)$$

The change in the thermodynamic potential due to the electron-phonon interaction, Eq. (12), at fixed electron chemical potential is the same as the change in the energy of the system at fixed electron number density since we consider only contributions of lowest order in the electron-phonon coupling:

$$\Delta\Omega(\mu) = \Delta E(n) \ . \quad (13)$$

Consequently, we may determine the contribution to the quasiparticle spectrum, $\Delta e_{\mathbf{p}}$, and the effective interaction between quasiparticles, $f_{\mathbf{p,p+q}}$ in the usual manner by functional differentiation, and we find:

$$\Delta e_{\mathbf{p}} = \frac{\delta \Delta E}{\delta n_{\mathbf{p}}} = \frac{1}{2}\sum_{\mathbf{q}} (1-2n_{\mathbf{p+q}})\frac{n_i |W|^2}{M_i}\frac{q^2}{(e_{\mathbf{p}}-e_{\mathbf{p+q}})^2-(c_L q)^2} \quad (14)$$

and

$$f_{\mathbf{p,p+q}} = \frac{\delta^2 \Delta E}{\delta n_{\mathbf{p}}\delta n_{\mathbf{p+q}}} = \frac{n_i |W|^2}{M_i}\frac{q^2}{(e_{\mathbf{p}}-e_{\mathbf{p+q}})^2-(c_L q)^2} \ . \quad (15)$$

Note that $f_{\mathbf{p,p+q}}$ is nonanalytic for $e_{\mathbf{p}}-e_{\mathbf{p+q}}\to 0$, just as it is in a normal Fermi liquid with two-body fermion-fermion interactions. We can now calculate the entropy from the quasiparticle spectrum. We are interested in the $T^3\ln T$ terms in the specific heat which come from the $(e_{\mathbf{p}}-e_{\mathbf{p+q}})^2/(c_L q)^2$ term in the effective interaction and we therefore evaluate Eq. (14) keeping just this dependence. We find

$$\Delta e_{\mathbf{p}} = -\frac{n_i |W|^2}{M_i c_L^2}(p-p_F)[v_F^{0^2}(p-p_F)^2 + \pi^2(k_B T)^2]\ln|p-p_F| \ . \quad (16)$$

One sees from the brackets of Eq. (16) that there is a contribution to the $T^3\ln T$ term in the entropy from the temperature dependence of the quasiparticle spectrum. The $T^3\ln T$ terms in the entropy are given by

$$\Delta S = \sum_{\mathbf{p}} \Delta e_{\mathbf{p}} \frac{\partial n_{\mathbf{p}}}{\partial T} = -\lambda^0 \frac{\pi^4}{16} n_e k_B^2 \left[\frac{v_F^0}{c_L}\right]^2 \left[\frac{T}{T_F^0}\right]^3 \left(\tfrac{7}{15}+\tfrac{1}{3}\right)\ln T = -\lambda^0 \frac{\pi^4}{20} n_e k_B^2 \left[\frac{v_F^0}{c_L}\right]^2 \left[\frac{T}{T_F^0}\right]^3 \ln T \ . \quad (17)$$

In the second of these equations, the $\frac{7}{15}$ comes from the $(p-p_F)^3$ term in Eq. (16) and the $\frac{1}{3}$ from the $(p-p_F)T^2$ term. Neglect of the temperature dependence of the quasiparticle spectrum results in a coefficient of the $T^3\ln T$ terms which is $\frac{7}{12}$ the actual one. Previous authors have underestimated the size of the $T^3\ln T$ contribution in this manner.

Danino and Overhauser[14] have considered the contribution of phonons to the specific heat of metals and they discovered a $T^3\ln T$ contribution due to the ω^2 dependence in the phonon spectral density, which they suggested might be additional to the one found earlier. However, this $T^3\ln T$ contribution is the same as that discussed above since the part of Eq. (7) which leads to the logarithmic terms may be rewritten in the form

$$\Delta\Omega = \int_0^\infty \frac{d\omega}{\pi} n_B(\omega)\omega F(\omega) \ , \quad (18)$$

where

$$F(\omega) = \frac{n_i |W|^2}{2M_i c_L^2}\sum_{\mathbf{q}} \frac{\mathrm{Im}\chi(\mathbf{q},\omega)}{\omega}\frac{q^2}{\omega^2-(c_L q)^2} \ , \quad (19)$$

which for small ω is proportional to $\omega^2\ln\omega$. Equation (18) is of the same form as Eq. (24) of Ref. 14, so the $T^3\ln T$ contributions to the specific heat due to the electron-phonon interaction may be regarded as either a Fermi or a Bose contribution. This point was also made previously[15] for the $T^3\ln T$ contributions to the specific heat of normal liquid ^{3}He from spin fluctuations. Note that Eq. (18) has the appearance of a contribution due to damping of phonons, which is proportional to $\mathrm{Im}\chi(\mathbf{q},\omega)$. We now consider the effect of other interactions on the $T^3\ln T$ term in the specific heat.

III. ELECTRON-ELECTRON INTERACTIONS AND BAND STRUCTURE

In metals it is important to take into account screening by electrons when determining the phonon-phonon interaction and the effective electron-phonon matrix element, as explained in Refs. 16 and 17. In addition, band-structure effects alter the quasiparticle effective mass. Calculations of the $T^3\ln T$ term due to the electron-phonon interaction may be carried out taking these effects into account, but since the calculations are a straightforward extension of earlier work, we shall not describe them in detail here. If one assumes that the Fermi surface is spherical to a good approximation when the effects of band structure are included, one finds

$$\Delta C = -\frac{3\pi^4}{20} n_e k_B^2 \lambda^C \left[\frac{v_F^C}{c_L}\right]^2 \left[\frac{T}{T_F^C}\right]^3 \ln\left[\frac{T}{T_0}\right] , \quad (20)$$

which is the same as Eq. (11), apart from the fact that λ, v_F, and T_F are now renormalized by the Coulomb interaction and band structure, as is indicated by the superfix C. In addition, the sound speed c_L must include these renormalization effects. T_0 is a cutoff temperature. With the inclusion of all vertex corrections, the long-wavelength electron-phonon matrix element that occurs in λ^C is given by

$$W^C = \frac{Z}{N^C} , \quad (21)$$

where Z is the valence of the ion, N^C is the density of states renormalized by the Coulomb interaction and band structure,[17] and the electron-phonon coupling parameter is given by

$$\lambda^C = N^C |W^C|^2 n_i / 2(M_i c_L^2) . \quad (22)$$

In making these calculations, we have neglected effects of order c_L^2/v_F^2 compared with the terms we include.

The coefficient of the $T^3 \ln T$ terms depends only on the properties of long-wavelength phonons, while T_0 depends on finite-wavelength properties. In the Debye model one finds

$$T_0 = \eta q_D c_L , \quad (23)$$

where

$$\eta = \exp\left[-\tfrac{1}{3} - \frac{15}{4\pi^4}\int_0^\infty dx \frac{x^4 \ln x e^x}{(e^x-1)^2}\right] \approx 0.169 . \quad (24)$$

IV. RENORMALIZATION AND QUANTITATIVE ESTIMATES

In our previous considerations we have neglected the effects of renormalization of quantities by the electron-phonon interaction. These come in two distinct classes: the first is due to the change in the quasiparticle effective mass and renormalization factor, a, which gives the spectral weight at the quasiparticle pole, and the second is due to modification of the electron-phonon vertex. The first of these is taken into account by multiplying the electron-phonon matrix element by the renormalization factor a, due to electron-phonon interaction, and by using a density of states and quasiparticle effective mass which include the phonon contribution.[18–20] The appropriately renormalized electron-phonon coupling parameter is thus

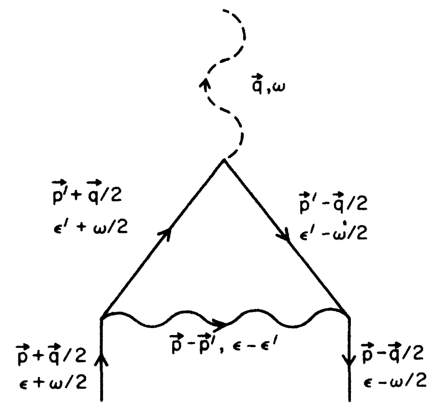

FIG. 1. The vertex correction, $\Delta\lambda(q,\omega)$ [Eq. (25)], due to the electron-phonon interaction.

$$\lambda = \frac{a^2 N^* |W^C|^2}{2M_i c_L^2} = a\lambda^C , \quad (25)$$

where we have used the result that the density of electron states, N^*, in the presence of the electron-phonon interaction is N^C/a.[18–20] Since the $T^3\ln T$ contribution to the specific heat varies as λ/v_F [see Eq. (20)], this is unchanged by these renormalization effects, because $m^* a = m_c^*$, where m^* is the effective mass calculated including band structure, electron-electron and electron-phonon interactions and m_c^* is the effective mass due solely to the electron-electron Coulomb interaction and band structure.

The second source of renormalization effects is the explicit renormalization of the electron-phonon interaction, by processes such as that shown in Fig. 1. Migdal[21] found that under most circumstances such vertex corrections are of order $(m/M_i)^{1/2}$ and therefore negligible. However the analysis which leads to this result breaks down for $q < \omega_D/\varepsilon_F$ and $\omega < \omega_D$, as has been discussed by Leggett.[18] The $T^3\ln T$ terms we are studying come from just such frequencies and wavenumbers. Leggett calculated the renormalization of the electron-phonon coupling, $\Delta\lambda$ due to the vertex correction shown in Fig. 1 and found

$$\Delta\lambda(q,\omega) = \sum_{\mathbf{p}',\epsilon'} G(\mathbf{p}'-\mathbf{q}/2, \epsilon'-\omega/2) \times G(\mathbf{p}'+\mathbf{q}/2, \epsilon'+\omega/2) |W(q)|^2 \times D(\mathbf{p}-\mathbf{p}', \epsilon-\epsilon') . \quad (26)$$

In what follows we drop the dependence of $\Delta\lambda$ on $\mathbf{p}$ and ϵ and assume that the electronic momenta and energies are equal to about p_F and ε_F. For $\omega \ll \omega_D$ one finds

$$\Delta\lambda(q,\omega) = -\int \frac{d\Omega}{4\pi} N^* a^2 \Pi(q,\omega) |W(q)|^2 D(p_F, \Delta\epsilon \ll \omega_D) + |W(q)|^2 \{a^2 N^* D(p_F, \Delta\epsilon \ll \omega_D) - a_c^2 N_c D(p_F, \Delta\epsilon \gg \omega_D)\} , \quad (27)$$

and for $\omega \gg \omega_D$

$$\Delta\lambda(q,\omega) = -\int \frac{d\Omega}{4\pi} N_c a_c^2 \, |W(q)|^2 \{D(p_F, \Delta\epsilon \gg \omega_D)\} , \tag{28}$$

where a typical momentum is of order p_F in the phonon propagator and the energies involved, $\Delta\epsilon = \epsilon - \epsilon'$, are either very much larger than ω_D or very much smaller. a_c is the renormalization constant associated with quasiparticles which are not renormalized by the electron-phonon interaction because their energies are higher than the characteristic phonon energies. One notes that for both $\omega \ll \omega_D$ and for $\omega \gg \omega_D$ there is no renormalization due to vertex corrections. This point has also been demonstrated by Prange and Kadanoff,[19] Heine, Nozières, and Wilkins,[17] and Prange and Sachs.[20] However for finite frequencies there is a renormalization of the electron-phonon coupling, whose contribution to second order in $\omega / q v_F^c$ is

$$\Delta\lambda(q,\omega) = N^* a^2 \left[\frac{\omega}{q v_F^c} \right]^2 |W(q)|^2 D(p_F, 0) . \tag{29}$$

This $(\omega/q)^2$ contribution to the electron-phonon vertex leads to an enhancement of the $T^3 \ln T$ terms in the specific heat. However, its coefficient is smaller by a factor $(c_L/v_F)^2$ than the $T^3 \ln T$ term found in the simple model of Sec. II. We also note that the sound speed is unaffected by the electron-phonon interaction, and thus to the order to which we are working the renormalization effects due to electron-phonon vertex corrections may be ignored. Hence, all electron-phonon renormalization effects on the $T^3 \ln T$ term in the specific heat vanish, and we can now use the expression derived earlier to compare its magnitude with the linear and cubic terms in Eq. (1).

The low-temperature specific heat is

$$C = \gamma T + \Gamma T^3 \ln \left[\frac{T}{T_0} \right] + \beta_{\rm ph} T^3 , \tag{30}$$

where

$$\gamma = \frac{\pi^2}{3} k_B^2 N^* = \frac{\pi^2}{2a} \frac{n_e}{T_F^C} k_B ,$$

$$\Gamma = -\frac{3\pi^4}{20} \lambda^C n_e k_B (T_F^C)^{-3} \left[\frac{v_F^C}{c_L} \right]^2 ,$$

and

$$\beta_{\rm ph} = \tfrac{12}{5} \pi^4 n_i k_B \Theta_D^{-3} .$$

Assuming that the valence of each ion is Z, one finds from charge neutrality that $q_D = (2/Z)^{1/2} p_F$ and so for the purposes of a rough comparison we shall take $q_D = p_F$. We shall also assume $\Theta_D = c_L p_F$. In order to compare the $T^3 \ln T$ term with the sum of the linear and cubic terms we will use values of the parameters for lithium taken from Tables 2.1, 3.1, and 11.1 of Ref. 22. These are $T_F = 55\,500$ K, $\Theta = 350$ K, and $\lambda = 0.4$. One finds that T_0 is approximately 59 K so that beyond this temperature the $T^3 \ln T$ term from the electron-phonon interaction no longer enhances the specific heat appreciably and that for lower temperatures the magnitude of the $T^3 \ln T$ term is less than 0.2% of the sum of the linear and cubic terms. We also consider the case of mercury, whose superconducting transition temperature is 4 K and for which T_F, Θ_D, and λ are 83 300 K, 84 K, and 1.6 respectively. With these values T_0 is approximately 14 K and the $T^3 \ln T$ term is much smaller than the cubic term even for very low temperatures. In general the $T^3 \ln T$ contribution from the electron-phonon interaction is smaller than the cubic term, $\beta_{\rm ph}$, by a factor of order (Θ_D / T_F). The other factors in the ratio between the two terms, including the $\ln T$ term, do not change this result by an order of magnitude. Looking at Tables 2.1 and 3.1 of Ref. 22 one sees that (Θ_D / T_F) is of order 10^{-4} for metallic elements. Previously it has been suggested[14,23] that measurements of the specific heat of indium[24,25] and of the heat of magnetization of mercury[26] show evidence for the $T^3 \ln T$ term due to the electron-phonon interaction. The above estimates indicate that it is unlikely that the features seen in the thermodynamic properties of these materials, which are of order 10% of $\beta_{\rm ph}$ at temperatures of about 3 K, are associated with this contribution.

V. CONCLUSION

In this paper we have shown that the $T^3 \ln T$ contributions to the specific heat due to the electron-phonon interaction may be regarded as having the same source as in Fermi liquids; namely the nonanalyticity of the effective interaction, $f_{\mathbf{p},\mathbf{p}+\mathbf{q}}$, for small q. Also this contribution may be cast in the form of a Fermi contribution or of a Bose contribution to the specific heat. We found that the coefficient of this $T^3 \ln T$ term is $\frac{12}{7}$ times larger than that found by previous authors and that this difference is due to the neglect in earlier calculations of the temperature dependence of the electron self energy. Although Migdal's theorem breaks down for the frequencies and wave numbers which give rise to the $T^3 \ln T$ terms, the renormalization of the $T^3 \ln T$ due to electron-phonon interaction corrections to the electron-phonon coupling vertex are negligible. We have compared the $T^3 \ln T$ term calculated here with the linear term and the cubic term arising from the excitation of thermal phonons. The $T^3 \ln T$ term is smaller by a factor of order $(\Theta_D / T_F) \ln(T_0 / T)$, which for temperatures above the superconducting transition temperature is negligible for ordinary metals because of the smallness of Θ_D compared to T_F. This result may be slightly sensitive to the calculation of T_0 which depends on the model used for finite wave-number properties of the electrons and phonons. However, a more detailed model for finite wave-number effects is unlikely to change this result qualitatively.

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