

# Quantum statistical mechanics of an array of resistively shunted Josephson junctions

Sudip Chakravarty, Gert-Ludwig Ingold,\* Steven Kivelson,<sup>†</sup> and Gergely Zimanyi<sup>‡</sup>

*Department of Physics, State University of New York at Stony Brook, Stony Brook, New York 11794*

(Received 11 September 1987)

We have constructed a fully quantum-mechanical model of an ordered array of resistively shunted Josephson junctions, and have determined the nature of the phase diagram as a function of the Josephson coupling,  $V$ , the capacitance,  $C$  (or, equivalently, the charging energy  $E_0 = 4e^2/C$ ), the shunt resistance,  $R$ , and the temperature,  $T$ . In order to treat the dissipative element ( $R$ ) in a quantum system, we have modeled it by a heat bath with spectral weight chosen to reproduce Ohmic resistance in the classical limit. Among other results, we find that in the extreme quantum limit,  $E_0 \gg V \gg k_B T$ , the onset of global phase coherence (superconductivity) in the array occurs only if  $R$  is less than a critical value  $R_c = Ah/e^2$ , where  $A$  is a number of order 1 which depends on the dimension and the lattice structure. The fact that the dissipation enters the thermodynamics at all is a consequence of the quantum nature of the transition. This transition is reminiscent of the results of recent experiments on thin films of granular superconductors.

## I. INTRODUCTION

It is often useful to describe the collective behavior of large numbers of particles in terms of the dynamics of a much smaller number of collective coordinates or coarse-grained fields such as the phase across a Josephson junction or the macroscopic wave function of a superconductor. Because of the collective nature of these variables, their dynamics is typically well approximated by classical, dissipative equations of motion, where the dissipation represents the effect of the coupling to the remaining microscopic degrees of freedom (the heat bath), e.g., the long-wavelength fluctuations of the superconducting order parameter can be well represented by the time-dependent Landau-Ginzburg equations. It is a characteristic of a classical system that the dynamics and the thermodynamics are completely separate, and thus the dissipation does not affect the thermodynamic properties of the system at all. Under some circumstances, it is possible to enter a regime of temperature and coupling constants where quantum fluctuations of the collective coordinates become important. Here, not only the effective dynamical mass of the collective coordinate, but the magnitude of the dissipation as well affect the thermodynamic properties of the system. For instance, it was recognized several years ago that in problems that involve a single collective coordinate  $\phi$  which is coupled to an Ohmic heat bath with resistance  $R$ , and which is otherwise free to tunnel between classical ground states in a symmetric double well<sup>1,2</sup> or a periodic potential,<sup>3,4</sup> there is a zero temperature "quantum-to-classical" phase transition as a function of  $R$ . The amount of dissipation is proportional to  $1/R$ . For  $R$  greater than a critical value of order  $h/(2e)^2$ ,  $\phi$  is free to tunnel between potential minima, i.e., quantum fluctuations restore the symmetry between the wells that are broken at the classical level. For  $R$  less

than the critical resistance  $\phi$  is localized in a single well and the quantum ground state has the same degeneracy as if  $\phi$  were a strictly classical variable.

Experiments have been carried out in recent years which confirm the validity of some aspects of this discussion. In particular, measurements of the decay of metastable persistent currents in Josephson junction devices have been performed<sup>5</sup> which are consistent with the theoretical descriptions of the system in terms of a single macroscopic coordinate with effective mass determined by the capacitance  $C$  and linear coupling to an Ohmic heat bath. However, no experiments have yet been performed on single Josephson-junction devices which observe a phase transition as a function of the dissipation. There have, however, been experiments performed on ultrathin films of granular superconductors which are highly suggestive of a transition as a function of dissipation.<sup>6,7</sup> These films are believed to be in the extreme quantum regime, since the small grain sizes imply charging energies  $E_0 = 4e^2/C$  large compared to the Josephson coupling  $V$  between nearby grains. It has been observed in a variety of films that if the normal-state sheet resistance is in excess of  $R_c \approx h/4e^2 \approx 6.45 \text{ k}\Omega$ , the films remain in a finite-resistance state down to  $T = 0$ , while for sheet resistance less than  $R_c$ , the films have an unmeasurably small resistance below a critical temperature. Within experimental uncertainties the critical value of the normal-state resistance appears to be  $h/4e^2$ , independent of sample geometry (implying insensitivity to the precise value of the charging energy  $E_0$ ), or whether the granular material is polycrystalline Sn (Ref. 6) or amorphous Ga,<sup>7</sup> or Pb.<sup>8</sup> This observation led Jaeger *et al.*<sup>6</sup> to suggest that the transition was similar to that predicted for single Josephson junctions: *a phase transition as a function of the dissipation*. This suggestion provided the original motivation for Chakravarty, Ingold, Kivelson, and Luther (CIKL) to study<sup>9</sup> the thermodynamic properties of a phenomenologically motivated model of an array of resistively shunted

Josephson junctions. From a somewhat different point of view the role of the Ohmic heat bath in the present context has also been discussed by Fisher.<sup>10</sup> However, it is important to stress at the outset that it is by no means definitively established that the dissipation, in the sense of a coupling to a heat bath, is, in fact, the important property of the system which is characterized by the normal-state resistance. It is possible that similar results could be obtained if one assumes that the resistance is a measure of the disorder. Moreover, if we imagine that the resistance is due to quasiparticle tunneling across a tunnel junction between grains, then the resistance is correlated with the Josephson coupling  $V$  between grains according to the relation  $VR = \hbar/8e^2\Delta_0$  at  $T=0$ , where  $\Delta_0$  is the superconducting gap of the grains. A theory based on this observation<sup>11</sup> appears to be in agreement with some aspects of the experiments. Thus, while there are strong analogies between the behavior of the model we have studied and the experiments on granular superconductors, the details of the relation between them is currently unclear. Nonetheless, it is important to explore in detail a well-defined model which, in our opinion, leads to many interesting results. Furthermore, it is not difficult to envision that in the near future such resistively shunted junction arrays can be artificially fabricated.

In our previous paper<sup>9</sup> we discussed the motivation for considering a simple model of a Josephson-junction array and obtained an approximate phase diagram for the model by use of a variational calculation. We showed that at low temperatures and for small capacitance, the system is highly quantum mechanical and that, therefore, the resistance has an important effect on the location of the phase boundary. In particular, at  $T=0$  and  $V/E_0$  less than a critical value, the onset of phase coherence in the ground state is determined entirely by the value of the resistance. This, in particular, is analogous to the experimental results in granular superconductors. (cf. Ref. 11.)

The present paper is a more complete and detailed treatment of the same model. In Sec. II we define the model. In Sec. III we discuss the variational solution to the model, including details of the calculation that were omitted in our previous paper. Sections IV and V contain new results. In Sec. IV we analyze the asymptotic behavior of the model in the limiting regions of the phase diagram without making a variational or mean-field approximation. In Sec. IV A we consider the limit of small dissipation  $\alpha = \hbar/4e^2R \ll 1$ , where, at finite temperatures, the effect of the heat bath is to produce small perturbative renormalization of the parameters; in Sec. IV B we discuss the case  $\alpha \gg 1$ , where the model becomes completely classical. In Sec. IV C we discuss the semiclassical limit  $V/E_0 \gg 1$ , where single phase slips can be treated as a dilute gas of instantons and in Sec. IV D we discuss the extreme quantum limit  $V/E_0 \ll 1$ , where the transition is driven by an infrared catastrophe characteristic of the Ohmic heat bath and which we study using a perturbative renormalization-group treatment. In Sec. V we obtain some rigorous bounds on the critical value of  $\alpha$ . Finally, in Sec. VI we summarize our most important results and discuss some unanswered questions which are currently being investigated.

## II. THE MODEL

We consider the model of an array of superconducting grains. Here we label each grain with an index  $j$ . We imagine that the largest Josephson coupling  $V$  is weak, i.e.,  $V \ll k_B T_c$ , where  $T_c$  is the bulk superconducting transition temperature, and we focus on temperatures  $T \lesssim V/k_B$ . Since  $T \ll T_c$  and the grains themselves are macroscopic, we can ignore the fluctuations of the magnitude of the order parameter,  $\Delta(T)$ ; but the phase  $\theta_j$  on each grain  $j$  remains a dynamical variable. A rather general Hamiltonian for such a system is  $(\Delta\theta_{ij} = \theta_i - \theta_j)$

$$H = \frac{1}{2} \sum_{i,j} 4e^2 n_i (C^{-1})_{ij} n_j + \sum_{\langle i,j \rangle} V_{ij} [1 - \cos(\Delta\theta_{ij})] + \sum_{\langle i,j \rangle} \Delta\theta_{ij} X_{ij} + \sum_{\langle i,j \rangle} h_{ij}(X_{ij}), \quad (1)$$

where  $C_{ij}$  is the mutual capacitance matrix, and  $n_j$  is the charge of the number of Cooper pairs on the grain  $j$  and is thus the canonical momentum conjugate to  $\theta_j$ :  $[n_i, \theta_j] = -i\delta_{ij}$ . The Josephson coupling energy between the grains  $i$  and  $j$  is given by  $V_{ij}$ . The sum  $\langle ij \rangle$  is over nearest-neighbor pairs. The first two terms constitute the widely studied standard phase Hamiltonian.<sup>12</sup> The third term represents the effect of dissipation in the junctions by coupling the phase difference across the junction to a heat bath. Following Caldeira and Leggett,<sup>13</sup> we determine the nature of the heat bath phenomenologically by requiring that it act like a shunt resistor (i.e., exhibits linear damping) in the classical limit. Thus,  $X_{ij}$  is a collective coordinate describing the coupling to the heat bath associated with the shunt resistor across the junction  $\langle ij \rangle$ , and  $h_{ij}(X_{ij})$  is the Hamiltonian of the  $\langle ij \rangle$ th heat bath. Since we will treat the coupling to the heat bath in linear-response theory, the effect of the heat bath is characterized by its zero-temperature spectral response function

$$J_{ij}(\omega) = \int \frac{dt}{2\hbar} e^{i\omega t} \langle X_{ij}(t) X_{ij}(0) \rangle,$$

which will reproduce Ohmic damping so long as

$$J_{ij}(\omega) = \frac{\hbar}{2\pi} \alpha_{ij} |\omega| \quad (2)$$

for  $\omega$  less than an upper cutoff frequency  $\omega_c$ , where  $\alpha_{ij} = \hbar/4e^2 R_{ij}$  and  $R_{ij}$  is the shunting resistance between grains  $i$  and  $j$ . For a disordered array,  $C_{ij}$ ,  $V_{ij}$ , and  $\alpha_{ij}$  are all random variables. This model is a slight generalization of the model considered by CIKL in that we have included the possibility of both mutual and single-grain contributions to the capacitance matrix, whereas CIKL assumed  $C_{ij}$  to be diagonal.

Since the effect of the heat bath is to introduce a time-retarded interaction between fluctuations of the phase, it is more convenient to treat this model in terms of a path-integral formalism. We are interested in the thermodynamic properties of the system, therefore we consider the effective Euclidean action functional obtained after the heat-bath degrees of freedom have been integrated out.<sup>14</sup>

$$\begin{aligned} \frac{S_{\text{eff}}}{\hbar} = & \int_0^\beta \frac{d\tau}{\hbar} \left[ \frac{1}{2} \sum_{\langle i,j \rangle} \left[ \frac{\hbar}{2e} \right]^2 \dot{\theta}_i C_{ij} \dot{\theta}_j \right. \\ & \left. + \sum_{\langle i,j \rangle} V_{ij} (1 - \cos \Delta \theta_{ij}) \right] \\ & + \frac{\beta}{4\pi} \sum_n |\omega_n| \sum_{\langle i,j \rangle} \alpha_{ij} |\Delta \tilde{\theta}_{ij}(n)|^2. \end{aligned} \quad (3)$$

The last term in  $S_{\text{eff}}$  represents the dissipative effect of the environment. For later convenience we have expressed it in the form of a series, where  $\omega_n = 2\pi n/\beta$  are the Matsubara frequencies and  $\Delta \tilde{\theta}_{ij}(n)$  is the Fourier transform of  $\Delta \theta_{ij}(\tau)$ .

To make progress on this problem, we shall simplify the model in several ways. Most importantly, we consider the problem of an ordered array with  $\alpha_{ij} = \alpha$ ,  $C_{ij} = C\delta_{ij}$ , and  $V_{ij} = V$ . For now, we will simply imagine that we are considering the properties of an artificial, ordered array. Secondly, we will assume that the capacitance matrix is diagonal. The generalization to a more general capacitance matrix is straightforward, though involved. However, our results do not depend sensitively on the detailed form of  $C_{ij}$ .

### III. VARIATIONAL CALCULATION

The model defined in Sec. II is still not exactly soluble, despite the simplifications we have made. In this section, as a first approximation, we shall pursue a variational approach to argue that an important aspect of the dissipative transition is associated entirely with the infrared property of the spectral density of the Ohmic heat bath ( $J_{ij}(\omega) \rightarrow |\omega|$  as  $\omega \rightarrow 0$ ) which leads to results similar to those obtained earlier in the context of macroscopic quantum effects in Josephson systems.<sup>1-4</sup> Moreover, as we shall see, in the regime of the phase diagram where we can make independent estimates which are highly plausible but not entirely rigorous, the variational calculation yields results that are semiquantitatively correct.

The simplifications discussed at the end of Sec. II lead us to study the following effective action:

$$\begin{aligned} \frac{S_{\text{eff}}}{\hbar} = & \int_0^\beta du \left[ \frac{1}{2E_0} \sum_i \left[ \frac{\partial \theta_i}{\partial u} \right]^2 \right. \\ & \left. + V \sum_{\langle i,j \rangle} [1 - \cos \Delta \theta_{ij}(u)] \right] \\ & + \frac{\alpha\beta}{4\pi} \sum_n |\omega_n| \sum_{\langle i,j \rangle} |\Delta \tilde{\theta}_{ij}(n)|^2. \end{aligned} \quad (4)$$

In the rest of the paper we shall focus our attention entirely on a  $d$ -dimensional hypercubic lattice. The generalization to any lattice for which all bonds are geometrically equivalent is straightforward and not particularly illuminating. In carrying out the path integral with the action defined by Eq. (4) one should keep in mind that Ohmic dissipation implies that the charges can be transferred continuously. Thus the identification  $\theta_i(0) = \theta_i(2\pi)$  is in fact inconsistent with this assumption.

Phase transitions implied by the effective action can be very complicated in detail. However, it is plausible that when  $\langle \theta_i^2 \rangle^{1/2}$  becomes of the order of unity the system will be disordered. It is also plausible that if there are important topological excitations of the system, they would become significant when  $\langle \theta_i^2 \rangle^{1/2} \sim 1$ . Thus an important aspect in locating the phase boundary is to understand when  $\langle \theta_i^2 \rangle^{1/2}$  becomes of the order of unity. This can be achieved by using a variational principle which has been discussed extensively by Feynman. The more common Rayleigh-Ritz procedure to find an upper bound of the ground-state energy is a special case of the present variational technique.<sup>14</sup> A variational estimate  $F'$  of the true free-energy  $F$  can be constructed by use of the Gibbs-Helmholtz inequality:<sup>14,15</sup>

$$F' = F_{\text{tr}} + \langle H - H_{\text{tr}} \rangle_{\text{tr}} \geq F, \quad (5)$$

where  $H$  is the true Hamiltonian, and  $F_{\text{tr}}$  and  $\langle \rangle_{\text{tr}}$  are the free energy and thermal average computed with the trial Hamiltonian  $H_{\text{tr}}$ .

We choose the trial Hamiltonian or equivalently the effective trial action to be the following:

$$\begin{aligned} \frac{S_{\text{tr}}}{\hbar} = & \frac{1}{2} \int_0^\beta du \left[ \frac{1}{E_0} \sum_i \left[ \frac{\partial \theta_i}{\partial u} \right]^2 + D \sum_{\langle i,j \rangle} \Delta \theta_{ij}(u)^2 \right] \\ & + \frac{\beta\alpha}{4\pi} \sum_n |\omega_n| \sum_{\langle i,j \rangle} |\Delta \theta_{ij}(n)|^2, \end{aligned} \quad (6)$$

where the variational parameter  $D$  is the spin-wave-stiffness constant. The underlying physical picture is that if the system has a global phase coherence, then the spin-wave stiffness will be finite; but if it does not,  $D$  will be zero. This is also in accordance with the discussion in the preceding paragraph. Thus the variational method gives a self-consistent criterion for the root-mean-square fluctuations of the phase variables  $\theta_i$ . Of course, in the present context, the global phase coherence in the spin language is also the same as the global phase coherence in the sense of superconductivity.

Given the trial action, it is easy to show that

$$\begin{aligned} \Delta F = F'(D) - F'(D=0) \\ = \frac{1}{2} \int_0^\beta d\mu \sum_{\langle i,j \rangle} \langle \Delta \theta_{ij}^2(0) \rangle_\mu - \frac{D}{2} \sum_{\langle i,j \rangle} \langle \Delta \theta_{ij}^2(0) \rangle_{\mu=D} \\ - V \sum_{\langle i,j \rangle} e^{-\langle \Delta \theta_{ij}^2(0) \rangle_{\mu=D}/2}, \end{aligned} \quad (7)$$

where  $\langle \rangle_\mu$  is the average computed with the action (6), with  $D$  replaced by the variable  $\mu$ . Requiring that  $\partial(\Delta F)/\partial D$  be zero leads to the equation

$$\frac{D}{V} = e^{-\langle \Delta \theta_{ij}^2(0) \rangle_{\mu=D}/2}, \quad (8)$$

where we have made use of the fact that all bonds of the chosen lattice are equivalent. However, Eq. (8) merely locates an extremum of the free energy, and not necessarily the minimum. In order to find out if the nontrivial solution of Eq. (8) represents the minimum, one has to explicitly check the free energy, which can be rewritten as

$$\frac{\Delta F}{N} = -z_0 D + \frac{1}{2N} \int_0^D d\mu \sum_{\langle i,j \rangle} \langle \Delta \theta_{ij}^2(0) \rangle_\mu - \frac{D}{2N} \sum_{\langle i,j \rangle} \langle \Delta \theta_{ij}^2(0) \rangle_{\mu=D}, \quad (9)$$

where we have made use of Eq. (8).  $N$  is the number of sites of the lattice, and the connectivity constant  $z_0$  is

given by

$$z_0 = \frac{\text{number of bonds}}{\text{number of sites}}. \quad (10)$$

For the hypercubic lattice considered in the present paper  $z_0$  is the dimensionality  $d$  of the lattice. It is easy to compute  $\langle \Delta \theta_{ij}^2(0) \rangle_\mu$ , which is given by

$$\langle \Delta \theta_{ij}^2(0) \rangle_\mu = \frac{2}{\beta N d} \sum_{\mathbf{k}} \sum_{n=0}^{\infty} \frac{z - \Delta(\mathbf{k})}{\omega_n^2/E_0 + (\mu + \alpha \omega_n/2\pi)[z - \Delta(\mathbf{k})]}. \quad (11)$$

Here  $z = 2z_0$  is the coordination number of the lattice which is  $2d$  for a hypercubic lattice.  $\Delta(\mathbf{k}) = \sum_i e^{i\mathbf{k} \cdot \mathbf{R}_i}$  is the nearest-neighbor structure factor. The  $\mathbf{k}$  sum runs over the first Brillouin zone.

Although we shall never need it, we write down, for the sake of completeness, the exact expression for the free energy  $F_{\text{tr}}(D=0)$ . This is given by

$$F_{\text{tr}}(D=0) = F_0 + \frac{\alpha}{2\pi\beta} \int_0^1 d\lambda \sum_{\mathbf{k}} \sum_{n=1}^{n_c} \frac{\omega_n[z - \Delta(\mathbf{k})]}{\omega_n^2/E_0 + (\lambda\alpha/2\pi)\omega_n[z - \Delta(\mathbf{k})]}, \quad (12)$$

where the cutoff in the sum over Matsubara frequencies arises from the cutoff of the Ohmic heat bath.  $F_0$  is the free energy of the model corresponding to the action

$$\frac{S_0}{\hbar} = -\frac{1}{2E_0} \int_0^\beta du \sum_i \left[ \frac{\partial \theta_i}{\partial u} \right]^2, \quad (13)$$

which can be calculated in any number of ways.

We will study the phase diagram obtained from Eq. (8) for a  $d$ -dimensional hypercubic lattice with  $z = 2d$ . We characterize the transition to be continuous or discontinuous depending on whether the nontrivial ( $D \neq 0$ ) solution of Eq. (8) appears continuously or discontinuously. In fact, whenever the transition is discontinuous, the solution of Eq. (8) does not correspond to the absolute minimum of  $F'$ ; the point at which a nontrivial transition first appears is analogous to a spinodal line and the true transition where  $F'(D) = F'(D=0)$  occurs elsewhere. We will ignore this complication for now, and will simply study the solutions of Eq. (8). The actual examination of the free energy will be made later in the present section.

The zero-temperature phase diagram is shown in Fig. 1 for  $d = 1, 2$ , and 3. The vertical boundary at  $\alpha = 1/z_0$  is a line of continuous transition. It is clear from the figure that for low dissipation, when  $V/E_0$  is sufficiently small, quantum fluctuations destroy the long-range phase coherence even at  $T = 0$ . The interesting point to note here is that for all values of  $V/E_0$  below a threshold the continuous transition takes place at  $\alpha = 1/z_0$  regardless of the value of  $V/E_0$ . Since the only place at which capacitance enters the problem is  $E_0$ , the capacitance is an irrelevant variable in this regime and the transition is entirely dictated by  $\alpha$ . When  $\alpha$  is greater than  $1/z_0$ , the dissipation suppresses the quantum fluctuations and permits global superconductivity. At finite temperature, the transition is always discontinuous in the sense described above. The transition temperature  $T^*(\alpha, V/E_0)$  is an increasing function of both of its arguments.

We now describe in more detail our calculations and

results in the novel regime of weak coupling and low temperatures,  $E_0 \gg 2\pi/\beta$ ,  $2\pi D/\alpha$ , where quantum effects are most important. In this regime  $E_0$  serves only to define a high-energy cutoff for the sum in Eq. (7) at  $n_0 = \beta E_0/2\pi$ ; for smaller  $n$ ,  $\omega_n^2/E_0$  can be ignored (i.e., taken to be approximately equal to zero). Within this approximation, the  $\mathbf{k}$  dependence of the numerator and denominator of the expression for  $\langle \Delta \theta_{ij}^2(0) \rangle_{\mu=D}$  [see Eq. (11)] cancel, and hence the  $\mathbf{k}$  sum is trivial. The self-consistent equation determining  $D$  now becomes

$$\frac{D}{V} = \left[ \frac{2\pi}{\beta E_0} \right]^{1/z_0 \alpha} \exp \left[ \frac{1}{z_0 \alpha} \psi \left( \frac{\beta D}{\alpha} \right) \right], \quad (14)$$

where  $\psi(x)$  is the digamma function. In the limit  $\beta \rightarrow \infty$  this equation leads to the solution

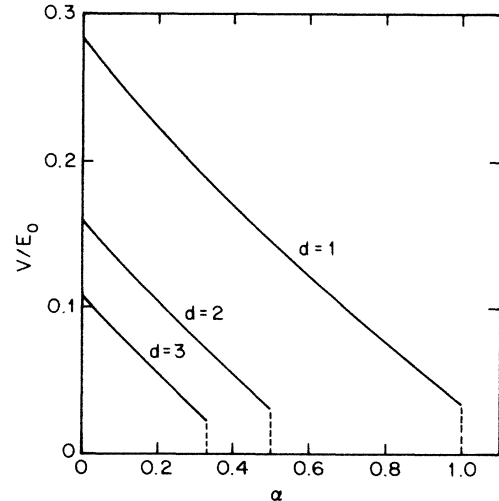


FIG. 1. The zero-temperature phase diagram.

$$D = E_0 \left[ \frac{2\pi}{\alpha} \right]^{1/(z_0\alpha-1)} \left[ \frac{V}{E_0} \right]^{z_0\alpha/(z_0\alpha-1)}, \quad \alpha > 1/z_0$$

$$D \equiv 0, \quad \alpha \leq 1/z_0. \quad (15)$$

Thus the transition takes place at  $\alpha = 1/z_0$  in  $d$  dimensions. As noted earlier, the capacitance enters through  $E_0$  which merely sets the energy scale, but the critical parameter  $\alpha$  depends only on the resistance. It is important to realize that this result is critically dependent on the fact that the heat bath we have considered is Ohmic [cf. Eq. (3)]. At low temperatures it is not difficult to show that the transition line between the normal and the superconducting phase in the  $\alpha$ - $T$  plane is given by

$$\frac{kT^*}{E_0} = \left[ \frac{\alpha x_0}{V/E_0} \left[ \frac{1}{2\pi} \right]^{1/z_0\alpha} \times \exp \left[ -\frac{1}{z_0\alpha} \psi(x_0) \right] \right]^{z_0\alpha/(1-z_0\alpha)}, \quad (16)$$

where  $T^*$  denotes the transition temperature and  $x_0$  is the solution of the equation

$$x_0 \psi'(x_0) = z_0 \alpha. \quad (16a)$$

The solid lines in Fig. 1, where  $D$  jumps discontinuously to zero, were obtained numerically. The calculations were simplified by approximating  $[z - \Delta(\mathbf{k})]$  by

$$z - \Delta(\mathbf{k}) \approx (\mathbf{k}a)^2, \quad (17)$$

where  $a$  is the lattice constant of the hypercubic lattice. Furthermore, the actual Brillouin zone was replaced by a spherical one whose radius  $k_0$  is such that in  $d$  dimensions

$$\frac{\Omega_d}{(2\pi)^d} \frac{(ak_0)^d}{d} = 1. \quad (18)$$

The  $d$ -dimensional solid angle  $\Omega_d$  is given by

$$\Omega_d = \frac{d\pi^{d/2}}{\Gamma(d/2+1)}. \quad (19)$$

The exact locations of the discontinuous transition lines, but not their existence, is quite model dependent; in particular, they depend on the precise form of the spin-wave spectrum for all  $k$  and on the precise shape of the Brillouin zone.

At zero temperature, the phase boundaries for small  $\alpha$  can be obtained analytically. Let  $\bar{V}_d^*(\alpha)$  be the minimal value of  $V/E_0$ , in dimension  $d$ , necessary for the existence of global phase coherence. Then it is not difficult to obtain the following results:

$$\bar{V}_1^*(\alpha) = \left[ \frac{\pi e}{16} \right]^2 \left[ 1 - \frac{32\alpha}{3\pi^2} \right] + O(\alpha^2), \quad (20)$$

$$\bar{V}_2^*(\alpha) = \left[ \frac{\sqrt{\pi} e}{12} \right]^2 \left[ 1 - \frac{18\alpha}{3\pi^2} \right] + O(\alpha^2), \quad (21)$$

$$\bar{V}_3^*(\alpha) = \left[ \frac{(6\pi^2)^{1/3} e}{32} \right]^2 \left[ 1 - \frac{128\alpha}{5\pi^2} \right] + O(\alpha^2). \quad (22)$$

For  $\alpha = 1/d - \epsilon$  ( $\epsilon \rightarrow 0$ ), a finite  $\bar{V}_d^*(\alpha)$  is required for superconductivity. For  $\alpha = 1/d + \epsilon$  ( $\epsilon \rightarrow 0$ ), the Josephson coupling energy necessary to achieve superconductivity can be arbitrarily small. The minimal  $\bar{V}_d^*(\alpha)$ 's are

$$[\bar{V}_1^*(1)]_m = \frac{1}{4e^2}, \quad (23)$$

$$[\bar{V}_2^*(\frac{1}{2})]_m = \frac{1}{4\pi e}, \quad (24)$$

$$[\bar{V}_3^*(\frac{1}{3})]_m = (6\pi^2/e)^{2/3}/36\pi^2. \quad (25)$$

Let  $\bar{D}_d^*(\alpha=0)$  be the jump in  $D/E_0$  at the transition points obtained from Eqs. (20)–(22) by setting  $\alpha=0$ . Then

$$\bar{D}_1^*(\alpha=0) = \left[ \frac{\pi}{16} \right]^2, \quad (26)$$

$$\bar{D}_2^*(\alpha=0) = \left[ \frac{\sqrt{\pi}}{12} \right]^2, \quad (27)$$

$$\bar{D}_3^*(\alpha=0) = \left[ \frac{(6\pi^2)^{1/3}}{32} \right]^2. \quad (28)$$

Some analytical results can also be obtained for finite temperatures. Going back to Eq. (16a) let us look at the behavior of  $x_0$  near  $\alpha = 1/z_0$ . Using the asymptotic expansion of  $\psi'(x)$  we immediately get

$$x_0 = \frac{1}{2(z_0\alpha-1)}, \quad \alpha - \frac{1}{z_0} \ll 1. \quad (29)$$

This leads using Eq. (14) to a finite jump  $D_0(\alpha)$  of  $D$  on the transition line:

$$D_0(\alpha) = \alpha x_0 / \beta^*, \quad (30)$$

where  $\beta^*$  is the inverse critical temperature given by Eq. (16). Therefore, it is only in the case of zero temperature that one has a smooth transition for  $D$  between the two phases. While discussing Eq. (16) one always has to keep in mind that  $E_0$  is the largest energy scale, assuring us that  $T^* \rightarrow 0$  as  $\alpha \rightarrow 1/z_0$ . In the vicinity of  $\alpha = 1/z_0$  we find that

$$\frac{kT^*}{E_0} \approx \frac{2z_0 V}{e^2 E_0} (z_0\alpha-1) \exp \left[ \frac{\ln(2\pi d V/E_0)}{z_0\alpha-1} \right]. \quad (31)$$

For large  $\alpha$  it is easy to show from Eqs. (14) and (16) that

$$\frac{z_0 T^*}{E_0} = \frac{z_0}{e} \left[ \frac{V}{E_0} \right]. \quad (32)$$

As promised earlier we now return to a discussion of the free energy. Equation (9) can be written more explicitly as

$$\begin{aligned} \frac{\Delta F}{N} = & -z_0 D + \int_0^D d\mu \frac{1}{N\beta} \sum_{\mathbf{k}} \sum_{n=0}^{\infty} \frac{z - \Delta(\mathbf{k})}{\omega_n^2/E_0 + [z - \Delta(\mathbf{k})](\mu + \alpha\omega_n/2\pi)} \\ & - \frac{D}{N\beta} \sum_{\mathbf{k}} \sum_{n=0}^{\infty} \frac{z - \Delta(\mathbf{k})}{\omega_n^2/E_0 + [z - \Delta(\mathbf{k})](D + \alpha\omega_n/2\pi)}. \end{aligned} \quad (33)$$

In this expression the cutoff on the Matsubara frequencies can be set to infinity with impunity. We shall examine this expression in the  $T=0$  limit. As before, to simplify our calculations, we assume a "spherical" Brillouin zone. We can now write

$$\begin{aligned} \frac{\Delta F}{N} = & -z_0 D + \frac{D\Omega_d}{(2\pi)^{d+1}} \left[ \frac{1}{D} \int_0^{ak_0} y^{d-1} dy \int_0^\infty d\omega \ln \left[ 1 + \frac{Dy^2}{\omega^2/E_0 + \alpha\omega y^2/2\pi} \right] \right. \\ & \left. - \int_0^{ak_0} y^{d+1} dy \int_0^\infty d\omega \frac{1}{\omega^2/E_0 + y^2(D + \alpha\omega/2\pi)} \right], \end{aligned} \quad (34)$$

where  $k_0$  is the radius of the spherical Brillouin zone and  $a$  the lattice constant. On making use of the self-consistent equation, we can also write it as

$$\frac{\Delta F}{N} = -z_0 D + z_0 D \ln \left[ \frac{D}{V} \right] + \frac{\Omega_d}{(2\pi)^{d+1}} \int_0^{ak_0} dy y^{d-1} \int_0^\infty d\omega \ln \left[ 1 + \frac{Dy^2}{\omega^2/E_0 + (\alpha\omega/2\pi)y^2} \right]. \quad (35)$$

In the limit  $V/E_0 \ll 1$ , i.e., for the vertical parts of the phase boundaries shown in Fig. 1,  $\Delta F/N$  simplifies and we find

$$\frac{\Delta F}{N} = -\frac{D}{\alpha}(\alpha z_0 - 1) + O \left[ \frac{D}{E_0} \right]. \quad (36)$$

Thus, in this case the nontrivial solution of the self-consistent equation also corresponds to lower free energy. Recall that the nontrivial solution appears for  $\alpha > 1/z_0$ .

Using numerical methods, one can see from Eqs. (35) and (8) that along the solid lines in Fig. 1, the nontrivial solution of Eq. (8) does not correspond to a lower free energy. The nontrivial solution corresponds to a lower free energy for somewhat higher value of  $V/E_0$ . Thus, the emergence of a nontrivial solution corresponds to a spinodal line in  $F'$ . For  $\alpha=0$ , this can be checked analytically. Let  $(V/E_0)_c$  be the actual transition point, i.e., the value of  $V/E_0$  for which the nontrivial solution of the self-consistent equation assumes a lower free energy. Then we find that

$$(V/E_0)_c = \begin{cases} \left[ \frac{\pi}{8} \right]^2 e, & \text{one dimension} \end{cases} \quad (37)$$

$$(V/E_0)_c = \begin{cases} \left[ \frac{\sqrt{\pi}}{6} \right]^2 e, & \text{two dimensions} \end{cases} \quad (38)$$

$$(V/E_0)_c = \begin{cases} \left[ \frac{(6\pi^2)^{1/3}}{16} \right]^2 e, & \text{three dimensions} \end{cases}. \quad (39)$$

Compare these results with those obtained from Eqs. (20)–(22) with  $\alpha$  set equal to 0. In all cases,  $(V/E_0)_c$  is larger by a factor of  $4/e$ . In contrast, it is not difficult to see that as  $\alpha$  approaches  $1/d$  the difference between  $\bar{V}^*(\alpha)$  and  $(V/E_0)_c$  vanishes.

We conclude with a few remarks concerning the variational phase diagram we have just discussed. Firstly, we

want to stress that a discontinuous jump in  $D$  does not necessarily signify a first-order transition. In  $d > 2$  it is likely that this jump is an artifact of our variational treatment of the problem. For  $d=2$  and at finite temperature we expect the transition to be of the classical  $XY$  variety. Thus, a discontinuous jump in the spin-wave stiffness constant is likely to occur, as in the Kosterlitz-Thouless<sup>16</sup> theory. At  $T=0$  we see that there is a tricritical point which separates the line of discontinuous transitions from the line of continuous transitions. This tricritical point occurs as a kink in the phase boundary. We shall argue below that there is another phase boundary which terminates at this point which is missed by our variational approach since it separates two phases in both of which  $D \neq 0$  (superconducting). One final remark concerns the position of the critical value of  $\alpha_c$ .  $\alpha_c$  is analogous to a critical temperature, and hence we do not expect it to be universal. As shown in Fig. 1,  $\alpha_c = 1/d$  for a square lattice.

#### IV. LIMITING REGIONS OF THE PHASE DIAGRAM

In this section we examine various limiting regions of the phase diagram where we can perform systematic calculations making use of one or another small parameter.

##### A. $\alpha \ll 1$ : Weak-dissipation limit

For  $\alpha=0$ , the system can be mapped onto a  $d$ -dimensional quantum  $XY$  model.<sup>12</sup> Thus the  $T=0$  transition belongs to the same universality class as the  $(d+1)$ -dimensional classical  $XY$  model, while the finite-temperature transition is characteristic of the  $d$ -dimensional classical  $XY$  model. This is because the system has a finite extent in the timelike dimension and therefore sufficiently close to the transition the quantum fluctuations may renormalize the parameters but will not modify the universality class of the corresponding classical system. For  $\alpha > 0$  the dissipation is an additional

source of quantum fluctuations, however at finite temperatures the universality class remains that of the  $d$ -dimensional  $XY$  model and the dissipation merely changes the quantum renormalization of the parameters. Thus we expect the  $\alpha \ll 1$  system to behave similarly to the  $\alpha=0$  model at finite temperatures. We would like to demonstrate that the additional renormalization of the parameters due to the dissipative quantum fluctuations can be computed to low order in  $\alpha$ .

To begin with, we consider expanding the partition function  $Z(V, E_0, \beta, \alpha)$  as a function of  $\alpha$ . Ultimately we would like to determine, when, if ever, such an expansion is convergent. We speculate that it is convergent in the disordered phase of the  $XY$  model but we have not succeeded in establishing this. Thus we will content ourselves with the observation that except at the critical points of the  $\alpha=0$  model there exists an asymptotic expansion in  $\alpha$ . While this does not prove that the  $\alpha=0$  phases continue to finite  $\alpha$  (as they do in the variational treatment), it gives evidence in support of this hypothesis. To see this, imagine computing the ratio  $Z(V, E_0, \beta, \alpha)/Z(V, E_0, \beta, 0)$  as a power series in  $\alpha$  (actually, we expect the logarithm of this quantity, i.e., the free energy to be better behaved so we exponentiate the series in a sort of linked cluster expansion):

$$Z(V, E_0, \beta, \alpha)/Z(V, E_0, \beta, 0) = \exp \left[ -\beta N \sum_{n=1}^{\infty} \alpha^n C_n \right]. \quad (40)$$

Here  $C_n$  is a  $2n$  phase correlation function of the  $\alpha=0$  model. For example, differentiating  $Z$  once with respect to  $\alpha$  we obtain

$$\begin{aligned} C_1 &= \frac{1}{2\pi} \frac{1}{\beta N} \sum_{\langle i,j \rangle} \int_0^\beta \int_0^\beta \frac{d\tau d\tau'}{(\tau - \tau')^2} \\ &\quad \times \langle [\Delta\theta_{ij}(\tau) - \Delta\theta_{ij}(\tau')]^2 \rangle \\ &= \frac{1}{\pi N} \sum_n \sum_{\langle i,j \rangle} \langle |\Delta\theta_{ij}(n)|^2 \rangle |\omega_n|, \end{aligned} \quad (41)$$

where the average should be taken with respect to the  $\alpha=0$  Hamiltonian. At finite temperatures  $C_1$  is finite, yielding the leading perturbative term for small  $\alpha$ .

$$\begin{aligned} \delta S[\{\phi_j\}] &= \int_0^{\beta\hbar} d\tau \left[ \frac{1}{2} \sum_j \dot{\phi}_j^2 / E_0 + V \sum_{\langle i,j \rangle} (\cos(\theta_i - \theta_j) \{1 - \cos[\phi_i(\tau) - \phi_j(\tau)]\} + \sin(\theta_i - \theta_j) \sin[\phi_i(\tau) - \phi_j(\tau)]) \right] \\ &\quad + \frac{\alpha}{4\pi} \sum_{\omega \neq 0} |\phi_i(\omega_n) - \phi_j(\omega_n)|^2 |\omega_n|. \end{aligned} \quad (44)$$

To make progress, integrate out  $\phi$  to obtain a renormalized effective action  $S^{\text{eff}}$  in terms of  $\{\theta_j\}$  alone. This can be done as a systematic expansion in the magnitude of the fluctuations by expanding  $\delta S$  in powers of  $\phi$ . This in turn produces an asymptotic expansion for  $S^{\text{eff}}$  in powers of  $\alpha^{-1}$ . By carrying out this procedure to second order, we obtain an expression for the effective action which is of the same form as  $S_0$  but with  $V$  replaced by

Given that the nature of the phase transition at small  $\alpha$  is the same as that for  $\alpha=0$ , one is tempted to try to compute the renormalized values of the parameters  $E_0(\alpha)$  and  $V(\alpha)$  such that the behavior of the  $\alpha \neq 0$  model with the original values of the parameters is the same as those of the  $\alpha=0$  model with renormalized values of the coupling constants. Away from the critical point this cannot be done in principle, since the form of the action is different in the two models. Near the critical point this procedure should work since both models are governed by the same fixed point Hamiltonian. Unfortunately, the model is too complicated for us to carry out this procedure. Nonetheless it follows from the nature of the small- $\alpha$  expansion that the renormalized coupling constants can be expanded in powers of  $\alpha$  at finite temperatures. Thus  $T_c = T_c(E_0, V, \alpha)$  should vary linearly with  $\alpha$  for small  $\alpha$ , as it does in the variational treatment as well.

At zero temperature the situation is more complex. Even the smallest amount of dissipation introduces a long-range interaction, proportional to  $(\tau - \tau')^{-2}$  along the temporal direction. According to a general result,<sup>17</sup> the correlation function of a system cannot fall off faster than the interaction itself. In particular, the spin-spin correlation function  $\langle e^{i\phi(\tau)} e^{i\phi(\tau')} \rangle$  cannot, in this case, decay faster than  $(\tau - \tau')^{-2}$ . It is therefore plausible to conclude that even for an arbitrarily small amount of dissipation the universality class of the transition is changed at zero temperature, i.e., it cannot belong to the same universality class as the  $(d+1)$ -dimensional  $XY$  model.

### B. $\alpha \gg 1$ : The classical $XY$ model

For  $\alpha \gg 1$  the system behaves as if it were classical. This is a consequence of the fact that for large enough  $\alpha$ , any time dependence of the phase produces a large action. To formalize this, we express

$$\theta_j(\tau) = \theta_j + \phi_j(\tau), \quad (42)$$

where the  $\theta_j$  is the average value of  $\theta_j(\tau)$ , and  $\phi$  is the fluctuating piece. We now express the action [see Eq. (4)] as the sum of a  $\phi$ -independent piece,

$$S_0 = \beta V \sum_{\langle i,j \rangle} [1 - \cos(\theta_i - \theta_j)], \quad (43)$$

and a fluctuating piece  $\delta S$ ,

$$V^{\text{eff}} = V \left[ 1 - \frac{X}{2} \right], \quad (45)$$

where

$$X = \langle [\phi_i(\tau) - \phi_j(\tau)]^2 \rangle \quad (46)$$

is given by the same expression as in Eq. (11), and  $\Delta(\mathbf{k})$  is

the structure factor defined in Sec. III.  $X$  is the same mean-squared phase fluctuation which was computed in Sec. III. For large  $\alpha$ , and  $\Delta(\mathbf{k})$  appropriate to a  $d$ -dimensional square lattice

$$X = \frac{1}{\alpha d} \ln \left| 1 + \left[ \frac{\alpha}{2\pi} \right] \right|. \quad (47)$$

Terms which come from expanding  $\delta S$  to higher order in  $\phi$  give rise to terms in  $S^{\text{eff}}$  which are higher order in  $\alpha^{-1}$ . These terms are not all of the same form as the terms in  $S_0$ . For instance, to second order in  $\alpha^{-1}$ , terms are generated coupling the spins on second-neighbor grains, as well as four spin couplings. For large enough  $\alpha$ , these terms can be ignored.

We conclude that for sufficiently large  $\alpha$ , the model behaves as if it were a classical  $XY$  model at the physical temperature, with an effective coupling constant  $V^{\text{eff}}$  which is somewhat reduced from its bare value by quantum fluctuations. Among the consequences of this are the fact that there are no phase transitions as a function of  $V/E_0$  so long as  $\alpha$  and  $\beta$  are sufficiently large. Thus, this aspect of the variational analysis is reconfirmed.

### C. $V/E_0 \gg 1$ : Dilute gas of phase slips

For  $V/E_0 \gg 1$  and  $k_B T \ll V$ , the zero-point fluctuations of the relative phase across a given junction are large compared to  $2\pi$ . Because  $V/E_0$  is large, the quantum zero-point fluctuations of the phase are large, and because  $k_B T$  is small compared to  $V$ , the concentration of vortices is low, and such vortices exist only as tightly bound vortex-antivortex pairs. Thus, the paths which make the largest contributions to the partition function have the property that most of the time and in most of the sample  $\Delta\theta_{ij} \equiv \theta_i - \theta_j$  is nearly equal to  $2n_{ij}\pi$  where  $n_{ij}$  is an integer and  $i$  and  $j$  are nearest-neighbor grains. Thus, in this limit two sorts of fluctuations about the classical ground state are important: small-amplitude spin waves and a dilute gas of phase-slip events in which the phase on a single grain changes by  $2\pi$ . The spin waves are innocuous in that they cannot drive a phase transition. We are left with the problem of computing the statistical mechanics of the gas of phase-slip events. A phase-slip event is specified by a grain number  $k$ , which labels the grain whose phase changes by  $2\pi$  (all other grains have their phase long before and long after the event unchanged) and an (imaginary) time  $T$  at which the event occurs. The phase-slip requires a characteristic tunneling time  $\tau_0$ ; however, as we shall see, the action associated with an isolated phase-slip diverges logarithmically at low temperature  $S^{\text{slip}} \sim 4\alpha^{\text{eff}} \ln(\beta\hbar/\tau_0)$ . Thus, even though the gas of phase slips is dilute, it does not form a noninteracting gas.

The action associated with a phase slip, and the interaction between phase-slip events can be readily estimated using the semiclassical approximation. To do this, we compute the solution to the classical equation of motion  $\delta S = 0$ , or

$$\ddot{\theta}_j = V \sum_i^{(j)} \sin(\theta_i - \theta_j) + \frac{\alpha}{4\pi^2} \int d\tau' \sum_i^{(j)} \left[ \frac{\Delta\theta_{ij}(\tau) - \Delta\theta_{ij}(\tau')}{(\tau - \tau')} \right] \quad (48)$$

subject to the boundary condition such that for  $\tau \gg \tau_0$

$$\begin{aligned} \theta_i(T + \tau) &= \theta_i(T - \tau) \quad \text{for } i \neq k, \\ &= \theta_i(T - \tau) \pm 2\pi \quad \text{for } i = k, \end{aligned} \quad (49)$$

and where the  $\sum^{(j)}$  is over the nearest-neighbor grains to grain  $j$ . Far from the event in space ( $|R_i - R_j| \gg a$ , where  $a$  is the lattice constant) or in time ( $|\tau - T| \gg \tau_0$ ) the phase differences across each junction are small (modulo  $2\pi$ ) and so the equations of motion can be linearized and the resulting equations can be solved for the asymptotic behavior of the classical path. From this solution, it is possible to determine the nature of the interactions between remote phase slips. The actual integrals involved are somewhat complicated. However, the interactions between phase slips on further than nearest-neighbor grains turn out to be short ranged in time and rather weak and so can safely be ignored. To see this consider the contribution to the interaction between two events at times  $T_1$  and  $T_2$  which comes from the fluctuations across a given junction  $\langle ij \rangle$ . We consider only the piece of the interaction which arises from the dissipative term in the action, since the interactions arising from the other terms are manifestly short ranged in time. Thus we suppose that

$$\Delta\theta_{ij}(\tau) = f_1(\tau - T_1) + f_2(\tau - T_2), \quad (50)$$

where  $f_j$  is the contribution to  $\Delta\theta_{ij}$  from event  $j$ . Thus,  $f_j(\tau)$  is zero except in a time interval of width of order  $\tau_0$  about  $\tau = 0$ . For  $|T_1 - T_2| \equiv T \gg \tau_0$ , it is easy to see that the interaction is independent of the detailed form of  $f_j$ . If neither grain  $i$  nor  $j$  is the grain on which the phase slip occurs in either event, then  $f_j(\tau) \rightarrow 0$  as  $|\tau|/\tau_0 \rightarrow \infty$  and

$$S^{\text{int}}(T) = -\frac{\alpha}{2\pi^2} \bar{f}_1 \bar{f}_2 \left[ \frac{\tau_0}{T} \right]^2 + O(\tau_0/T)^3, \quad (51)$$

where

$$\bar{f}_j = \int_{-\infty}^{+\infty} \frac{d\tau}{\tau_0} \dot{f}_j(\tau). \quad (52)$$

If one of the events, which we choose arbitrarily to be event 1, involves grain  $i$  or  $j$ , then  $f_1(\tau) \rightarrow F_1 \neq 0$  as  $\tau/\tau_0 \rightarrow +\infty$ , and hence

$$S^{\text{int}}(T) = -\frac{\alpha}{2\pi^2} \bar{f}_2 F_1 \left[ \frac{\tau_0}{T} \right] + O(\tau_0/T)^2. \quad (53)$$

Finally, if both phase slips occur on one of the two relevant grains,

$$S^{\text{int}}(T) = -\frac{\alpha}{2\pi^2} F_1 F_2 \ln(T/\tau_0) + O(\tau_0/T). \quad (54)$$



Thus there are long-ranged (logarithmic)—in time—interactions between phase slips on the same or nearest-neighbor grains, and short-ranged interactions between phase slips on further grains. Moreover, for large  $V$ , we expect that the disturbances in the phase on remote grains will be small ( $\bar{f}_j \ll 2\pi$  for  $|R_j - R_k| \gg a$ ) so the magnitude of the interaction between spatially well separated phase slips will also be small.

In light of the long-ranged nature of the interaction in the time direction, and its short-ranged character in the spatial direction, it is likely that any phase transition which occurs as a function of  $\alpha$  in this regime of parameters will be essentially one dimensional in character. Thus, we consider a simplified model in which we consider only phase slips on a single grain. We expect the interaction with phase slips on other grains will not have a large effect on the nature of the transition. Moreover, the resulting analysis is similar to that which goes into establishing bounds on the critical value of  $\alpha$  in Sec. V.

Since the simplified model is now truly one dimensional, we can immediately conclude that no transition can occur except at zero temperature (at finite temperature the “size” of the one-dimensional system,  $\beta\hbar$ , is finite). The zero-temperature transition is exactly analogous to the transition in a single Josephson junction that was studied by Schmid<sup>3</sup> and others.<sup>4</sup> The only difference is that the effective coupling to the heat bath is  $\alpha^{\text{eff}} = z\alpha$  as opposed to  $\alpha$  in the single-junction case, and the effective Josephson coupling  $V^{\text{eff}}$  and the effective charging energy  $E_0^{\text{eff}}$  are similarly renormalized due to the presence of the neighboring grains. (The exact nature of the renormalization depends on the solution of the classical equations of motion, but for large  $V$  we expect  $V^{\text{eff}} < zV$  and  $E_0^{\text{eff}} < E_0$ .) For large  $V^{\text{eff}}/E_0^{\text{eff}}$ , Schmid<sup>3</sup> showed that the phase transition occurs as a function of  $\alpha^{\text{eff}}$  alone. For  $\alpha^{\text{eff}} < 1$ , the phase fluctuations are unbounded, while for  $\alpha^{\text{eff}} > 1$ , phase slips occur only in flip-antiflip pairs and hence the phase is “localized.” Thus, the critical value of  $\alpha$  is  $1/z$ .

The presence of a zero-temperature phase transition at large  $V/E_0$  is inconsistent with the variational phase diagram as we mentioned in Sec. III. As we suggested there, we suspect that this is due to the fact that both the free phase slip and the localized phase-slip phases have a nonzero spin-wave-stiffness constant in this regime since there is zero density of free vortices. However, we have not yet proven that this is the case.

#### D. $V/E_0 \ll 1$ : Infrared catastrophe and the quantum-to-classical transition

Finally, we consider what is probably the most novel segment of the phase diagram, where  $V/E_0 \ll 1$  and  $k_B T$  is very small. (We will make this second criterion precise below.) In this regime, we can treat the problem using a perturbative renormalization group scheme, where we compute the renormalization-group flows in powers of

the nonlinear coupling constant  $V$ . Our treatment is analogous to the scaling analysis of the sine-Gordon model<sup>18</sup> and is very similar to the treatment of the single-junction problem of Schmid<sup>3</sup> and others.<sup>4</sup> The renormalization-group transformation involves a systematic coarse graining in the time direction. We begin by reviewing the philosophy of the perturbative renormalization-group treatment as it applies to this problem.

The first step is to separate the fluctuating phases into a fast ( $\theta_j^f$ ) and slow ( $\theta_j^s$ ) piece:

$$\theta_j^s(\tau) = \sum_{\substack{|\omega_n| \\ |\omega_n| < \omega_c}} \tilde{\theta}_j(\omega_n) e^{i\omega_n \tau}, \quad (55a)$$

$$\theta_j^f(\tau) = \sum_{\substack{\omega_n \\ |\omega_n| \geq \omega_c}} \tilde{\theta}_j(\omega_n) e^{i\omega_n \tau}, \quad (55b)$$

where  $\tilde{\theta}_j$  is the Fourier transform of  $\theta$  and  $\omega_c$  is the cutoff frequency which separates the slow from the fast components of  $\theta_j$ . (There are some pathologies associated with the sharp cutoff employed above. These can be avoided by use of a smooth cutoff function, but this is merely a technical complication, and easily handled.) Next we integrate out the fast components to obtain an expression for an effective action which depends on the slow components alone. Since we cannot do this exactly, we do it perturbatively to low order in  $V/E_0$ . Finally, we bring this effective action into the original form, by expressing the time integral in terms of a rescaled unit of time chosen so that  $\omega_c = 1$ . We then repeat this procedure by defining a new cutoff frequency  $\omega'_c < \omega_c$ , integrating out the modes with frequency between  $\omega'_c$  and  $\omega_c$ , and finally rescaling our unit of time so that  $\omega'_c = 1$ . If we make the difference  $\omega_c - \omega'_c \ll \omega_c$ , we generate in this way a differential equation for the coupling constants in the effective action as a function of  $\omega_c$ . So long as the nonlinear coupling constant  $V$  does not flow to large values, the approach is reliable. However, when  $V/E_0$  gets to be of order one, the perturbative expressions for the renormalization-group flows cease to be valid.

Here, we carry out this procedure only to first order in  $V/E_0$ . This turns out to be particularly simple, since no new couplings are generated to this order in perturbation theory; in other words the effective action is of the same form as the bare action.

The kinetic energy and the dissipative terms in the action are quadratic in  $\theta_j$  and so separate exactly into a sum of terms that depend only on the fast components and a term which depends only on the slow components. Thus, we can define an unperturbed action  $S_0$  obtained by setting  $V=0$ , and can formally integrate out the fast modes to obtain an expression for the partition function in terms of the slow modes alone

$$Z = \mathcal{N} \int \mathcal{D}\theta^s \exp[-S_0(\theta^s)/\hbar] \left\langle \exp \left[ \frac{V}{\hbar} \sum_{\langle i,j \rangle} \int_0^{\beta\hbar} d\tau \cos[\Delta\theta_{ij}^s(\tau) + \Delta\theta_{ij}^f(\tau)] \right] \right\rangle_f, \quad (56)$$

where the angular brackets represent the thermal average over the fast modes with respect to the unperturbed action  $S_0[\theta']$ . This average cannot, in general, be evaluated exactly. However, we can expand the exponential in a cumulant expansion in powers of  $V$  and evaluate the resulting expressions term by term. Since the averages are taken with respect to a quadratic action, there is no reason, other than the (quite severe) limits on our algebraic fortitude, that this cannot be done to arbitrary order in  $V$ . Here, we will carry out the procedure only to first order in  $V$ . The result is

$$\left\langle \exp \left[ \frac{V}{\hbar} \sum_{(i,j)} \int_0^{\beta\hbar} d\tau \cos[\Delta\theta_{ij}^s(\tau) + \Delta\theta_{ij}^f(\tau)] \right] \right\rangle_f = \exp \left[ \frac{V}{\hbar} \sum_{(i,j)} \int_0^{\beta\hbar} d\tau \cos[\theta_i^s(\tau) - \theta_j^s(\tau)] e^{-X_{ij}/2} \right], \quad (57)$$

where

$$X_{ij} = \langle [\theta_i^f(\tau) - \theta_j^f(\tau)]^2 \rangle_f \quad (58)$$

$$= \frac{1}{Nz} \sum_{\omega_n, k_n} \frac{\Delta(\mathbf{k})}{\omega_n^2/E_0 + (\alpha/2\pi) |\omega_n| \Delta(\mathbf{k})} \quad (59)$$

$\omega'_c \leq |\omega_n| < \omega_c$

is the mean-squared fast variation in the phase difference across a Josephson junction. The calculation of  $X$  is made simpler by the following observation: As we saw previously in the quantum regime which we are studying here, where  $E_0$  is large, the kinetic-energy term solely serves to produce a high-frequency cutoff into the sum over frequencies  $\omega_n$ . Since we are varying an explicit high-frequency cutoff in our renormalization procedure, we can drop the kinetic-energy term entirely. Thus, we can ignore the  $\omega_n^2$  term in the denominator of Eq. (5). The  $\mathbf{k}$  sum can then be done trivially. For  $\omega_c - \omega'_c \equiv d\omega_c \ll \omega_c$ , it follows that  $X_{ij} = (1/\alpha)d\omega_c/\omega_c$ . Finally, we rescale time, and the potential so that the resulting action is of the same form as it was before we integrated out the fast modes. Thus, we require that

$$V' d\tau' = V d\tau, \quad (60)$$

where  $d\tau'\omega'_c = d\tau\omega_c$ . The result is a differential equation for  $V$  as function of the cutoff:

$$d \ln V / d \ln \omega_c = \left[ 1 - \frac{1}{z_0 \alpha} \right] + O(V^2). \quad (61)$$

None of the other parameters renormalize to this order. It is immediately clear from this that for  $\alpha < 1/z_0$ ,  $V$  scales to zero as we systematically reduce the cutoff frequency as

$$V \sim V_0 (\omega_c / \omega_0)^{(1-z_0\alpha)/z_0\alpha}, \quad (62)$$

where  $V_0$  is the bare value of  $V$  and  $\omega_0$  is a microscopic cutoff frequency, typically  $\omega_0 \sim \alpha E_0$ . We conclude that for  $\alpha < 1/z_0$ , the phases are disordered. For  $\alpha > 1/z_0$  and  $T=0$ ,  $V$  scales to large values as  $\omega_c \rightarrow 0$ , and hence scales out of the regime in which our perturbative treatment is valid. We imagine that the system approaches a strong-coupling fixed point, in which there is long-ranged phase coherence. At finite temperature, we can estimate the temperature dependence of the phase boundary by the following simple argument: We expect that where  $V$  is of order one, the system will order. We imagine that up to  $V$  of order one, the renormalization-group flows do not differ dramatically from those deduced from lowest-order perturbation theory. Thus, we can follow Eq. (61) out to

$V=1$ . Since the lowest frequency in the problem is  $\omega_c = k_B T / \hbar$ , we can obtain an estimate of the critical temperature  $T_c$  by inverting Eq. (62) with  $\omega_c = k_B T_c / \hbar$ , and setting  $V=1$ . The result is

$$k_B T_c = \hbar \omega_0 (\hbar \omega_0 / V_0)^{-z_0 \alpha / (z_0 \alpha - 1)}. \quad (63)$$

Since in the extreme quantum regime considered here,  $\hbar \omega_0 \gg V_0$ , it follows that  $T_c$  is a rapidly increasing function of  $\alpha$  for  $\alpha > 1/z_0$ . This analysis clearly breaks down when  $k_B T \sim \hbar \omega_0$  when the size of the system in the time direction is comparable to the microscopic frequency.

In summary, we have confirmed the existence of a low-temperature phase transition as a function of  $\alpha$  for low temperatures and in the extreme quantum regime. The transition is caused by an overlap catastrophe due to the high density of low-frequency modes in the heat bath. The transition is formally similar to that which occurs in the Kondo problem, or in the single-junction problem. The dependence of the transition on dimensionality is rather trivial, in that it only affects the critical value of  $\alpha$ . For a non-Ohmic heat bath, with a lower density of low-frequency modes, the nature of the transition, if any, would be entirely different.

## V. BOUNDS ON THE ZERO-TEMPERATURE CLASSICAL-TO-QUANTUM PHASE BOUNDARY

By arguments similar to those used in Sec. IV D, it is possible to prove that at zero temperature and for any value of  $V/E_0$  there exists a critical value of  $\alpha$  at which the system undergoes a phase transition which we have characterized as a quantum-to-classical transition. We consider fluctuations in the phase difference across a given junction. Since the voltage drop across the junction is proportional to the time derivative of the phase difference across the junction, and hence, by Ohm's law, the current through the shunt resistor is as well, this quantity is related to the net dissipation in the junction. Thus it is a natural question to ask whether the phase fluctuations are bounded or unbounded. For the purposes of this discussion, it makes very little difference that the phase is only defined as modulo  $2\pi$ , and so, for convenience, we will treat it as an extended variable

which is also consistent with the Ohmic nature of the dissipative heat bath. Thus, we will characterize the quantum state as one in which the phase fluctuations are unbounded, that is a state which can be considered to be a gas of free phase slips. The classical state is one in which the phase fluctuations are bounded, that is one in which the phase slips are confined.

To obtain a lower bound on the critical value of  $\alpha$ , we consider a model in which we completely suppress the fluctuations on all grains other than grain 1. By doing this, we have systematically reduced the magnitude of all fluctuations. We are left with a model with a single fluctuating phase  $\theta_1$ . The resulting model is formally exactly equivalent to the problem of a single resistively shunted Josephson junction with Josephson coupling  $V^{\text{eff}} = zV$ , capacitance  $C$ , and effective coupling to the heat bath  $\alpha^{\text{eff}} = z\alpha$ . As mentioned previously, this model has been studied extensively by Schmid<sup>3</sup> and others.<sup>4</sup> It is known that for large and small  $V/E_0$  it has a localization transition at  $\alpha = 1$ , and it has been argued that this is the critical value of  $\alpha$  for all  $V/E_0$ . Thus, we can conclude that for  $\alpha < 1/z$  the fluctuations of the phase are certainly unbounded, since they are unbounded even if all other fluctuations are suppressed.

To obtain an upper bound on the critical value of  $\alpha$ , we consider another simplified model in which we set the coupling between grains 1 and 2 and all other grains equal to zero. Since the coupling to other grains tends to restrain the fluctuations on grains 1 and 2, we thus systematically overestimate the magnitude of the fluctuations. The resulting model can be expressed in terms of two decoupled degrees of freedom, the average phase  $\bar{\theta} = \frac{1}{2}(\theta_1 + \theta_2)$ , and the relative phase  $\Delta\theta = \theta_1 - \theta_2$ . The average phase obeys a free Schrödinger equation, but the relative phase is, again, equivalent to the phase across a single Josephson junction (which, indeed, it is) with the effective mutual capacitance  $C^{\text{eff}} = \frac{1}{2}C$ , the effective Josephson coupling, and coupling to the heat bath equal to the bare values. Since this model implies that the phase is localized (bounded fluctuations) for  $\alpha > 1$ , it follows that this must be an upper bound on the true critical value of  $\alpha$ .

We conclude that for any value of  $V/E_0$ , and at  $T = 0$ , there exists a critical value of  $\alpha$ ,  $\alpha_c(V/E_0)$ , such that for  $\alpha < \alpha_c$  the phase fluctuations are unbounded, while for  $\alpha > \alpha_c$  the phases across the junctions are bounded. Bounds on the  $\alpha_c(V/E_0)$  can be established from an analysis of the single-junction problem. If we take as true the assertion that in the single-junction problem  $\alpha_c(V/E_0) = 1$  for all  $V/E_0$ , it follows that in the array  $1/z \leq \alpha_c(V/E_0) \leq 1$  for all  $V/E_0$ .

## VI. CONCLUSIONS

We have studied the quantum statistical mechanics of an array of resistively shunted Josephson junctions. We have uncovered a rich phase diagram, and shown that, because of the quantum nature of the system, the dissipation plays a central role in the thermodynamics. While our results are quite similar to those observed experimen-

tally in ultrathin films of granular superconductors, it is important to stress that it is not definitively established that the dissipation in these materials is well represented by an Ohmic heat bath, as it would if the junctions were shunted with a macroscopic resistor as briefly mentioned in the Introduction. Thus, we do not expect that the theory we have developed above will account for the behavior of these granular films in any rigorous sense. Nonetheless, it seems likely that in these materials the superconducting transition shares many important features with the model we have studied. (1) The transition temperature is below the bulk transition temperature and hence the superconducting order parameter has a well-developed magnitude on each grain; the superconducting state results when global coherence is established between the phases of the order parameter on different grains. (2) This ordering is strongly renormalized by quantum fluctuations of the phase. (3) Dissipation plays an important role in suppressing quantum fluctuations and hence stabilizing the superconducting state. Thus, it is not completely accidental that the observed properties of the transition resemble those that we have derived here. Given that the granular films can be made which are truly in the quantum regime, it is to be hoped that in the near future, artificial arrays can be fabricated with built in shunt resistors, so that the predictions of the theory can be compared directly to experiment.

There remain a number of interesting open questions which warrant further study. Since in a quantum system, the dynamics and the thermodynamics are not separate it seems likely that the present model will exhibit interesting critical dynamics. For instance, it would be interesting to determine the conductivity of the system from the same Hamiltonian as the phase diagram was derived. One of the features that emerges from our model is the occurrence of a zero-temperature transition from a superconducting state which contains a finite concentration of phase slips, to one in which they are bound in slip-antislip pairs due to the dissipation. It is not clear what experimental signature of this transition would be accessible. Finally, there is a suggestion due to Jaeger *et al.*<sup>6</sup> for which there is increasingly good experimental evidence,<sup>7,8</sup> namely that the transition to a superconducting state in granular superconductors actually occurs at a universal value of the normal-state resistance corresponding to  $\alpha = 1$ , independent of material or sample history. Such a universal value cannot be explained in any straightforward way by any of the models considered to date; in all the models considered to date the critical value of  $\alpha$  plays a role analogous to the critical temperature in a classical statistical-mechanics model, and is hence not universal (although it may be relatively insensitive to microscopic details). If this observation is confirmed, it will require a deeper theoretical analysis than any that have been carried out to date.

## ACKNOWLEDGMENTS

We would like to acknowledge conversations with A. M. Goldman, B. I. Halperin, B. G. Orr, H. Jaeger, A.

Luther, A. Schmid, and G. Schön. This work was supported in part by the National Science Foundation through Grant Nos. DMR-86-01908 (S.C.) and DMR-83-18051 (S.K.), and PHY82-17853, supplemented by funds from the National Aeronautics and Space Adminis-

tration. S.K. would like to acknowledge support from Alfred P. Sloan Foundation. G.T.Z. would like to acknowledge support from IBM. G.I. would like to acknowledge support from the Studienstiftung des deutschen Volkes.

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\*Present address: Institut für Theoretische Physik, Universität Stuttgart, D-7000 Stuttgart 80, Federal Republic of Germany.

†Present address: Institute for Theoretical Physics, University of California at Santa Barbara, Santa Barbara, CA 93106.

‡On leave from Central Research Institute for Physics, H-1525 Budapest, Hungary.

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