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Scaling and crossover in a fermion-boson mixture

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Thermodynamic behavior of a mixture of weakly interacting fermions and bosons is investigated in $4-\epsilon$ dimensions by the renormalization-group method with the purpose of studying scaling and crossover properties of the system in the tricritical region. Conventional tricritical scaling, first found to break down for a classical infinite-component model, is seen to do so more spectacularly in the case of the mixture. Whereas in the infinite-component model, conventional scaling holds in the ordered and disordered phases separately (i.e., with different tricritical exponents), it is impossible in either of the phases of the mixture. The breakdown of scaling in the mixture is associated with the dimensionless strength v_6 of the six-point interaction in the effective Hamiltonian that causes the parameters of the renormalized Hamiltonian to depend on *two* combinations of scaling fields rather than one. The strength v_6 is a quantum-mechanical parameter, being proportional in three dimensions to $b^3/\lambda_T^4 K_F$, where λ_T, K_F, and b denote, respectively, the boson thermal wavelength, the Fermi momentum of the fermion component, and the scattering length associated with the fermion-boson interaction. The square root of this quantity agrees with the nonuniversality parameter, which was found to characterize tricritical amplitude ratios in three dimensions in an earlier work.

I. INTRODUCTION

Until now there has been only a limited number of renormalization-group (RG) studies of tricritical behavior exhibited by ^{3}He-^{4}He mixtures. The few that have been carried out[1,2] employ classical spin models of the Landau-Ginzburg-Wilson type and are incomplete in the sense that they are either confined to the behavior of the model near an unstable Gaussian fixed point[1] or ignore[2] the important ϕ^6-interaction term corresponding to the M^6 term in the classical Landau expansion[3] in powers of the order parameter M. They are, consequently, unable to give a correct account of crossover from critical to tricritical behavior in the neighborhood of the tricritical point.

As the mixtures are composed of quantum fluids, the relevance of classical models in their study needs to be called into question. Moreover, an exact solution by Sarbach and Fisher[4,5] of an infinite-component classical spin model has revealed the presence of nonuniversal effects in tricritical behavior. One expects these effects to be dependent on the classical or quantum nature of the system.

A natural starting point for a theoretical study of ^{3}He-^{4}He mixtures is a system of interacting fermions and bosons. However, as general methods for dealing with strongly interacting or nondilute quantum many-body systems are not available, Goswami and the present author have recently attempted[6,7] to develop a theory of tricritical behavior in a mixture of weakly interacting fermions and bosons. The attitude adopted in this work was essentially that of the RG approach,[8,9] and sought to eliminate the fermion amplitudes and the short-wavelength boson amplitudes to arrive at an effective, low-momentum boson Hamiltonian.[6] A treatment of fluctuations of the order of parameter in the effective Hamiltonian in the Hartree-Fock approximation produced[7] an improved version of the classical Landau theory,[3] especially in the normal phase of the mixture. A more refined approximation for the self-energy parts in a Green's-function formulation gave results[10] for tricritical amplitude ratios similar to those obtained for an infinite-component, classical spin model[5] but characterized by a nonuniversal quantum parameter.

In this paper the above work is carried to its logical conclusion by subjecting the effective Hamiltonian of the mixture in $4-\epsilon$ dimensions to RG transformations. The aim is to study the scaling properties and crossover behavior of the mixture in the neighborhood of the tricritical point (TCP). The analysis follows the lines of an earlier work[11,12] concerned with application of the RG approach to a system of bosons with repulsive two-body interactions. The effective Hamiltonian of the mixture differs from that case by the presence of a six-point interaction term (in the zeroth order this term gives M^6 term of the classical theory). The main points of the earlier work are reviewed in Sec. II in the context of the RG transformation of the effective Hamiltonian.

The essential point of our study already emerges in Sec. III, where recursion relations crucial to the study of crossover from critical behavior to tricritical behavior are derived. These relations show that the renormalized parameters which characterize the Hamiltonian after the RG transformation cannot be written in terms of just one combination of scaling fields u and t (which may be regarded as measures of small deviations from the TCP). A second combination ($v \mid t \mid ^{1-\epsilon}$) which represents the renormalized strength of the six-point interaction enters these relations in an essential manner. Consequently, conventional or orthodox tricritical scaling[13,14] does not hold for $\epsilon<1$. Also, v being a nonuniversal parameter (in particular it is dependent on the fermion-boson interaction strength), tricritical scaling is not universal. Calculations of crossover scaling functions in Sec. IV demonstrate these facts.

In Sec. V results for the mixture are compared with those obtained by Sarbach and Fisher[4] for the infinite-component classical model. It is pointed out that apart from the fact that the nonuniversal parameter v for the mixture is of a quantum mechanical origin, the violation of orthodox scaling found in the normal and ordered phases for the mixture, does not occur in the infinite-component model. In the latter case, orthodox scaling is obeyed in either of the phases although with different sets of scaling exponents. Implications of the results for a real three-dimensional system have been pointed out.

II. EFFECTIVE HAMILTONIAN AND RENORMALIZATION-GROUP TRANSFORMATION IN FIRST ORDER

In this section we recall the effective boson Hamiltonian[6] for a fermion-boson mixture and derive formally exact equations for the order parameter and the coexistence line. The perturbation theoretic renormalization-group approach developed for a Bose system[11,12] is reviewed in the context of the effective Hamiltonian and recursion relations are derived for the parameters of the Hamiltonian to first order in the interaction strengths.

The system under consideration is a mixture of fermions of mass $(m_3)/2$ and spin $\frac{1}{2}$ and bosons of mass $(m_4)/2$ and spin zero contained in a box of volume V with periodic boundary conditions. The particles are assumed to interact via two-body short-range potentials. The strengths of the fermion-fermion, fermion-boson, and boson-boson interactions are denoted, respectively, by u_3, u_{34}, and u_4. The partial chemical potentials of the fermions and boson are denoted, respectively, by μ_3 and μ_4. Elimination of the fermion field amplitudes and the short-wavelength boson field amplitudes belonging to the momentum range $p>p_c$, where p_c is small compared with the boson thermal momentum $(4\pi\beta/m_4)^{-1/2}$, yields the approximate effective boson Hamiltonian[6] (in units such that $\hbar=1$)

$$H_e=C_0+\sum_k\left[\frac{k^2}{m_4}-\mu_4'\right]b_k^\dagger b_k+h_4+h_6-\tfrac{1}{2}hV^{1/2}(b_0+b_0^\dagger)\,, \tag{1}$$

$$h_4=\frac{u_4'}{V}\sum_{k_1,k_2,k} b_{k_1}^\dagger b_{k_2}^\dagger b_{k_1-k}b_{k_2+k}\,, \tag{2}$$

$$h_6=\frac{u_6}{V^2}\sum_{k_1,\ldots,k_3'} b_{k_1}^\dagger b_{k_1-k_1'}b_{k_2}^\dagger b_{k_2-k_2'}\times b_{k_3}^\dagger b_{k_3-k_3'}\delta_{\mathrm{Kr}}(k_1'+k_2'+k_3')\,. \tag{3}$$

Here C_0 is a c-number function of the free-fermion density n_3^F and the free-boson density n_4' for the momentum range $p>p_c$,

$$\mu_4'=\mu_4-u_{34}n_3^F-4u_4'n_4'+(u_{34}^2n_4'+u_3u_{34}n_3^F)\frac{\partial n_3^F}{\partial\mu_3}\,, \tag{4}$$

$$u_4'=u_4-\tfrac{1}{2}u_{34}^2\frac{\partial n_3^F}{\partial\mu_3}+O(u_{34}^3)\,, \tag{5}$$

$$u_6=\frac{u_{34}^3}{6}\frac{\partial^2 n_3^F}{\partial\mu_3^2}\,, \tag{6}$$

and each k summation has an upper cutoff p_c. The last term in (1) represents the symmetry breaking term appropriate to a system at rest with h denoting the field conjugate to the real part of the order parameter $(b_0/V^{1/2})$.

One may use the Bogolubov prescription[15] to replace $(b_0/\sqrt{V})$ by a c number M. The effective Hamiltonian can then be written in the form

$$H_e=C_0'+\sum_k\left[\frac{k^2}{m_4}-\mu_4'\right]b_k^\dagger b_k+V_2'+V_3'+V_4'+V_5'+V_6'\,, \tag{7}$$

where

$$C_0'=C_0+V(-\mu_4'M^2+u_4'M^4+u_6M^6)-hM\,, \tag{8}$$

$$V_2'=(4u_4'M^2+9u_6M^4)\sum_k b_k^\dagger b_k+(u_4'M^2+3u_6M^4)\sum_k(b_k^\dagger b_{-k}^\dagger+b_{-k}b_k)\,, \tag{9}$$

$$V_3'=V^{-1/2}(2u_4'M+9u_6M^3)\sum_{k_1,k_2}(b_{k_1}^\dagger b_{k_2}^\dagger b_{k_1+k_2}+\text{H.c.})+\frac{u_6M^3}{V^{1/2}}\sum_{k_1k_2k_3}(b_{k_1}^\dagger b_{k_2}^\dagger b_{k_3}^\dagger+\text{H.c.})\delta_{\mathrm{Kr}}(k_1+k_2+k_3)\,, \tag{10}$$

$$V_4'=\frac{1}{V}(u_4'+9u_6M^2)\sum_{k_1,\ldots,k_4} b_{k_1}^\dagger b_{k_2}^\dagger b_{k_3}b_{k_4}\delta_{\mathrm{Kr}}(k_1+k_2-k_3-k_4)+\frac{3u_6M^2}{V}\sum_{k_1,\ldots,k_4}(b_{k_1}^\dagger b_{k_2}^\dagger b_{k_3}^\dagger b_{k_4}+\text{H.c.})\delta_{\mathrm{Kr}}(k_1+k_2+k_3-k_4)\,, \tag{11}$$

$$V_5'=\frac{3u_6M}{V^{3/2}}\sum_{k_1,\ldots,k_5}(b_{k_1}^\dagger b_{k_2}^\dagger b_{k_3}^\dagger b_{k_4}b_{k_5}+\text{H.c.})\delta_{\text{Kr}}(k_1+k_2+k_3-k_4-k_5-k_6)\ , \tag{12}$$

$$V_6'=\frac{u_6}{V^2}\sum_{k_1,\ldots,k_6}b_{k_1}^\dagger b_{k_2}^\dagger b_{k_3}^\dagger b_{k_4}b_{k_5}b_{k_6}\delta_{\text{Kr}}(k_1+k_2+k_3-k_4-k_5-k_6)\ . \tag{13}$$

Each k summation excludes the value $k=0$ and H.c. denotes the Hermitian conjugate. The thermodynamic potential per unit volume, Ω, associated with H_e is

$$\Omega=\Omega'(T,\mu_3,\mu_4,M)-hM\ , \tag{14}$$

$$\Omega'=-(\beta V)^{-1}\ln Z\ , \tag{15}$$

$$Z=\text{Tr}\exp(-\beta H_e')\ , \tag{16}$$

where H_e' denotes the effective Hamiltonian minus the symmetry breaking term.

The unknown quantity M, which we identify as the order parameter of the mixture, is determined by the requirement that Ω be minimum with respect to variations in M. On setting the first derivative of Ω equal to zero, one gets the equation of state

$$h=\frac{\partial\Omega'}{\partial M}=V^{-1}\left\langle\frac{\partial H_e'}{\partial M}\right\rangle\ , \tag{17}$$

where $\langle\ \rangle$ denotes thermodynamic average calculated with the H_e'.

Use of (7) in (17) gives

$$\frac{h}{2M}=r_n\ , \tag{18}$$

$$\begin{aligned}r_n=&-\mu_4'+2u_4'M^2+3u_6M^4+(4u_4'+18u_6M^2)n'+(2u_4'+12u_6M^2)Y\\&+\frac{1}{MV^{3/2}}(2u_4'+27u_6M^2)\sum_{k_1k_2}\langle b_{k_1}^\dagger b_{k_2}^\dagger b_{k_1+k_2}\rangle+\frac{3u_6M}{V^{3/2}}\langle b_{k_1}^\dagger b_{k_2}^\dagger b_{k_3}^\dagger\rangle\delta_{\text{Kr}}(k_1+k_2+k_3)\\&+\frac{9u_6}{V^2}\sum_{k_1,\ldots,k_4}\langle b_{k_1}^\dagger b_{k_2}^\dagger b_{k_3}b_{k_4}\rangle\delta_{\text{Kr}}(k_1+k_2-k_3-k_4)\\&+\frac{6u_6}{V^2}\sum_{k_1,\ldots,k_4}\langle b_{k_1}^\dagger b_{k_2}^\dagger b_{k_3}^\dagger b_{k_4}\rangle\delta_{\text{Kr}}(k_1+k_2+k_3-k_4)\\&+\frac{3u_6}{MV^{5/2}}\sum_{k_1,\ldots,k_5}\langle b_{k_1}^\dagger b_{k_2}^\dagger b_{k_3}^\dagger b_{k_4}b_{k_5}\rangle\delta_{\text{Kr}}(k_1+k_2+k_3-k_4-k_5)\ ,\end{aligned} \tag{19}$$

where

$$n'=V^{-1}\sum_k\langle b_k^\dagger b_k\rangle\ , \tag{20}$$

$$Y=V^{-1}\sum_k\langle b_kb_{-k}\rangle\ , \tag{21}$$

and we have used the fact that all the averages are real quantities.

It is easy to see that r_n is a function of M^2 only. The difference between the thermodynamic potentials of the ordered phase ($M\neq0$, $h\to0$) and the normal phase ($h\to0$, $M\to0$, $h/M\neq0$) at a given (T,μ_3,μ_4) is thus given by

$$\Omega'(M^2,T,\mu_3,\mu_4)-\Omega'(0,T,\mu_3,\mu_4)=\int_0^{M^2}dM_1^2r_n(M_1^2)\ , \tag{22}$$

with M^2 determined by (18). It follows that at fixed μ_3 (or μ_4), the ordered phase and the normal phase can coexist along the line given by

$$\int_0^{M^2}dM_1^2r_n(M_1^2)=0\ . \tag{23}$$

Our aim is to study the effective Hamiltonian H_e in $d=(4-\epsilon)$ dimensions by the RG method. The quantity of primary interest is then the dimensionless Hamiltonian $H_0=\beta H_e'$. It is convenient to introduce the dimensionless parameters[11]

$$s=\beta p_c^2\,|\,m_4,\quad r=-m_4\mu_4'p_c^{-2},\quad v_4=\beta u_4'p_c^ds^{-2},\tag{24}$$

$$v_6=\beta u_6p_c^{2d}s^{-3}\ , \tag{25}$$

and write H_0 as

$$H_0=E_0+s\sum_k(k^2p_c^{-2}+r)b_k^\dagger b_k+(V_2+V_3+V_4+V_5+V_6)\ , \tag{26}$$

$$E_0(M)=C+V(rsM^2+v_4s^2p_c^{-d}M^4+v_6p_c^{-2d}s^3M^6)\ , \quad (27)$$

$$V_i=\beta V_i',\quad i=2,3,\ldots 6\ , \quad (28)$$

$$C=\beta C_0\ . \quad (29)$$

A RG transformation is performed by dividing the k space into two subspaces h_0 and h_1 where h_0 comprises momenta $0<|q|<p_c\zeta^{-1}$ and h_1 comprises momenta $p_c\zeta^{-1}<|p|<p_c$, ζ being an arbitrary number large compared to unity. The boson amplitudes b_p, $b_p^\dagger$ are eliminated through a partial trace procedure by writing Z in the form[11]

$$Z=Z_0\operatorname*{Tr}_{(h_0)}\left[\exp(-E_0+H_F^{(0)})\left[1+\sum_{n=1}^{\infty}\frac{(-1)^n}{n!}\int_0^1 d\tau_1\cdots\int_0^1 d\tau_n\left\langle PU(\tau_1)\cdots U(\tau_n)\right\rangle_{H_F^{(1)}}\right]\right]\ , \quad (30)$$

where

$$Z_0=\operatorname{Tr}\exp(-H_F^{(1)})\ , \quad (31)$$

$$H_F^{(0)}=s\sum_q(q^2p_c^{-2}+r)b_q^\dagger b_q\ , \quad (32)$$

$$H_F^{(1)}=s\sum_p(p^2p_c^{-2}+r)b_p^\dagger b_p\ , \quad (33)$$

$$U(\tau)=\exp[\tau(H_F^{(0)}+H_F^{(1)})]U\exp[-\tau(H_F^{(0)}+H_F^{(1)})]\ . \quad (34)$$

U denotes the sum of the interaction terms V_i in (26), P the time-ordering operator and, $\langle\ \rangle_{H_F^{(1)}}$ denotes thermodynamic average calculated with $H_F^{(1)}$. The trace in (30) is to be calculated over the momentum subspace h_0, i.e., over a complete set of states constructed from the operators $\{b_q,b_q^\dagger\}$. The vertices contained in $U(\tau)$ have been shown graphically in Fig. 1.

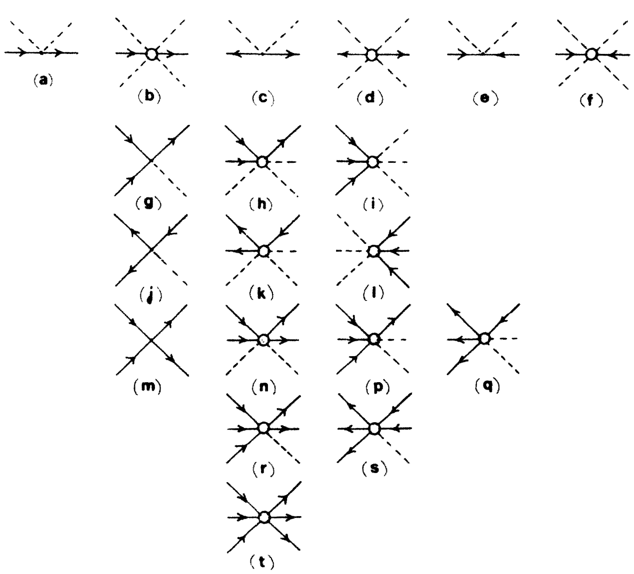

FIG. 1. Diagrammatic representation of vertices V_2 through V_6 appearing in the effective boson Hamiltonian [cf. Eqs. (9)–(13)]. A dot represents a four-point vertex of strength u_4', while a circle represents a six-point vertex of strength u_6. An ingoing (outgoing) solid line represents creation (annihilation) operator of a boson of momentum $k\neq 0$. A broken line represents a c-number factor M.

The first-order term in (30) involves averages of $V_i(\tau)$. Using the fact that

$$b_p(\tau)=\exp(\tau H_F^{(1)})b_p\exp(-\tau H_F^{(1)})=b_p\exp[-\tau\epsilon(p)]\ , \quad (35)$$

$$\tilde b_p(\tau)=\exp(\tau H_F^{(1)})b_p^\dagger\exp(-\tau H_F^{(1)})=b_p^\dagger\exp[\tau\epsilon(p)]\ , \quad (36)$$

where

$$\epsilon(p)=s(p^2p_c^{-2}+r)\ , \quad (37)$$

one finds

$$\langle V_2(\tau)\rangle=V_2(\tau,q)+(4\beta u_4'M^2+9\beta u_6M^4)s^{-1}Vp_c^dI_1(r,\zeta)\ . \quad (38)$$

Here, and in what follows, $V_i(\tau,q)$ denotes the operator obtained from $V_i(\tau)$ by replacing all k summations by q summations, and

$$I_1(r,\zeta)=\int_{\zeta^{-1}}^1\frac{s\,d^dq}{\exp[s(q^2+r)]-1}\simeq A_0(d)\int_{\zeta^{-1}}^1\frac{q^{d-1}dq}{(q^2+r)}\ , \quad (39a)$$

$$A_0(d)=[2^{d-1}\Pi^{d/2}\Gamma(d/2)]^{-1}\ . \quad (39b)$$

The approximation implied by the second equation in (39a) is justified because s is a small quantity by virtue of

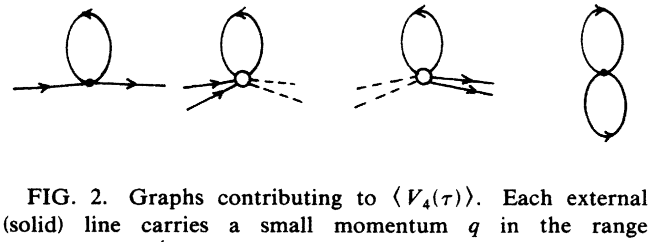

FIG. 2. Graphs contributing to $\langle V_4(\tau)\rangle$. Each external (solid) line carries a small momentum q in the range $0<|q|<p_c\zeta^{-1}$, whereas each internal line (contraction) has a large momentum p in the range $p_c\zeta^{-1}<|p|<p_c$.

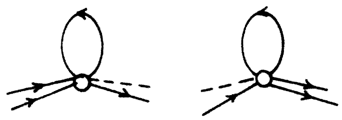

FIG. 3. Graphs contributing to $\langle V_5(\tau)\rangle$.

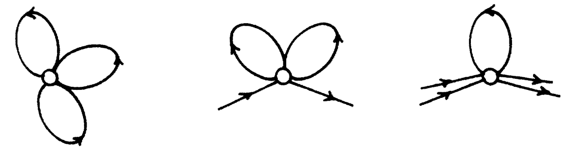

FIG. 4. Graphs contributing to $\langle V_6(\tau)\rangle$.

the definition of the cutoff p_c.

As regards V_3, its average is simply $V_3(\tau,q)$. Contributions to the averages of V_4, V_5, and V_6 can be represented diagrammatically as in Figs. 2, 3, and 4. Each external line carries a momentum q, while each internal line (contraction) has a momentum p; a dashed line indicates a factor M. The results for the averages are

$$\langle V_4(\tau)\rangle = V_4(\tau,q)+(4\beta u_4'+36\beta u_6 M^2)\left[\frac{I_1 p_c^d}{s}\right]\sum_q \tilde{b}_q(\tau)b_q(\tau)$$

$$+9\beta u_6 M^2\left[\frac{I_1 p_c^d}{s}\right]\sum_q [\tilde{b}_q(\tau)\tilde{b}_{-q}(\tau)+b_{-q}(\tau)b_q(\tau)]+V(2\beta u_4'+18\beta u_6 M^2)\left[\frac{I_1^2 p_c^{2d}}{s^2}\right], \quad (40)$$

$$\langle V_5(\tau)\rangle = V_5(\tau,q)+\frac{18\beta u_6 M}{V^{1/2}}\left[\frac{I_1 p_c^d}{s}\right]\sum_{q_1,q_2}[\tilde{b}_{q_1}(\tau)\tilde{b}_{q_2}(\tau)b_{q_1+q_2}(\tau)+\tilde{b}_{q_1+q_2}(\tau)b_{q_2}(\tau)b_{q_1}(\tau)], \quad (41)$$

$$\langle V_6(\tau)\rangle = V_6(\tau,q)+6\beta u_6\left[\frac{I_1 p_c^d}{s}\right]^3+18\beta u_6\left[\frac{I_1 p_c^d}{s}\right]^2\sum_q \tilde{b}_q(\tau)bq(\tau)$$

$$+\frac{9\beta u_6}{V}\left[\frac{I_1 p_c^d}{s}\right]\sum_{q_1,\ldots,q_4}\tilde{b}_{q_1}(\tau)\tilde{b}_{q_2}(\tau)b_{q_3}(\tau)b_{q_4}(\tau)\delta_{\mathrm{Kr}}(q_1+q_2-q_3-q_4). \quad (42)$$

For the moment we ignore the second- and higher-order terms in the expansion (30). Then

$$Z=Z_0 \operatorname*{Tr}_{(h_0)} \exp[-H_1(q)], \quad (43)$$

$$H_1(q)=E_0(M)+H_F^{(0)}+\sum_{i=2}^{6} V_i(q), \quad (44)$$

where $V_i(q)$ is obtained from the expression for $\langle V_i(q,\tau)\rangle$ by replacing $b_q(T),\tilde{b}_q(T)$ by $b_q,b_q^\dagger$, respectively.

If one collects all the terms containing $(b_q^\dagger b_q)$ in (44), one gets

$$s\sum_q (q^2 p_c^{-2}+r')b_q^\dagger b_q+4(\beta u_4'+9\beta u_6 I_1 p_c^d s^{-1})M^2 \times\sum_q b_q^\dagger b_q+9\beta u_6 M^4\sum_q b_q^\dagger b_q, \quad (45)$$

where

$$r'=r+4\beta u_4' p_c^d s^{-2} I_1+18\beta u_6 p_c^{2d} s^{-3} I_1^2. \quad (46)$$

The advantage of the parameters v_4 and v_6 defined by (24) and (25) is obvious.

To restore the original momentum range in the various terms in H_1, we rescale the q's according to

$$\zeta q=k. \quad (47)$$

The new Hamiltonian H_1 is then easily seen to take exactly the same form as H_0 given by (26). In place of C and the parameters (s,M,V,r,v_4,v_6) of H_0 one obtains C_1 and $(s_1,M_1,V_1,r_1,v_4^{(1)},v_6^{(1)})$, where

$$C_1=C+Vp_c^d(2v_4 I_1^2+6v_6 I_1^3), \quad (48)$$

$$s_1=\zeta^{-2}s, \quad (49)$$

$$V_1=\zeta^{-d}V, \quad (50)$$

$$M_1=\zeta^{d/2}M, \quad (51)$$

$$r_1=\zeta^2(r+4v_4 I_1+18v_6 I_1^2), \quad (52)$$

$$v_4^{(1)}=\zeta^{4-d}(v_4+9v_6 I_1), \quad (53)$$

$$v_6^{(1)}=\zeta^{6-2d}v_6. \quad (54)$$

The above first-order recursion relations have only the Gaussian fixed point

$$s^*=M^*=r^*=v_4^*=v_6^*=0, \quad (55)$$

which does not give a correct account of critical behavior for $d<4$.[8] Before proceeding to second-order calculations, however, it is useful to note down the expression for the first-order scaling fields associated with the recursion relations (49)–(54).

For small r, the integral I_1 has the value

$$I_1=A_1-A_1\zeta^{-d/2}-\frac{A_0}{\epsilon}(\zeta^{\epsilon}-1)r+O(r^2)\ , \tag{56}$$

$$A_1=A_0/(d-2)\ . \tag{57}$$

Up to first-order terms the relations (52) and (53) can consequently be written as

$$a_2^{(1)}=\zeta^2 a_2\ , \tag{58}$$

$$a_4^{(1)}=\zeta^{\epsilon}a_4\ , \tag{59}$$

where

$$a_2=r+4A_1v_4+18A_1^2v_6\ , \tag{60}$$

$$a_4=v_4+9A_1v_6\ , \tag{61}$$

and $a_2^{(1)},a_4^{(1)}$ denote the same quantities after the RG transformation. In terms of the initial parameters (μ_4',u_4',u_6), the scaling fields a_2,a_4 are

$$a_2=s^{-1}(-\beta\mu_4'+4\beta u_4'I_0+18\beta u_6I_0^2)\ , \tag{62}$$

$$a_4=s^{-2}p_c^d(\beta u_4'+9\beta u_6I_0)\ , \tag{63}$$

where

$$I_0=m_4A_1p_c^{d-2}\beta^{-1}\simeq A_0\int_0^{p_c}\frac{q^{d-1}dq}{\exp(\beta q^2/m_4)-1}\ . \tag{64}$$

It is interesting to note that for $d=3$, apart from multiplicative factors (βs^{-1}) and $\beta p_c^ds^{-2}$, the scaling fields a_2 and a_4 are the same as those appearing in the equation of state in the Hartree-Fock approximation [cf. Eq. (42) of Ref. 7].

III. RECURSION RELATIONS TO SECOND ORDER

In this section we calculate contributions of second order to the recursion relations. The calculations are rather tedious, but the results can be presented in a simple manner through graphs.

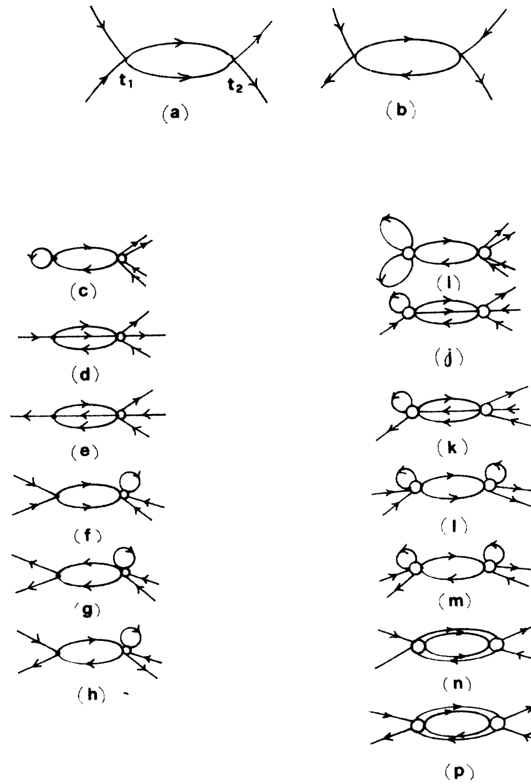

FIG. 5. Second-order graphs contributing to renormalization of the strength v_4 of the four-point vertex (m) of Fig. 1.

In Fig. 5 are exhibited all the connected graphs arising from the second-order term in (30) which can contribute to the four-point vertex (m) in Fig. 1. The contribution of a graph is typically calculated as follows: Consider, for example, the graph 5(b). It carries a numerical factor 2^4 corresponding to two ways of choosing each external (q-momentum) line. The internal lines contribute a factor $n_p(1+n_p)$, where n_p is the Bose distribution factor $[\exp s(p^2/p_c^2+r)-1]^{-1}$. The smallness of s enables n_p as well as $(1+n_p)$ to be approximated by $s(p^2/p_c^2+r)$. Integrations over the imaginary times τ_2 and τ_1 give a factor $\frac{1}{2}$ making the total contribution $C(b)$ of the graph as

$$C(b)=8\left[\frac{s^2p_c^{-d}v_4}{V}\right]^2\sum_p[s(p^2p_c^{-2}+r)]^{-2}\sum_{q_1,\dots,q_4}b_{q_1}^{\dagger}b_{q_2}^{\dagger}b_{q_3}b_{q_4}\delta_{\mathrm{Kr}}(q_1+q_2-q_3-q_4)\ . \tag{65}$$

This adds to $v_4^{(1)}$ in Eq. (53) the term

$$-8A_0v_4^2\zeta^{\epsilon}\ln\zeta+O(\epsilon v_4^2)+O(rv_4^2)\ . \tag{66}$$

All other graphs with two internal lines connecting the vertices in Fig. 5 also give contributions proportional to $\zeta^{\epsilon}\ln\zeta$. The total contribution C_2 of all the two-line graphs to $v_4^{(1)}$ in (53) is seen to be

$$C_2=-10(v_4+qv_6I_1)^2A_0\zeta^{\epsilon}\ln\zeta-36v_4v_6I_1A_0\zeta^{\epsilon}\ln\zeta-162v_6^2I_1^2A_0\zeta^{\epsilon}\ln\zeta\ . \tag{67}$$

The second and third terms represent, respectively, the contribution of graphs 5(c) and 5(i).

The three-line graphs 5(d), 5(e), 5(j), and 5(k) also make contributions to the recursion relation (53). They add to $v_4^{(1)}$ a contribution C_3 given by

$$C_3 = -\zeta^{\epsilon}(72 v_4 v_6 + 648 v_6^2 I_1) I_3 \ , \tag{68}$$

$$I_3 = \int \int d^d q_1 d^d q_2 (q_1^2 q_2^2 \, | q_1 + q_2 |^2)^{-1} \ , \tag{69}$$

where the integration is to be carried over the domain $\zeta^{-1} < (|q_1|, |q_2|) < 1$ subject to the restriction that $|q_1 + q_2|$ must also lie between ζ^{-1} and 1. The restriction makes the exact calculation of I_3 very difficult. However, as regards the dependence of I_3 on ζ one finds to zeroth order in ϵ (Ref. 16)

$$I_3 = \frac{A_0^2}{8}[a(1-\zeta^{-2}) - b\zeta^{-2}\ln\zeta] \ , \tag{70}$$

where a and b are pure numbers.

The four-line graphs (n) and (p) in Fig. 5 give contributions to $v_4^{(1)}$ of order ζ^{8-3d}. As will be seen below, these contributions are of no importance.

Upon adding (C_2) and (C_3) to the first-order contribution to $v_4^{(1)}$, we find in place of (53) the relation

$$v_4^{(1)} + 9A_1 v_6^{(1)} = \zeta^{4-d}[a_4 - 10 a_4^2 A_0 \ln\zeta - 9 v_6 a_2 \ln\zeta] + O(\zeta^{8-3d} v_6^2 \ln\zeta) \ , \tag{71}$$

and

$$v_6^{(1)} = \zeta^{6-2d}[v_6 - (24+2b) v_4 v_6 A_0 \ln\zeta - (216+18b) v_6^2 A_0 A_1 \ln\zeta] \ . \tag{72}$$

In analogy with (54), one expects $v_6^{(1)}$ given by (72) to be the renormalized v_6 up to second order. The graphs contributing to v_6 in second order are shown in Fig. 6. Evaluation of the graphs gives, for $v_6^{(1)}$, the expression (72) provided b equals 12. Since b is hard to evaluate directly, we shall use the value 12 for b wherever it occurs. The left-hand side of (71) is consequently $a_4^{(1)}$.

It is not difficult to see that if we had included an eight-operator vertex in the effective Hamiltonian, the last term in (71) would contribute to the renormalization of the strength v_8 of such a vertex. With v_6 positive, the v_8 vertex is not of any consequence. Terms proportional to $\zeta^{(8-3d)}$ will, therefore, be ignored throughout.

We turn next to the recursion relation for r. The diagrams contributing to r in second order are displayed in Fig. 7. The contribution D_2 of the two-line graphs 7(a), 7(b), 7(c), and 7(d) to $r^{(1)}$ is

$$D_2 = -4\zeta^2(4 v_4^2 I_1 + 54 v_4 v_6 I_1^2 + 162 v_6^2 I_1^3) A_0 \ln\zeta \ . \tag{73}$$

The contribution D_3 arising from the three-line graphs 7(e), 7(f), 7(g), and 7(h) is

$$D_3 = -\zeta^2(8 v_4^2 + 144 v_4 v_6 I_1 + 648 v_6^2 I_1^2) I_3 \ . \tag{74}$$

The four-line graph 7(i) gives a contribution proportional to ζ^{8-3d} while the five-line graph 7(j) gives a contribution proportional to ζ^{10-4d}. These are ignored in view of the remarks above concerning terms involving ζ^{8-3d}. Adding D_2, D_3 to the first-order contributions, we find in place of (58)

$$a_2^{(1)} = \zeta^2 a_2 \left[1 - \frac{a_4}{2\pi^2}\ln\zeta\right] \ . \tag{75}$$

In obtaining (75) we have taken $b = 12$ as before, and replaced $A_0(d)$ by $A_0(4)$.

Equations (71) and (72) can be rewritten as

$$a_4^{(1)} = \zeta^{\epsilon} a_4 \left[1 - \frac{5}{4\pi^2} a_4 \ln\zeta - \frac{9}{8\pi^2}\frac{v_6 a_2}{a_4}\ln\zeta\right] \ , \tag{76}$$

$$v_6^{(1)} = \zeta^{-2+2\epsilon} v_6 \left[1 - \frac{6}{\pi^2} a_4 \ln\zeta\right] \ . \tag{77}$$

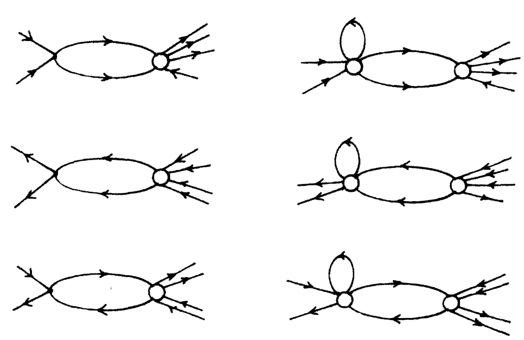

FIG. 6. Second-order graphs contributing to renormalization of the strength v_6 of the six-point vertex (t) in Fig. 1.

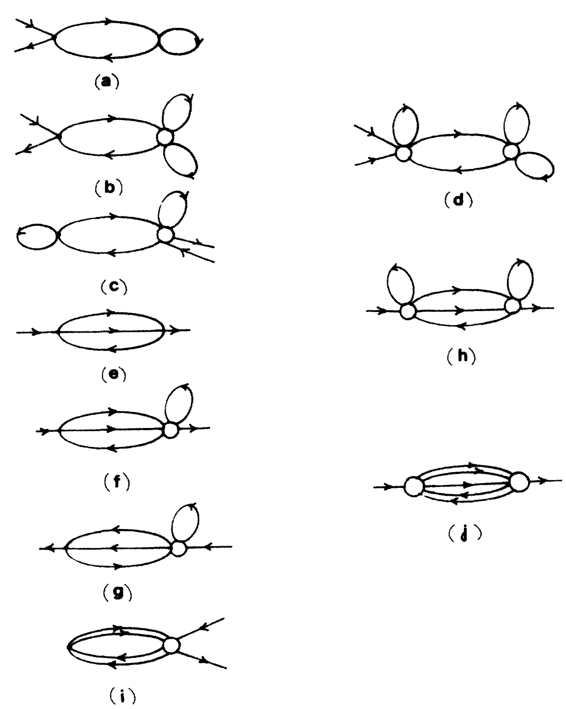

FIG. 7. Second-order graphs contributing to renormalization of the effective chemical potential r [cf. Eq. (26)].

The fixed point of the recursion relations (75) through (77) to first order in ϵ is

$$a_2^*=0,\quad v_6^*=0,\quad a_4^*=4\pi^2\epsilon/5\ , \tag{78}$$

and is reached provided (a_2,a_4) satisfy the initial conditions

$$a_2=0\ , \tag{79}$$

$$a_4>0\ . \tag{80}$$

For $a_4<0$ to begin with, one does not reach the fixed point. In view of the definitions (60) and (61), the fixed point (78) implies

$$v_4^*=4\pi^2\epsilon/5,\quad r^*=-\epsilon/5\ . \tag{81}$$

These are the fixed point values for the parameters of a weakly interacting Bose system with purely repulsive interactions.[11] As will be seen below, the critical behavior of the mixture near its λ line is the same as the critical behavior of a pure Bose system.

Linearization of the recursion relations near the fixed point (78) shows that the only relevant scaling field is a_2 and corresponds to eigenvalue $(2-2\epsilon/5)$. The λ line for the mixture is thus $a_2=0$ as long as $a_4>0$.

It was shown in Ref. 7 that, at fixed μ_4, the lines $a_2=0$ and $a_4=0$ have the general form in μ_3-T plane exhibited in Fig. 8. The point of intersection of the lines marks the end point of the λ-line in the region $a_4>0$ and defines the tricritical point. In the region $a_4<0$, a λ line cannot exist, but normal and ordered phase may coexist on the line given by (23).

The tricritical region is by definition the neighborhood of the tricritical point. In this region one can distinguish two limiting types of behavior: critical behavior corresponding to $a_2\to 0$, $a_4>0$ and tricritical behavior corresponding to $|a_2|>0$, $a_4\to 0$. Our aim is to study scaling in the tricritical region and to calculate scaling functions in order to exhibit crossover from critical to tricritical behavior. The usual linearization of recursion relations near the fixed point in RG theory is obviously not suitable for this purpose. One requires to use the full recursion relations.

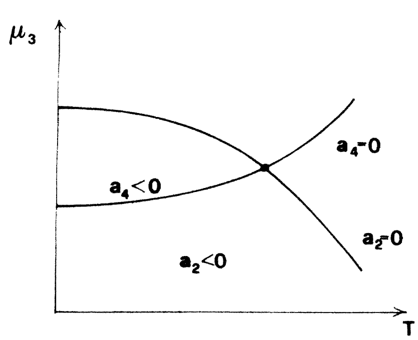

FIG. 8. Qualitative plots of the curves $a_2=0$ and $a_4=0$, where a_2, a_4 are given, respectively, by Eqs. (62) and (63). The concave side of the curve $a_2=0$ represents the region $a_2<0$ and the concave side of $a_4=0$ represents the region $a_4<0$. The λ line $(a_2=0)$ exists in the region $a_4>0$ only, terminating at the intersection of the curves $a_2=0$ and $a_4=0$. The intersection is defined as the tricritical point of the system.

We define

$$u_4=a_4/a_4^*\ , \tag{82}$$

$$u=u_4/(1-u_4)\ . \tag{83}$$

The recursion relation for a_2 can then be written as

$$a_2^{(1)}=\zeta^2 t/(1+\zeta^\epsilon u)^{2/5}\ , \tag{84}$$

$$t=a_2/(1-u_4)^{2/5}\ . \tag{85}$$

To order ϵ, (84) and (75) are identical. On choosing $\zeta^2=|t|^{-1}$, we have

$$a_2^{(1)}=\pm(1+x)^{-2/5}\ , \tag{86}$$

where

$$x=u/|t|^{\epsilon/2}\ , \tag{87}$$

and $\pm$ denotes the sign of t

In a similar manner, (76) and (77) can be written as

$$a_4^{(1)}/a_4^*=x/(1+x\pm w|t|^{1-\epsilon}/x)\ , \tag{88}$$

$$v_6^{(1)}=v|t|^{1-\epsilon}/(1+x)^{24/5}\ , \tag{89}$$

where

$$v=v_6/(1-u_4)^{24/5}\ , \tag{90}$$

$$w=\frac{9}{10}\frac{v}{(a_4^*)^2}(1-u_4)^{16/5}(1-|t|^{\epsilon/2})\ , \tag{91}$$

and $\pm$ in (88) again denotes the sign of t.

Evidently, (u,t,v,w) are the nonlinear scaling fields associated with our linear (first order) scaling fields (u_4,a_2,v_6). Note, however, that (u,t,v) are identical with (u_4,a_2,v_6) for small values of (u_4,a_2,v_6) in which we are primarily interested, and w differs from v in that case by a pure number.

As will be seen in Sec. IV, the recursion relations (86), (88), and (89) determine the crossover from critical behavior to tricritical behavior. These relations do not involve a *single* combination of scaling fields; two such combinations, viz., x and $vt^{1-\epsilon}$ appear. The phenomenological tricritical scaling hypothesis thus fails for the fermion-boson mixture for $\epsilon<1$.

IV. CROSSOVER FUNCTIONS

In this section we derive the equation of state for the mixture and discuss its scaling property.

After the RG transformation, the thermodynamic potential Ω' defined by (15) is given by

$$\Omega'=\Omega_0'-\frac{\zeta^{-d}}{\beta V_1}\ln \mathrm{Tr}\exp(-H_1)\ , \tag{92}$$

where Ω_0' is independent of M, and H_1 is obtained from H_0 by the replacement $(s,M,V,r,v_4,v_6)\to(s_1,M_1,V_1,r_1,v_4^{(1)},v_6^{(1)})$. The equation of state (17), accordingly, can be written as

$$\beta h_1 = \frac{1}{V_1}\left\langle \frac{\partial H_1}{\partial M_1} \right\rangle , \tag{93}$$

where

$$h_1 = \zeta^{d/2} h , \tag{94}$$

and the thermodynamic average $\langle\ \rangle$ is calculated with the Hamiltonian H_1. Comparison of (93) and (17) shows that the equation of state is formally invariant under the RG transformation.

An approximation to the equation of state, correct to first order in the interactions, is obtained by using a generalized Hartree-Fock factorization of the four-operator averages in the expression (19) for the inverse susceptibility r_n. The result is[10]

$$r_n = -\mu_4' + 4u_4' n' + 18u_6 (n')^2 + 2(u_4' + 9u_6 n')M^2 + 3u_6 M^4 + (2u_4' + 12u_6 M^2 + 18u_6 n')Y + 9u_6 Y^2 , \tag{95}$$

$$n' = A_0 \int_0^{p_c} dq\, q^{d-1} \left[\frac{(q^2/m_4 + r_n + r_s)}{(\exp(\beta\epsilon_q) - 1)\epsilon_q} - \frac{1}{2} + \frac{q^2/m_4 + r_n + r_s}{2\epsilon_q} \right] , \tag{96}$$

$$Y = -A_0 \int_0^{p_c} dq\, q^{d-1} \left[\frac{r_s}{[\exp(\beta\epsilon_q) - 1]\epsilon_q} - \frac{r_s}{2\epsilon_q} \right] , \tag{97}$$

$$r_s = 2(u_4' + 9u_6 n' + 9u_6 Y)M^2 + 6u_6 M^4 + 2(u_4' + 9u_6 n')Y , \tag{98}$$

$$\epsilon_q^2 = (q^2/m_4 + r_n)(q^2/m_4 + r_n + 2r_s) . \tag{99}$$

Here r_s denote anomalous self-energy $\Sigma_{02}(0,0)$;[12] for a Bose system it is the analogue of the inverse transverse susceptibility of a classical spin system. For small M and small h/M, r_n, r_s are both small quantities permitting for n' and y the approximations

$$n' = I_0 + \frac{A_0 m_4^2}{\beta\epsilon p_c^\epsilon}(r_n + r_s) - \frac{a_1}{2}[r_n^{1-\epsilon/2} + (r_n + 2r_s)^{1-\epsilon/2}] , \tag{100}$$

$$Y = \frac{A_0 m_4^2 r_s}{\beta\epsilon p_c^\epsilon} + \frac{a_1}{2}[r_n^{1-\epsilon/2} - (r_n + 2r_s)^{1-\epsilon/2}] , \tag{101}$$

where I_0 is given by (64) and

$$a_1 = A_0 K(d) m_4^{d/2}/\beta , \tag{102}$$

$$K(d) = \frac{1}{\epsilon} + O(\epsilon) . \tag{103}$$

Substitution from (100) and (101) into (95) and (98) gives

$$r_{n0} = a_2 + 2a_4 m^2 + 3v_6 m^4 + \frac{A_0}{\epsilon}[(4a_4 + 18v_6 m^2)r_{n0} + (6a_4 + 30v_6 m^2)r_{s0}] - \frac{A_0}{\epsilon}[(a_4 + 3v_6 m^2)r_{n0}^{1-\epsilon/2} + (3a_4 + 15v_6 m^2)(r_{n0} + 2r_{s0})^{1-\epsilon/2}] , \tag{104}$$

$$r_{s0} = 2a_4 m^2 + 6v_6 m^4 + \frac{A_0}{\epsilon}(18m^2 v_6 r_{n0} + 2a_4 r_{s0} + 36m^2 v_6 r_{s0}) + KA_0[a_4 r_{n0}^{1-\epsilon/2} - (a_4 + 18m^2 v_6)(r_{n0} + 2r_{s0})^{1-\epsilon/2}] , \tag{105}$$

where

$$m^2 = s p_c^{-d} M^2 , \tag{106}$$

$$r_{n0} = s^{-1}\beta r_n , \tag{107}$$

$$r_{s0} = s^{-1}\beta r_s , \tag{108}$$

and a_2, a_4, v_6 are given, respectively, by (60), (61), and (25). It is gratifying to note that one obtains exactly the same scaling fields in the approximate, direct calculation of the equation of state as in the RG approach.

Consider first the equation of state in the normal phase, i.e., $h \to 0$, $M \to 0$, $h/M \neq 0$. The invariance of the equation of state referred to above implies

$$r_n^{(1)} = a_2^{(1)} + \frac{a_4^{(1)}}{2\pi^2\epsilon}[r_n^{(1)} - (r_n^{(1)})^{1-\epsilon/2}] , \tag{109}$$

where

$$r_n^{(1)} = \beta s_1^{-1} h_1 / 2M_1 = \zeta^2 r_{n0} . \tag{110}$$

Taking, as before, $\zeta^2 = |t|^{-1}$ and defining the scaling function y for the susceptibility by

$$r_{n0}^{-1} = |t|^{-1} y , \tag{111}$$

we obtain on using the recursion relations (86) and (88) the following equation for y:

$$1 = y/(1+x)^{2/5} - \frac{\epsilon}{5} x \ln y / (1 + x + w|t|^{1-\epsilon}/x) . \tag{112}$$

To ensure a positive susceptibility in the normal phase, t (or a_2) has been assumed positive in the normal phase.

Equation (112) shows that the scaling function y is in general a function of two variables, viz., x and $(wt^{1-\epsilon})$. Thus orthodox tricritical scaling[13,14] does not hold for the normal phase of the fermion-boson mixture for $\epsilon < 1$. In the infinite component model investigated by Sarbach and Fisher,[4] orthodox scaling holds for the normal phase with Gaussian tricritical exponents. It may be pointed out that investigation of a classical one-component field theoretic model by techniques of renormalized perturbation theory[17] by Lawrie[18] did not give results different from those of Sarbach and Fisher.[4]

What kind of crossover behavior does (112) imply? In the critical regime ($t \to 0$, $u \neq 0$) Eq. (112) has the approximate solution

$$y = x^{2/5}\left[1+\frac{2\epsilon}{25}\ln x\right] \simeq x^{(2/5+2\epsilon/25)} . \quad (113)$$

This implies that the susceptibility exponent γ has a value identical with that of a pure Bose system,[12] viz.

$$\gamma = 1-\frac{\epsilon}{5}+0(\epsilon^2) . \quad (114)$$

In the tricritical regime ($t \neq 0$, $u \to 0$), (112) gives

$$y = 1+\tfrac{2}{5}x-\tfrac{3}{25}x^2+\frac{8x^3}{125}+2\epsilon x^3/25wt^{1-\epsilon}+\cdots . \quad (115)$$

The tricritical exponent γ_t thus is 1 as in the classical theory.[6,19]

Note that in the critical regime as well as in the tricritical regime the dependence of y on the variable $(wt^{1-\epsilon})$ is small. It is significant only in the region $x \sim (wt^{1-\epsilon}) \sim 1$.

We consider next the full equation of state given by (104) and (105). Under the RG transformation m^2 defined by (106) scales as $(\zeta^{2-\epsilon}m^2)$. Choosing again $\zeta^2 = |t|^{-1}$, we find

$$r_n^{(1)} = \pm 1/(1+x)^{2/5}+2m_1^2X_4+m_1^4\alpha X_6|t|^{1-\epsilon} + \frac{\epsilon}{20}(X_4+3m_1^2\alpha X_6|t|^{1-\epsilon})r_n^{(1)}\ln r_n^{(1)} + \frac{\epsilon}{20}(3X_4+15m_1^2\alpha X_6|t|^{1-\epsilon}) \times (r_n^{(1)}+2r_s^{(1)})\ln(r_n^{(1)}+2r_s^{(1)}) , \quad (116)$$

$$r_s^{(1)} = 2m_1^2X_4+2m_1^4\alpha X_6|t|^{1-\epsilon}-\frac{\epsilon}{20}X_4r_n^{(1)}\ln r_n^{(1)} + \frac{\epsilon}{20}(X_4+6m_1^2\alpha X_6|t|^{1-\epsilon}) \times (r_n^{(1)}+2r_s^{(1)})\ln(r_n^{(1)}+2r_s^{(1)}) , \quad (117)$$

where $r_n^{(1)}$ given by (110) is

$$r_n^{(1)} = \frac{\beta h}{2m_1}\left[\frac{sp_c^d}{a_4^*}\right]^{-1/2}|t|^{(-3/2+\epsilon/4)} , \quad (118)$$

$$m_1^2 = m^2a_4^*|t|^{-1+\epsilon/2} , \quad (119)$$

$$\alpha = 3v/(a_4^*)^2 , \quad (120)$$

$$X_4 = a_4^{(1)}/a_4 , \quad (121)$$

$$X_6 = (1+x)^{-24/5} . \quad (122)$$

It is evident that in the ordered phase ($h \to 0$, $m_1 \neq 0$, $t<0$)

$$m^2a_4^* = |t|^{1-\epsilon/2}Y_2(x,\alpha|t|^{1-\epsilon}) , \quad (123)$$

where Y_2 denotes the solution of (116) and (117) for m_1^2 when $r_n^{(1)}=0$.

Thus, in the ordered phase also, orthodox tricritical scaling is not valid. In addition to its occurrence in X_4, the variable $v|t|^{1-\epsilon}$ now also appears in the coefficients of the terms involving the anomalous self-energy r_s. For the infinite component model, on the other hand, classical scaling holds in the ordered phase.

The exponent $(1-\epsilon/2)$ in (123) cannot be identified with either the critical exponent 2β or the tricritical exponent $2\beta_t$ for the square of the order parameter without a knowledge of the crossover function Y_2. In the critical and tricritical limits the forms of Y_2 are, however, easily determined. In the critical limit, Eqs. (116) and (117) give

$$3r_s^{(1)} = 4m_1^2+x^{-2/5} , \quad (124)$$

$$x^{-2/5} = 2m_1^2+\frac{3\epsilon}{10}r_s^{(1)}\ln(2r_s^{(1)}) . \quad (125)$$

For small ϵ, the solution for m_1^2 is

$$Y_2 = x^{(-2/5+3\epsilon/25)}/2^{(1+3\epsilon/10)} , \quad (126)$$

which implies that the critical exponent β has the value

$$\beta = \frac{1}{2}-\frac{3\epsilon}{20}+O(\epsilon^2) . \quad (127)$$

In the tricritical limit ($t \neq 0$, $x \to 0$), (116) and (117) give, after some simplification,

$$r_s^{(1)} = \tfrac{2}{5}+\tfrac{8}{5}m_1^4\alpha|t|^{1-\epsilon} , \quad (128)$$

$$1 = m_1^4\alpha|t|^{1-\epsilon}+\frac{3\epsilon}{2}m_1^2\alpha|t|^{1-\epsilon}r_s^{(1)}\ln(2r_s^{(1)}) . \quad (129)$$

The solution of these equations for small ϵ is

$$m_1^2 = (\alpha|t|^{1-\epsilon})^{-1/2}[1-3(\alpha|t|^{1-\epsilon})^{1/2}\ln 2] \quad (130)$$

which implies that the tricritical order parameter exponent β_t is $\frac{1}{4}$ as in the classical theory.[7,19]

V. CONCLUSION

In this paper we have carried out RG analysis in $(4-\epsilon)$ dimensions of the effective Hamiltonian of a mixture of weakly interacting fermions and bosons with a view to study scaling and crossover effects which may occur in such a model. The earlier RG studies[1,2] of tricritically have been confined to classical spin models, and on account of imperfect treatments were unable to provide proper accounts of scaling and crossover from critical to tricritical behavior.

The essential results for the mixture are contained in the recursion relations derived in Sec. III which reveal that for $\epsilon<1$ the renormalized thermodynamic fields $a_2^{(1)}$, $a_4^{(1)}$, and $v_6^{(1)}$ cannot be written in terms of a single combination of scaling fields. Two combinations, viz., x and $v|t|^{1-\epsilon}$ characterize these relations. As the renormalized fields are important ingredients of crossover behavior both in the normal as well as the ordered phase, conventional, orthodox scaling[12,13] does not hold in either of the phases in the tricritical region. Calculations of crossover functions for the susceptibility in the normal phase and for the order parameter in the ordered phase in Sec. IV demonstrate this explicitly.

It is interesting to compare the results of the RG treatment of the fermion-boson with those obtained by Sarbach and Fisher[4] for the infinite-component classical spin model which provided the first example of breakdown of orthodox scaling. As pointed out in Sec. IV, study of an Ising-type model[18] by the methods of renormalized perturbation theory did not give different results. The first point worthy of notice is that breakdown of scaling in the classical model is not as direct or transparent as in the case of the mixture. Whereas no scaling in the orthodox sense is possible either in the normal or in the ordered phase of the mixture, in the case of the infinite component model orthodox scaling holds for each phase though separately, i.e., with different sets of critical exponents. Secondly, while the failure of scaling in the classical model is associated with a nonuniversal parameter which involves the range of the spin-spin interaction, for the mixture it is associated with a quantum parameter α given by (120). In the high density fermion limit, α can be written in view of (6), (25), and (90) as

$$\alpha = N(d)\frac{m_4}{m_3}(p_c^{2d-6}/\lambda_T^4 k_F^{4-d})(2m_3 u_{34})^3 , \tag{131}$$

where λ_T denotes the boson thermal wavelength, k_F the Fermi momentum of the fermion component and $N(d)$ is a pure number.

Although the analysis of this paper is not applicable to a real three-dimensional system, it illuminates the results obtained in Ref. 10 connected with nonuniversality of scaling in three dimensions. The recursion relation (54) for v_6 shows that, for $d=3$, v_6 is a marginal variable. The breakdown of scaling found in $4-\epsilon$ dimensions may, therefore, be expected to be absent in three dimensions. The equation of state, however, will still contain v_6 as a nonscaling parameter and unless this can be scaled away by a suitable redefinition of the various quantities, universality of classical scaling will be violated. The equation of state (104) is, in fact, valid for $d=3$ provided that one replaces A_0/ϵ by $1/4\pi$. It is not difficult to see that if one tries to eliminate v_6 from this equation by defining $m^2 v_6$ as m_1^2, $r_{n0}v_6$ as r_{n1}, one finds an equation which does not contain v_6 except that the last term now carries a multiplicative factor $v_6^{1/2}$. Classical scaling in $d=3$ thus lacks universality. The consequences of this lack of universality have been investigated in Ref. 10. It is interesting to note that the nonuniversality parameter in that treatment (denoted by α) is nothing but $v_6^{1/2}$. As discussed in detail in Ref. 10, certain tricritical amplitude ratios (such as Q_1 which is a measure of the deviation of the upper coexistence line from the λ line in $T-x$ plane, x denoting ^{3}He concentration) are directly connected with the nonuniversality parameter $v_6^{1/2}$ which, in view of (131), is proportional to $(b^3/\lambda_T^4 k_F)$, b denoting the scattering length associated with the fermion-boson interaction u_{34}. An accurate measurement of these ratios can provide a test of nonuniversality of scaling in three dimensions.

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