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Landauer-type formulation of quantum-Hall transport: Critical currents and narrow channels

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Landauer-type formulas are derived for two-dimensional quantum-Hall transport in the extreme quantum limit ($\hbar\omega_c \to \infty$) for an arbitrary current through an electron system confined to a channel. For narrow channels the problem is mapped into a one-dimensional ordinary ($B=0$) transport problem. An expression for the critical current and a description of the breakdown of the dissipationless transport are obtained. We find that it is possible to have quantization of the Hall resistance even when the magnetoresistance is nonzero.

In one dimension (1D), i.e., when the electrons enter or leave the sample only through leads with a single quantum channel, the resistance arising from the elastic scattering with the obstacle (sample) is very simply related to the transmission T: $R=(h/e^2)(1-T)/T$. This insightful formula, known as the Landauer formula,[1] and its multichannel generalizations have been used extensively[2] in the study of transport in 1D and quasi-1D systems.

In this Rapid Communication we consider two-dimensional (2D) quantum-Hall transport[3] in the extreme quantum limit $\hbar\omega_c \to \infty$ so that the electronic motion is always confined to the lowest Landau level. Any attempt at a "Landauer-type" description must take into account the following important facts. (i) The problem is necessarily 2D. (ii) The current depends on the potential drop along the current V_{xx} as well as on the transverse (Hall) potential V_{xy}. In particular, a current can flow with $V_{xx}=0$, in which case ρ_{xy} is quantized according to $\rho_{xy}=V_{xy}/I=h/e^2$. (iii) A magnetic field breaks time-reversal symmetry, and therefore scattering processes are qualitatively different than in the absence of a field. The kinetic energy of the electron is frozen out; its classical trajectory consists of motion along an equipotential contour[4,5] with velocity proportional to the gradient of the potential. With this picture in mind, we see in Fig. 1 that an electron can get transmitted across a barrier[6] either by following a semiclassical path (e.g., trajectory 1) or by tunneling through the saddle point (e.g., from trajectory 2 to 2′). The transmission coefficient must be calculated quantum mechanically including both these possibilities.

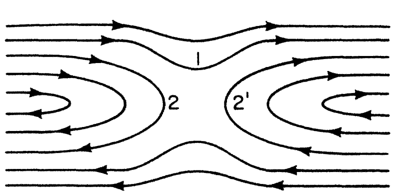

FIG. 1. Shows equipotential contours in the vicinity of a barrier.

We start by considering a model of an ideal sample in which the electrons are confined to a channel centered about the x axis by a potential

$$V=V_0(y)\,. \tag{1}$$

For simplicity we assume that $V_0(y)$ is a monotonically increasing function of $|y|$ with its minimum at $y=0$. We will choose units such that $\hbar=1$ and the magnetic length $l=\sqrt{\hbar c/eB}=1$. For magnetic field in the positive z direction the electrons with $y>0$ move towards the right and the electrons with $y<0$ move towards the left. It is convenient to parametrize the electronic states in terms of a positive coordinate y and consider two sheets: the R sheet with R electrons moving rightward, and the L sheet with L electrons moving leftward. For periodic boundary conditions, the electronic eigenstates are separated by $\delta y=2\pi/L_x$ (L_x is the sample length). This can be used to convert the sum over states into an integral over y. If we assume that all the R states below $\mu_R=V_0(y_R)$ and the L states below $\mu_L=V_0(y_L)$ are occupied (choose $\mu_R>\mu_L$), then the net current moving toward right is given by

$$I=\frac{e}{h}\int_{y_L}^{y_R} dy\, V_0'(y)\,, \tag{2}$$

which yields the correct quantized value of the Hall resistance.

Now introduce a potential barrier $V_B(x,y)$ with finite extent in the x direction, as shown in Fig. 2. On either side there is a particle reservoir. The reservoir on the left emits R electrons up to an energy μ_R, and is perfectly absorbing for L electrons, while the one on the right emits L

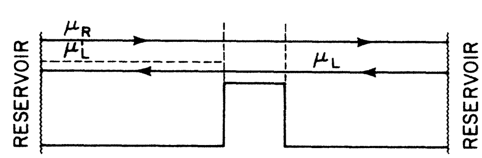

FIG. 2. Shows schematic of Hall transport in the presence of a barrier.

electrons up to an energy μ_L and absorbs R electrons. Because this implies that all states with energy below μ_L will be occupied, only the R electrons with energies above μ_L can get reflected at the barrier. Thus, the current is

$$I = \frac{e}{h}\int_{y_L}^{y_R} dy\, T(y) V_0'(y) \tag{3}$$

where $T(y)$ is the transmission coefficient at energy $V_0(y)$. The distribution of L electrons on the left-hand side of the barrier is modified due to the backscattering of R electrons from the barrier. Since our model is completely elastic, the distribution is not thermal. However, since we expect T to be a basically increasing function of the energy of the electrons, the occupation probability f for L electrons will be a generally decreasing function of energy with $f=1$ for E less than μ_L and $f=0$ for E greater than μ_R. We assume that despite the fact that the distribution is only quasithermal, we can define a chemical potential $\mu_L' \equiv V_0(y_L')$ by requiring that the number of electrons (occupied states) with $E > \mu_L'$ is equal to the number of holes (unoccupied states) with $E < \mu_L'$, as one would if the distribution were thermal. The result is

$$y_L' = y_R - \int_{y_L}^{y_R} dy\, T(y) \ . \tag{4}$$

Of course, the chemical potential of the L sheet on the right-hand side of the barrier remains unperturbed. From the voltage drop along the sample we get[6]

$$\rho_{xx} = (\mu_L' - \mu_L)/eI \ , \tag{5}$$

while from the voltage drop across,

$$\rho_{xy} = (\mu_R - \mu_L')/eI \ . \tag{6}$$

We note a subtlety of this analysis which is that the voltage drops must be defined relative to the chemical potential potential for L electrons on the left-hand side of the barrier μ_L' rather than the chemical potential potential μ_R' defined analogously for R electrons on the right-hand side of the barrier. The reason is that the distribution of R electrons on the right-hand side of the barrier is not even remotely thermal; for T an increasing function of E, the occupation probability is also an increasing function of E for $\mu_L < E < \mu_R$. Thus, the chemical potential is ill defined; rather, under appropriate circumstances with very long inelastic lifetimes, this inverted nonequilibrium distribution could lead to hot-electron effects. We also note here that the only essential ingredients in our theory are that the ground state of the electron gas is an incompressible fluid, and that the excitations in the *ideal region* of the sample can be treated in terms of scattering states of quasiparticles. Thus, the analysis can be readily extended to include the effect of electron-electron interaction, and in particular to treat the fractional quantum-Hall effect (QHE).

We now study the consequences of Eqs. (5) and (6) in various situations. Consider first the limit $I \to 0$ or $\mu_R \to \mu_L^+$, so that $T(y)$ in Eqs. (3) and (4) is approximately constant over the range of integration. In this case it is easy to show that $\rho_{xx} = (h/e^2)(1-T)/T$, and $\rho_{xy} = h/e^2$ for arbitrary T. Thus, for small currents, ρ_{xy} remains quantized even though ρ_{xx} may be quite different from zero.[7] This situation is realized, for example, when the barrier is so high that the sample breaks into two classically disconnected regions. As a result of the small overlap between the wave functions on either side of the barrier, the transmission coefficient and hence the current is small and the above considerations apply.

The transmission coefficient can be evaluated from the knowledge of the Green's function

$$G(r_f,t;r_i) \equiv \langle r_f | e^{-iHt} | r_i \rangle \ , \tag{7}$$

which can be written as a coherent-state path integral $G = \int \mathcal{D}(r) \exp(iS)$ where[8]

$$S = -\int_0^t dt\,[\dot{x} y + V(x,y)] \ . \tag{8}$$

The action in Eq. (8) is the action for an electron whose motion is restricted to the lowest Landau level. It can also be obtained from the standard action[9] by taking the limit $\hbar\omega_c \to 0$ (or $m \to 0$). The term $\dot{x}y$ in Eq. (8) obtains from the usual $(e/c)\dot{\mathbf{r}}\times\mathbf{A}$ with the Landau gauge choice for the vector potential $\mathbf{A} = -By\hat{\mathbf{x}}$. In order to simplify the calculation of the transmission coefficient we specialize to narrow channels with parabolic confinement

$$V(x,y) = \tfrac{1}{2}Ky^2 + V_B(x) \ , \tag{9}$$

so that the y coordinate can be integrated away,[8] which leaves us with

$$G(x_f,x_i;t) = \int \mathcal{D}(x)\, e^{iS_x[x]} \ , \tag{10}$$

$$S_x[x] = \int_0^t dt \left[\frac{1}{2K}\dot{x}^2 - V_B(x) \right] . \tag{11}$$

Thus we have mapped the problem of the motion of an electron in the 2D potential $\frac{1}{2}Ky^2 + V_B(x)$ in the extreme quantum limit ($\hbar\omega_c \to \infty$) into the problem of the motion of a particle of mass K^{-1} in a 1D potential $V_B(x)$ with no magnetic field. This is not surprising considering that the scattering process is effectively one dimensional here since for a given energy there is only one R state and one L state on either side of the barrier. The evaluation of the transmission coefficient for this new 1D problem is an elementary exercise in quantum mechanics. (Away from the extreme quantum limit, when more than one Landau level is occupied, the problem reduces in an analogous manner to a multichannel problem, one channel per relevant Landau level.)

Now we show some numerical results. The width of the strip containing electrons is fixed for a given number of electrons, i.e., $y_L + y_R = 2y_F$, where $\frac{1}{2}Ky_F^2$ is the Fermi energy in the absence of any current. We vary the potential μ_L and calculate μ_R, I, μ_L', ρ_{xx}, and ρ_{xy}. To be specific, we choose the barrier to be $V_B(x) = \frac{1}{2}Ky_B^2$ in a region of thickness t and $V_B(x) = 0$ elsewhere. Then the transmission coefficient of a particle with energy $\frac{1}{2}Ky^2$ is given by

$$T(y) = \left[1 + \frac{y_B^4 \sin^2(t\sqrt{|y^2 - y_B^2|})}{4y^2 |y^2 - y_B^2|} \right]^{-1} , \tag{12}$$

for $y > y_B$, and for $y < y_B$ the sin is to be replaced by sinh. In Fig. 3 we show ρ_{xx} and ρ_{xy} as a function of current with the parameters $t=3$, $y_B=2$, $y_F=20$. The Hall resis-

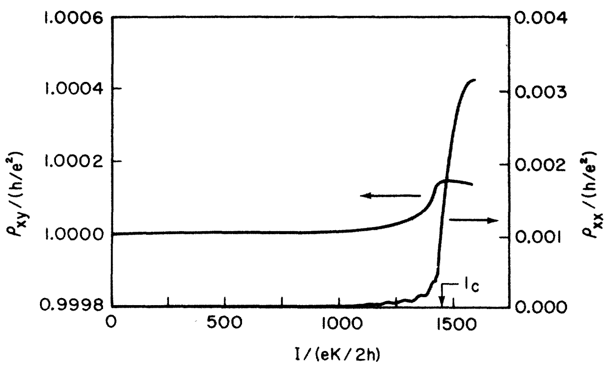

FIG. 3. Shows ρ_{xx} and ρ_{xy} as a function of current.

tance ρ_{xy} remains almost quantized, and ρ_{xx} remains almost zero until the current reaches a critical value, at which point the quantization breaks down.

The phenomenon of current-induced breakdown of the quantized-Hall effect is experimentally well known[10] with somewhat sample specific critical currents. There have been a number of theoretical attempts[5,11] to explain this phenomenon, most of which consider inelastic scattering with phonons as the underlying mechanism for the breakdown. In the present model, however, we consider only elastic processes and the breakdown occurs when the current is increased to the extent that the potential at one edge μ_L drops below the level of the barrier, so that the transmission of the R electrons at energy μ_L changes from nearly perfect to nearly zero. This happens because for energies above the barrier $(E > \frac{1}{2}Ky_B^2)$ transmission occurs as on trajectory 1 in Fig. 1, while for energies below the barrier, the transmission is classically forbidden and the electron must tunnel as from 2 to 2′ in Fig. 1. The model predicts that in narrow channels, an increase in the electron density will result in a higher critical current. A broader channel translates into a heavier effective mass, and hence for a given Fermi energy, the wider the channel, the more sharply defined is the critical current.

There is some structure in ρ_{xx} at the breakdown, which comes from the oscillatory part (sin) in Eq. (12). This sort of structure is a common occurrence for narrow channels[12,13] as well as for regular 2D QHE experiments.[10] In narrow channels,[14] ρ_{xx} and ρ_{xy} have been found to show sample-specific and reproducible noiselike aperiodic structure both in the quantized region and away from it. These fluctuations are presumably a result of scattering with very many different scatterers along the current path. Within our model, we expect the fluctuations to be small in the quantized region because of near perfect transmission, but large in the nonquantized region because then μ_L is of the order of the impurity potential fluctuations and the transmission coefficient is extremely sensitive to changes in the current or the magnetic field. This is in agreement with the experimental observations.[14]

Within the formalism described above, it is natural to describe the quantum-Hall transport as a function of the current. However, one might as easily vary the magnetic field, which translates into a variation of the Fermi energy $V_0(y_F)$. The physics of breakdown of quantization remains exactly the same.

In conclusion, we have developed a Landauer formalism for ρ_{xy} and ρ_{xx} for 2D quantum-Hall transport in the extreme quantum limit. For narrow channels, the problem reduces to an ordinary 1D transport problem, which has been studied in great detail in the literature.[1,2] Within this formalism there is also a natural critical current for the breakdown of quantum-Hall effect.

Note added. After submitting this manuscript we became aware of the work by Streda, Kucera, and MacDonald[15] in which the Landauer formulas were derived along similar lines. However, the emphasis is on very different physics in that work. They study the *zero current* multichannel problem and obtain a sum rule which is the multichannel generalization of

$$\rho_{xx}+\rho_{xy}=\frac{h}{e^2}\left(\frac{1-T}{T}\right)+\frac{h}{e^2}=\frac{h}{e^2}\frac{1}{T}\ . \tag{13}$$

We, on the other hand, emphasize the *finite current* nonlinear behavior, and, in particular, describe a mechanism for the current-induced breakdown of the Hall quantization. We also explicitly evaluate the high-field transmission coefficient within a simple model.

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