Anderson localization and interactions in one-dimensional metals

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A one-dimensional interacting electron gas in a random potential exhibits a localized-delocalized transition for increasingly attractive interactions. We develop here a renormalization-group approach to study this transition. Our treatment allows us to obtain the phase diagram and the exponents of the correlation functions in the delocalized regime. The boundary between the two regimes is found to depend both on disorder and the strength of the interactions. For (nearly) spin-isotropic interactions the delocalized phase is dominated by superconducting fluctuations of either singlet or triplet type. The temperature dependence of the conductivity in the delocalized phase is also obtained and a nonuniversal power-law behavior is found. A description of the crossover towards the localized phase is given and the localization length is computed. An analogous description is developed for the localized-superfluid transition of a one-dimensional boson gas. In this case the transition to the localized regime occurs for increasingly repulsive interactions. We suggest a phase diagram with two different localized phases. Finally, we discuss some possible implications of our model for real quasi-one-dimensional metals.

I. INTRODUCTION

In one- or two-dimensional metals the presence of disorder entails a localization of all electronic states and, therefore, a vanishing static conductivity at zero temperature.[1,2] For one-dimensional systems this fact can be shown by exact calculations.[3–5] The existence of interactions among electrons changes this behavior qualitatively:[6–9] For sufficiently attractive interactions, there is a competition between superconducting fluctuations and disorder which can lead to delocalization. Previous methods[6,8,9] developed to study the localized-delocalized transition in one-dimensional systems are valid only in the limit of vanishing disorder, and generally do not take into account the renormalization of interactions by the disorder.

We will consider in this paper the influence of interactions on an impure one-dimensional system. Such a study can be useful to gain insight into the influence of disorder on real quasi-one-dimensional metals. For example, alloying experiments on compounds with tetramethyltetraselenafulvalene (TMTSF) or tetramethytetrathiafulvalene (TMTTF) chains, e.g., $(TMTTF)_2X$ and $(TMTSF)_2X$, have revealed interesting phenomena at low temperatures.[10,11] In intrinsically disordered compounds like quinoline-di(tetracyanoquinodimethane) $[Qn(TCNQ)_2]$ localization has been claimed to be important up to temperatures of the order of 200 K.[12] All these organic compounds are narrow-band systems (typical bandwidth $\simeq 1$ eV), and electron-electron interactions therefore are certainly important.

However, the present study has also a purely theoretical interest: As one-dimensional electron-electron interactions can be treated exactly, here we have a model system to study the not yet fully understood question of the competition between superconductivity and Anderson localization, and more generally of the interplay between localization and electron-electron interactions. Moreover, our treatment straightforwardly extends to the localized-superfluid transition in a one-dimensional interacting boson system. These questions are at present of considerable interest in two and three dimensions,[13,14] and a detailed investigation of the one-dimensional situation may therefore be helpful in understanding the higher-dimensional cases.

We have developed a renormalization-group method to treat both the disorder and the interactions and to allow for their mutual renormalization effects. A similar method was previously used only for the considerably simpler case of spinless fermions.[7] Our renormalization scheme fits perfectly into the previously developed frameworks of the interacting one-dimensional electron gas without disorder.[15] It allows a consistent description of the delocalized regime, as well as the zero-temperature transition to the localized phase and finite-temperature crossover effects.

The plan of the paper is as follows. In Sec. II the model of the one-dimensional electron gas with disorder is introduced. The effect of forward scattering by impurities is discussed and shown not to lead to localization. Section III is concerned with impurity backward scattering, which can lead to localization. Renormalization equations are derived and the zero-temperature phase diagram is obtained and discussed. In Sec. IV the renormalization equations are used to study the localization length, the influence of nonzero temperature on the conductivity in the delocalized phase, and the crossover to the localized phase. In Sec. V we discuss the analogy with a system of one-dimensional bosons undergoing a localized-superfluid transition. A comparison with some experimental results and a discussion of the limits of our treatment are in Sec. VI. Some mathematical details are to be found in the Appendices.

II. DEFINITION OF THE MODEL AND EFFECTS OF IMPURITY FORWARD SCATTERING

We use the standard "g-ology"[15] description of the one-dimensional interacting electron system: As the only important processes are those near the Fermi surface, the energy spectrum is linearized around k_F and the interactions are parametrized by the constants $g_{1\perp}$, $g_{1\parallel}$, $g_{2\perp}$, $g_{2\parallel}$ for processes with momentum transfer close to zero ($g_{2\perp}$,$g_{2\parallel}$) and $2k_F$ ($g_{1\perp}$,$g_{1\parallel}$). The complete Hamiltonian of the pure system is

$$H=\sum_{k,\sigma,r} v_F(rk-k_F)a^{\dagger}_{rk\sigma}a_{rk\sigma}+L^{-1}\sum_{k_1,k_2,p,\sigma,\sigma'}(g_{1\parallel}\delta_{\sigma,\sigma'}+g_{1\perp}\delta_{\sigma,-\sigma'})a^{\dagger}_{+,k_1\sigma}a^{\dagger}_{-k_2,\sigma'}a_{+,k_2+p,\sigma'}a_{-k_1-p,\sigma}$$
$$+L^{-1}\sum_{p,\sigma,\sigma'}(g_{2\parallel}\delta_{\sigma,\sigma'}+g_{2\perp}\delta_{\sigma,-\sigma'})\rho_{+,\sigma}(p)\rho_{-,\sigma'}(-p)\ , \tag{2.1}$$

where

$$\rho_{r\sigma}(p)=\sum_k a^{\dagger}_{r,k+p,\sigma}a_{r,k,\sigma}\ ,$$
$$\psi_{r\sigma}(x)=L^{-1/2}\sum_k a_{rk\sigma}e^{ikx}\ , \tag{2.2}$$

$r=\pm$ denotes right- and left-going fermions, $\sigma=\pm$ indicates spin up and down, and $a_{rk\sigma}$ ($a^{\dagger}_{rk\sigma}$) is the annihilation (creation) operator for a fermion in state (r,σ) with momentum k.

We now use the boson representation[16,17] of fermion operators, introduce charge- (ρ) and spin- (σ) density operators in the standard way,[15] and define the phase fields,

$$\phi_\nu(x),\theta_\nu(x)=\mp\frac{i\pi}{L}\sum_{p\neq 0}\frac{1}{p}e^{-\alpha|p|/2-ipx}[\nu_+(p)\pm\nu_-(p)]\ , \tag{2.3}$$

$\nu=\rho$ or σ .

In $A,B=\cdots$ the upper sign refers to A, and α is a short-range cutoff parameter of the order of the lattice constant. In terms of boson operators the Hamiltonian is expressed by

$$H=H_\rho+H_\sigma+\frac{2g_{1\perp}}{(2\pi\alpha)^2}\int dx\cos(\sqrt{8}\phi_\sigma)\ , \tag{2.4}$$

where H_ρ and H_σ are defined by

$$H_\nu=\frac{1}{2\pi}\int dx\left[(u_\nu K_\nu)(\pi\Pi_\nu)^2+\left[\frac{u_\nu}{K_\nu}\right](\partial_x\phi_\nu)^2\right]\ ,$$
$$u_\nu=\left[v_F^2-\frac{(g_\nu)^2}{4\pi^2}\right]^{1/2}\ ,$$
$$K_\nu=\left[\frac{2\pi v_F+g_\nu}{2\pi v_F-g_\nu}\right]^{1/2}\ , \tag{2.5}$$
$$g_\rho,g_\sigma=g_{1\parallel}-g_{2\parallel}\mp g_{2\perp}\ ,$$

and Π_ν is the momentum-density conjugate to ϕ_ν,

$$[\Pi_\nu(x),\phi_\mu(x')]=-i\delta_{\nu,\mu}\delta(x-x')\ . \tag{2.6}$$

As is well known, in one dimension an ordered ground state with a broken continuous symmetry cannot exist. However, the tendency of the system to order manifests itself in divergent correlations of $2k_F$ charge-density wave (CDW), spin-density wave (SDW), or singlet (SS) and triplet (TS) superconducting Cooper-pairing type. These fluctuations are described by the correlation functions:[10,15]

$$R_i(x,t)=-\langle[T_\tau O_i(x,\tau)O_i^{\dagger}(0,0)]\rangle\ , \tag{2.7}$$

where T_τ is the time-ordering operator for the imaginary time τ, and

$$O_{\rm CDW}(x,\tau)=\frac{e^{2ik_Fx}}{\pi\alpha}\exp[-i\sqrt{2}\phi_\rho(x,\tau)]$$
$$\times\cos[\sqrt{2}\phi_\sigma(x,\tau)]\ ,$$
$$O_{\rm SDW}(x,\tau)=\frac{e^{2ik_Fx}}{\pi\alpha}\exp[-i\sqrt{2}\phi_\rho(x,\tau)] \tag{2.8}$$
$$\times\cos[\sqrt{2}\theta_\sigma(x,\tau)]\ .$$

$O_{\rm SS}$ and $O_{\rm TS}$ are obtained with $k_F=0$ and replacing $\phi_\rho\to\theta_\rho$. For the triplet-type phases (SDW, TS) only one of the three possible spin orientations is exhibited.

The asymptotic behavior of the correlation functions is determined by the coefficient K_ν in (2.5). One has

$$R_i(x,\tau)\sim\max(x,u_\nu\tau)^{-2+\alpha_i}$$

with

$$\alpha_{\rm CDW}=2-K_\rho-K_\sigma^*,\quad \alpha_{\rm SDW}=2-K_\rho-1/K_\sigma^*\ ,$$
$$\alpha_{\rm SS}=2-1/K_\rho-K_\sigma^*,\quad \alpha_{\rm TS}=2-1/K_\rho-1/K_\sigma^*\ . \tag{2.9}$$

Here K_σ^* is the renormalized value of K_σ, containing all corrections due to the $g_{1\perp}$ interaction. For spin-independent interactions one has $K_\sigma^*=1$ ($g_{1\perp}>0$) or $K_\sigma^*=0$ ($g_{1\perp}<0$). One should note that the relation between the K_ν and the microscopic constants given in (2.5) depends on the specific cutoff procedure used in the boson representation[17] and is universal only to first order in g_1,g_2. On the other hand, the low-energy properties of any interacting one-dimensional fermion system are, in the absence of commensurability effects, described correctly by (2.9), with the parameters K_ν,u_ν functions of the microscopic coupling constants, and $K_\sigma^*=1$ (or 0) for spin-independent interactions.[15,18] Specifically, the relations between different exponents following from (2.9) are expected to be universally valid.

As pointed out by Abriksov and Ryzhkin,[5] in the limit of a weak impurity potential, the interaction between the electrons and the impurities can be parameterized by two uncorrelated Gaussian random fields η and ξ. These two fields, respectively, describe the forward and backward scattering by the impurities (see Fig. 1) and have the probability distributions

$$P_\eta = \exp\left[-D_\eta^{-1}\int \eta^2(x)dx\right],$$
$$P_\xi = \exp\left[-D_\xi^{-1}\int \xi^*(x)\xi(x)dx\right], \tag{2.10}$$

where $D_\nu = v_F/\tau_\nu$, and τ_ν is the scattering time associated with each process. The field η is real, whereas ξ is complex, ξ and ξ^* being associated with momentum transfer $2k_F$ and $-2k_F$, respectively. The interaction Hamiltonian with the impurities is given by

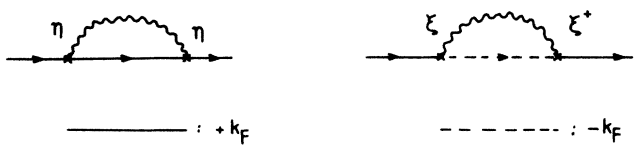

FIG. 1. Diagrams representing respectively, the forward and backward scattering by impurities. The solid (dashed) lines indicate fermions with momentum $+k_F$ ($-k_F$).

$$H_f = \sum_\sigma \int dx\, \eta(x)[\psi^\dagger_{+\sigma}(x)\psi_{+\sigma}(x)+\psi^\dagger_{-\sigma}(x)\psi_{-\sigma}(x)] = -\frac{\sqrt{2}}{\pi}\int dx\, \eta(x)\partial_x\phi_\rho ,$$
$$H_b = \sum_\sigma \int dx[\xi(x)\psi^\dagger_{+\sigma}(x)\psi_{-\sigma}(x)+\xi^*(x)\psi^\dagger_{-\sigma}(x)\psi_{+\sigma}(x)] \tag{2.11}$$
$$= \frac{1}{\pi\alpha}\int dx\{\xi(x)e^{i(\sqrt{2}\phi_\nu(x)+2k_Fx)}\cos[\sqrt{2}\phi_\sigma(x)]+\text{H.c.}\} .$$

The forward scattering due to impurities can thus be absorbed in the free part of the Hamiltonian which becomes

$$H'_\rho = \frac{1}{2\pi}\int dx\left[u_\rho K_\rho(\pi\Pi_\rho)^2+\left[\frac{u_\rho}{K_\rho}\right]\{\partial_x[\phi_\rho(x)+\tilde\eta(x)]\}^2\right]-\frac{K_\rho}{\pi u_\rho}\int dx\,\eta^2(x) ,$$
$$\tilde\eta(x) = -\frac{\sqrt{2}K_\rho}{u_\rho}\int^x dz\,\eta(z) . \tag{2.12}$$

We will only consider in the following the case of quenched disorder. In order to treat the backward scattering we use the replica trick,[19] and after averaging over the field ξ, the complete action of the replicated system is

$$S = \sum_i \int d\tau\left[L^{\prime i}_\rho + L^i_\sigma + \frac{2g_{1\perp}}{(2\pi\alpha)^2}\int dx\cos[\sqrt{8}\phi^i_\sigma(x,\tau)]\right]$$
$$-\frac{D_\xi}{(\pi\alpha)^2}\sum_{i,j}\int\int\int d\tau\, d\tau'\, dx\cos[\sqrt{2}\phi^i_\sigma(x,\tau)]\cos[\sqrt{2}\phi^j_\sigma(x,\tau')]\cos\{\sqrt{2}[\phi^i_\rho(x,\tau)-\phi^j_\rho(x,\tau')]\} , \tag{2.13}$$

where i,j are the replica indices, the L_i are the Lagrangians associated with the Hamiltonians H_i, and integration over τ,τ' is from 0 to $1/T$. The free energy F of the system at temperature T is given by

$$Z(n) = \int \prod_{i=1}^n \mathcal{D}\phi_i e^{-S(\phi_i)} ,$$
$$F(T) = -T\lim_{n\to 0}\frac{1}{n}[Z(n)-1] , \tag{2.14}$$

where n is the number of replicas. To eliminate the forward scattering we introduce the field $\tilde\phi = \phi_\rho(x)+\tilde\eta(x)$. The whole Hamiltonian can be expressed in terms of the new field since

$$\cos\{\sqrt{2}[\phi^i_\rho(x,\tau)-\phi^j_\rho(x,\tau')]\}$$
$$= \cos\{\sqrt{2}[\tilde\phi^{\,i}_\rho(x,\tau)-\tilde\phi^{\,j}_\rho(x,\tau')]\} . \tag{2.15}$$

The forward scattering due to impurities can thus be completely absorbed in the redefinition of the ϕ_ρ field.

The conductivity is given by the Kubo formula

$$\sigma(\omega) = \frac{1}{\omega}\int_0^{+\infty} dt\, e^{i\omega t}\int dx\langle[j(x,t)j(0,0)]\rangle . \tag{2.16}$$

If we use the Matsubara frequencies we have

$$\sigma(\omega) = \frac{-i}{\omega}\left[\int_0^{1/T} d\tau\, e^{i\omega_n\tau}\Xi(\tau)\right]_{i\omega_n\to\omega+i\delta} , \tag{2.17}$$

where

$$\Xi(\tau) = \int dx\langle T_\tau j(x,\tau)j(0,0)\rangle . \tag{2.18}$$

Using the boson representation Ξ becomes

$$\Xi(\tau) = \frac{1}{\pi^2}\int dx\langle T_\tau[\partial_\tau\phi_\rho(x,\tau)][\partial_\tau\phi_\rho(0,0)]\rangle . \tag{2.19}$$

When expressed in terms of the $\tilde{\phi}$ field this gives (we do not explicitly rewrite the ϕ_σ part which is not affected by the transformation)

$$\sigma(\omega)=\frac{-i}{\omega\pi^2}\langle T_\tau[\partial_\tau\tilde{\phi}_\rho(x)][\partial_\tau\tilde{\phi}_\rho(x)]\rangle_{q=0,i\omega_n\to\omega+i\delta} ,$$

$$R_{\mathrm{CDW,SDW}}(x,t)\propto e^{i\sqrt{2}[\tilde{\eta}(x)-\tilde{\eta}(0)]}\langle e^{i\sqrt{2}\tilde{\phi}_\rho(x,t)}e^{-i\sqrt{2}\phi_\rho(0,0}\rangle ,$$

$$R_{\mathrm{SS,TS}}(x,t)\propto\langle e^{i\sqrt{2}\theta_\rho(x,t)}e^{-\sqrt{2}\tilde{\theta}_\rho(0,0)}\rangle . \tag{2.20}$$

Thus neither the conductivity nor the pairing fluctuations are affected by the forward scattering. This agrees with previous results for the noninteracting case.[5,20] For the CDW and SDW fluctuations the average over the field η is easily done and we get

$$R_{\mathrm{CDW,SDW}}(x,t)=e^{-D_\eta(K_\rho/u_\rho)^2|x|}R_{\mathrm{CDW,SDW}}(x,t)|_{\eta\equiv0} . \tag{2.21}$$

In the absence of disorder the correlation functions have a power-law decay in real space which leads to a power-law behavior of the Fourier transformed functions for small q and ω

$$R_i(q,\omega)\propto[\omega^2+(v_Fq)^2]^{-\alpha_i/2} , \tag{2.22}$$

The exponents α_i depend on the interactions [cf. Eq. (2.9)]. This behavior is modified in the presence of disorder. If we compute the Fourier transform of (2.21), we get

$$R_{\mathrm{CDW,SDW}}(k_0=v_Fq,\omega)=\int_{-\infty}^{+\infty}dk\frac{2m}{m^2+(k_0-k)^2}(k^2+\omega^2)^{-\alpha_{\mathrm{CDW,SDW}}/2} , \tag{2.23}$$

where $m=D_\eta(K_\rho/u_\rho)^2$. For $\alpha_i<1$ the integral is convergent, and thus the CDW and SDW are not divergent. If $\alpha_i>1$, we have for $k_0=0$,

$$R_i(0,\omega)\sim\frac{2\Gamma(1/2)\Gamma(\alpha_i-1)}{m\Gamma(\alpha_i/2)}\omega^{1-\alpha_i} . \tag{2.24}$$

The forward scattering suppresses all divergences for CDW and SDW from the space integration. The only source of divergence remains the time integration. Thus the forward scattering reduces the exponent of the CDW and SDW correlation functions.

The above discussion should be modified in the case of a half-filled band. One can then have electron-electron umklapp scattering which in terms of the boson fields is given by a term:[15]

$$\frac{2g_3}{(2\pi\alpha)^2}\int dx\cos[\sqrt{8}\phi_\rho(x)] . \tag{2.25}$$

The change from ϕ_ρ to $\tilde{\phi}_\rho$ is then no longer trivial. This point necessitates a separate study and will not be considered in the following.

III. IMPURITY BACKWARD SCATTERING: RENORMALIZATION EQUATIONS AND PHASE DIAGRAM

The effect of electron-impurity backward scattering, as described by the random field ξ, is much more drastic than forward scattering: In the noninteracting case this term leads to localization.[5] More generally, both impurity backscattering and the $g_{1\perp}$ interaction give rise to divergent terms in a perturbation calculation. We therefore use a perturbative development in the disorder D_ξ and in the interaction $g_{1\perp}$ to generate the renormalization equations under a change of the length scale $\alpha\to e^l\alpha$. We will limit ourselves to a first-order development in D_ξ and a second-order development in $g_{1\perp}$. To this order the renormalization equations can be derived without using replicas, and we will omit the replica indices in the following. To this order, the renormalization equations are therefore identical for quenched and annealed disorder. This point and its physical consequences will be discussed in greater detail in Sec. VI. The part of the action due to the backward scattering is, in the fermion representation,

$$S_{D_\xi}=-D_\xi\sum_{\sigma,\sigma'}\int d\tau\,d\tau'\,dx(\psi^\dagger_{+,\sigma}\psi_{-,\sigma})_{x,\tau}\times(\psi^\dagger_{-,\sigma'}\psi_{+,\sigma})_{x,\tau'} . \tag{3.1}$$

Such a process is purely elastic due to the double integration over time. In order to derive the renormalization equations, we have to split the integral in two parts by using the cutoff $u_\sigma|\tau-\tau'|\le\alpha$. If we denote by $\oint$ the integration for $u_\sigma|\tau-\tau'|\ge\alpha$, we have

$$S_{D_\xi}=-D_\xi\sum_{\sigma,\sigma'}\oint d\tau\,d\tau'\,dx(\psi^\dagger_{+,\sigma}\psi_{-,\sigma})_{x,\tau}(\psi^\dagger_{-,\sigma'}\psi_{+,\sigma})_{x,\tau'} - \frac{2D_\xi\alpha}{u_\sigma}\sum_{\sigma,\sigma'}\int d\tau\,dx(\psi^\dagger_{+,\sigma}\psi_{-,\sigma})_{x,\tau}(\psi^\dagger_{-,\sigma'}\psi_{+,\sigma})_{x,\tau} . \tag{3.2}$$

The second term in (3.2) is equivalent to a g_1 process and can be absorbed in the definition of the $g_{1\perp}$ and $g_{1\parallel}$ constants. Thus the initial values of the g are modified. If we call g^{in} the original constants of the g-ology representation, we have

$$g_{1\parallel,1\perp}=g^{\mathrm{in}}_{1\parallel,1\perp}-\frac{2D_\xi\alpha}{u_\sigma} . \tag{3.3}$$

Details of the derivation of the renormalization equations are left for Appendix A, and we give here only the resulting equations. The renormalization equations are expressed in terms of the modified g constants (3.3):

$$\frac{dK_\rho(l)}{dl}=-\frac{1}{2}\left[\frac{K_\rho^2 u_\rho}{u_\sigma}\right]\mathcal{D}(l)\ ,$$
$$\frac{dK_\sigma(l)}{dl}=-\tfrac{1}{2}[\mathcal{D}(l)+y(l)^2]K_\sigma^2\ ,$$
$$\frac{du_\rho(l)}{dl}=-\frac{u_\rho^2 K_\rho}{2u_\sigma}\mathcal{D}(l)\ ,$$
$$\frac{du_\sigma(l)}{dl}=-\frac{u_\sigma K_\sigma}{2}\mathcal{D}(l)\ , \tag{3.4}$$
$$\frac{dy(l)}{dl}=[2-2K_\sigma(l)]y(l)-\mathcal{D}(l)\ ,$$
$$\frac{d\mathcal{D}(l)}{dl}=[3-K_\rho(l)-K_\sigma(l)-y(l)]\mathcal{D}(l)\ ,$$

with the dimensionless quantities

$$\mathcal{D}=\frac{2D_\xi\alpha}{\pi u_\sigma^2}\left[\frac{u_\sigma}{u_\rho}\right]^{K_\rho},\quad y=g_{1\perp}/(\pi u_\sigma)\ . \tag{3.5}$$

The renormalizations of K_ν and u_ν are of first order in $\mathcal{D}$ and y^2, and consequently can be neglected on the right-hand sides of the first three equations. Contrary to the pure case, charge and spin degrees of freedom are now no longer decoupled.

The last equation governs the disorder effects. If $(3-K_\rho-K_\sigma-y)>0$, $\mathcal{D}$ will increase under renormalization. The solution flows to a strong-coupling regime where our first-order renormalization equations are no longer valid. The physical properties of this strong-coupling regime will be strongly dependent on whether we have quenched or annealed disorder, although the first-order renormalization equations are not. Let us consider quenched disorder (a short discussion about annealed disorder will be given in Sec. VI). The strong coupling ($\mathcal{D}\to\infty$) regime contains the noninteracting point, which we know[3-5] is localized. Assuming that there is no fixed point at intermediate coupling, we can identify the whole $\mathcal{D}\to\infty$ phase with a localized phase. If $3-K_\rho-K_\sigma-y<0$, $\mathcal{D}$ will decrease, the disorder is irrelevant, and the system is delocalized. K_ρ and K_σ are functions of the interactions [cf. (2.5)] and K_ρ increases for increasingly attractive interactions. Therefore, a localized-delocalized transition is driven by the interactions. In the limit of extremely weak disorder ($\mathcal{D}\to 0$) the boundary between the two regimes is given by the following condition:

$$3-K_\rho-K_\sigma^*=0\ . \tag{3.6}$$

where K_σ^* is the value of K_σ renormalized by the $g_{1\perp}$ interaction [$K_\sigma^*=K_\sigma(g_{\sigma^*})$ with $g_{\sigma^*}=(g_{\sigma^2}-g_{1\perp}^2)^{1/2}$]. In this limit, our renormalization equations reproduce previous results. The case of spinless fermions (formally $K_\sigma^*=K_\rho=K_\sigma$ and $y=0$) was studied by Chui and Bray[6] using a series expansion, and by Apel[7] by a renormalization-group method. In this particular case our equations reduce to those of Apel, and the result of Chui and Bray corresponds to a zeroth-order renormalization in our case (i.e., only the renormalization of the disorder is considered). We recover also the result of Apel and Rice[9] obtained by using scaling arguments on finite-length systems. All these previous results [except Ref. (7)] neglect all the renormalization effects on the interactions due to a finite disorder. They are thus valid *only* in the limit $\mathcal{D}\to 0$. Another method used is the self-consistent harmonic approximation (SCHA) of Suzumura and Fukuyama.[8] The SCHA allows us to treat the $g_{1\perp}$ interaction but neglects both renormalization effects from the disorder and from the $g_{1\parallel}$ term. This method is thus valid only in the limit $\mathcal{D},y\to 0$.

Close to the noninteracting point (which means $g^{\rm in}=0$) one has to check that our renormalization equations does not generate $g^{\rm in}$ terms. In order to do so, we have at a scale l to separate in $K(l)$ the part coming from inelastic processes and the part coming from the second part of Eq. (3.2). Close to the noninteracting point, if we write the Hamiltonian as a sum of inelastic processes ($g^{\rm in}$) and a purely elastic process like Eq. (3.1), we have, according to (3.2),

$$g_{1\parallel,1\perp}(l)=g_{1\parallel,1\perp}^{\rm in}(l)-\frac{2D_\xi(l)\alpha}{u_\sigma}\ . \tag{3.7}$$

Our renormalization equations are at the noninteracting point ($g_1^{\rm in}=g_2^{\rm in}=0$),

$$\frac{1}{\pi v_F}\frac{d(g_1-2g_2)}{dl}=\mathcal{D}(l)\ ,$$
$$\frac{1}{\pi v_F}\frac{dg_1}{dl}=-\mathcal{D}(l)\ , \tag{3.8}$$
$$\frac{d\mathcal{D}}{dl}=\mathcal{D}(l)\ ,$$

where we have neglected terms of order $\mathcal{D}^2$ as should be done, and used the fact that the equations for $g_{1\perp}$ and $g_{1\parallel}$ are identical. At the noninteracting point we also have $\mathcal{D}=2D_\xi\alpha/\pi v_F^2$. Using (3.7) and $y=g_{1\perp}/\pi v_F$ we get $y_1=y_1^{\rm in}-\mathcal{D}$. And thus from (3.8) we have

$$\frac{1}{\pi v_F}\frac{d(g_1^{\rm in}-2g_2^{\rm in})}{dl}=0\ ,$$
$$\frac{1}{\pi v_F}\frac{dg_1^{\rm in}}{dl}=0\ , \tag{3.9}$$
$$\frac{d\mathcal{D}}{dl}=\mathcal{D}(l)\ .$$

Thus the renormalization equations do not generate any inelastic processes if we start from the noninteracting point. Of course if we start from the transition point where the couplings $g^{\rm in}$ are nonzero, inelastic processes are renormalized.

To obtain the phase diagram for finite disorder, the renormalization of K_ρ and K_σ by $\mathcal{D}$ must be taken into account. As the spin-anisotropy is usually small, we will consider the following:

$$g_{1\perp}=g_{1\parallel}=g_1 ,$$
$$g_{2\perp}=g_{2\parallel}=g_2 , \tag{3.10}$$

and to be consistent with our development in $g_{1\perp}$

$$K_\sigma=1+\frac{y}{2}+O(g_{1\parallel}^2) ,$$
$$u_\sigma=v_F+O(g_{1\parallel}^2) , \tag{3.11}$$

where $y=g_1/(\pi v_F)$. Inserting this into Eq. (3.4) the second and fourth equations coincide and we obtain

$$\frac{dK_\rho(l)}{dl}=-\frac{1}{2}\left[\frac{K_\rho^2 u_\rho}{v_F}\right]\mathcal{D}(l) ,$$
$$\frac{dy(l)}{dl}=-\mathcal{D}(l)-y(l)^2 , \tag{3.12}$$
$$\frac{d\mathcal{D}(l)}{dl}=[2-K_\rho(l)-\tfrac{3}{2}y(l)]\mathcal{D}(l) .$$

From these renormalization equations different kinds of behavior are obtained.

(1) *$\mathcal{D}$ and y flow to zero*: This is a fixed line of equations, parametrized by K_ρ. As $\mathcal{D}=0$, the system is delocalized, and charge and spin degrees of freedom are asymptotically decoupled. We recover power laws for the correlation functions but with renormalized exponents α_i^* [see Eq. (2.22)]:

$$\alpha_{\mathrm{CDW,SDW}}=1-K_\rho^* ,$$
$$\alpha_{\mathrm{SS,TS}}=1-(K_\rho^*)^{-1} , \tag{3.13}$$

where the asterisk indicates fixed-point quantities. The complete study of the equation has to be done numerically. Some curves for the renormalized exponents are shown in Fig. 2. The stability of the fixed line requires $K_\rho^*\geq 2$, so that we have superconducting fluctuations in the delocalized phase. The nature of the dominant fluctuation depends on the spin part of the correlation functions. Formally, SS and TS are equally divergent. However, the fixed line can only be reached if initially $g_1>0$, and therefore, as in the pure case,[15] there are logarithmic corrections which lead to dominant triplet fluctuations.

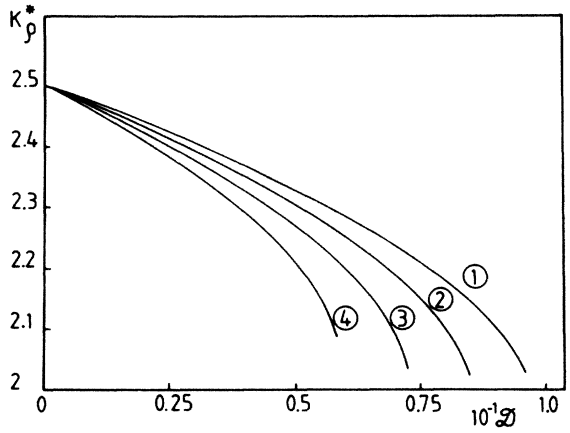

FIG. 2. Renormalized value of K_ρ as a function of the disorder $\mathcal{D}$. The curves end at $\mathcal{D}_{\mathrm{crit}}$ for which the localized-delocalized transition takes place. Curves 1,2,3,4 are for $y=0.5,0.4,0.3,0.2$, respectively.

(2) *$\mathcal{D}$ flows to zero and y to $-\infty$*: If g_1 is initially small or negative, y flows to large negative values, indicating, as in the pure case, that the spin degrees of freedom are frozen. If $\mathcal{D}$ initially increases much more slowly than y, i.e., for large K_ρ, one is in a situation where ϕ_σ fluctuations are essentially frozen out. The renormalization equations become in this case (cf. Appendix A)

$$\frac{dK_\rho(l)}{dl}=-\tfrac{1}{2}K_\rho^2\mathcal{D}(l) ,$$
$$\frac{du_\rho(l)}{dl}=-\frac{u_\rho K_\rho}{2}\mathcal{D}(l) , \tag{3.14}$$
$$\frac{d\mathcal{D}(l)}{dl}=[3-K_\rho(l)]\mathcal{D}(l) ,$$

with

$$\mathcal{D}=\frac{2C_\sigma D_\xi\alpha}{\pi u_\rho^2} , \tag{3.15}$$

and C_σ is a constant of order unity coming from the ϕ_σ correlations in the perturbation expansion. In this case the exponents are given by

$$\alpha_{\mathrm{CDW}}=2-K_\rho^* ,$$
$$\alpha_{\mathrm{SS}}=2-(K_\rho^*)^{-1} . \tag{3.16}$$

As $K_\rho^*\geq 3$, singlet-pairing fluctuations dominate in this region.

(3) *$\mathcal{D}$ flows to infinity*: The solution of the equations flows towards a regime where the renormalization equations are not valid. If we suppose that there is no intermediate fixed point we scale to a strongly disordered system. The system can then be described by a Hamiltonian of localized particles. The magnetic properties of the model will depend again on the renormalized value of g_1. If $g_1^*<0$ we will have a nonmagnetic system of localized pairs of spins. This is equivalent to a charge-density wave pinned by the impurities[21] (if the pinning is strong enough the CDW will adjust itself on the impurities and the $2k_F$ modulation will not matter very much). We will denote this state by the somewhat imprecise term "charge-density glass," or pinned charge-density wave (PCDW). On the other hand, for $g_1^*>0$, there is a repulsion between electrons of the same spin. The electrons thus avoid localizing pairwise in the same state. The system of localized electrons will consist mostly of isolated spins localized around randomly distributed sites. Due to the randomness, the exchange interaction between adjacent electrons will also be random, and we therefore expect properties typical of a random antiferromagnet (RAF) (Ref. 22).

The phase diagram for a fixed value of disorder is given in Fig. 3 together with the usual diagram of one-dimensional systems without disorder. The limits between the different regimes depend both on disorder and

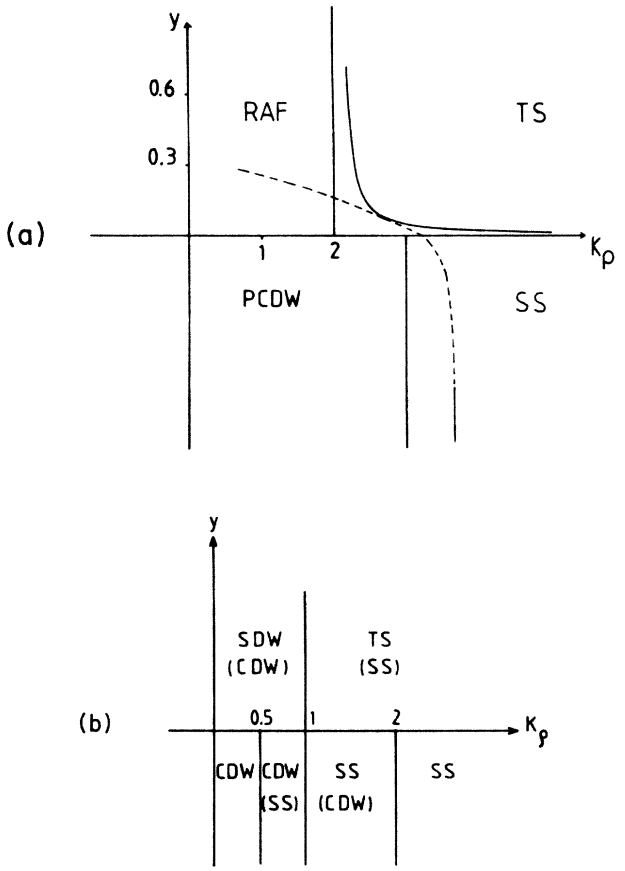

FIG. 3. Phase diagram for electrons in the y-K_ρ plane. (a) is for finite disorder, $\mathcal{D}=0.05$, and (b) is the usual one without disorder. In one dimension, "phase diagram" means a diagram of the dominant fluctuation. The fluctuations less divergent than the dominant one are within parentheses. Dashed lines are for parts of the phase diagram that cannot be obtained precisely by the present method. In (a) the two vertical lines are the limit of the phase diagram when $\mathcal{D}\to 0$.

interactions. In general, the different phase boundaries have to be determined numerically. However, for large K_ρ we find that the SS-TS boundary is given asymptotically by $y_c=\mathcal{D}/(K_\rho-2)$.

Along the SS-localized and TS-localized limits the localization length diverges. A more detailed discussion will be given in Sec. IV. In the SS phase where there is a gap Δ_σ in the spin-excitation spectrum,[15] we can define a spin-correlation length u_σ/Δ_σ. As along the TS-SS limit, this spin-correlation length is also divergent; the three curves meet on a multicritical point. From our method we can obtain only the TS-localized limit, the TS-SS limit, and the asymptotic form of the SS-localized limit for large negative y (solid lines in Fig. 3). Along the TS-localized limit we find a nonuniversal value of K_ρ^*.

We cannot give the location of the multicritical point, but we can estimate the SS-localized curve if the spin-correlation length is smaller than the localization length. We start with Eqs. (3.12). If we are close to the SS-localized transition the disorder starts to decrease, whereas y becomes negative and increases in absolute value. Thus the spin degrees of freedom are increasingly frozen while the disorder remains weak. We stop the renormalization when y has reached a given value $y_0=y(l_0)$. As the disorder is still weak, we can use the values $K_\rho(l_0)$ and $\mathcal{D}(l_0)$ as initial values in (3.14) and resume the renormalization. This picture breaks down when we get too close to the multicritical point, where the two lengths diverge simultaneously. This argument gives for the SS-localized line $K_\rho(l=0)>3$; thus we would expect the multicritical point to be also at $K_\rho(l=0)>3$. As the spin degrees of freedom are always frozen in the SS phase, the renormalized value K_ρ^* on the critical line SS-localized is always 3, as can be seen from (3.14).

Our first-order equations imply that for the localized phase ($\mathcal{D}\to\infty$) one should have $g_i\to-\infty$, i.e., a nonmagnetic behavior of the PCDW type, regardless of the strength of the original interaction g_1. This unphysical behavior of the model arises because we consider here only first-order impurity effects. In the pure case ($\mathcal{D}=0$), for $y>0$, the exponents of the charge- and spin-density fluctuations are the same, but logarithmic corrections favor the SDW fluctuation.[15] For $y>0$, the system is thus dominated by SDW fluctuations with coexisting CDW fluctuations. In our calculation, we have limited ourselves to the first order in the development in the impurity Hamiltonian. The coupling between the CDW and the impurities is of the first order in the impurity potential, and the CDW gains energy by adjusting to the impurity potential.[21] On the other hand, the magnetic (SDW) fluctuations interact with the disorder via second-order terms only. Thus, to the order of the present calculation, the only phase which can take advantage of the impurities is the CDW. This sufficiently favors the CDW fluctuation to lead to a PCDW phase for all values of y. For a proper description of the magnetic properties in the localized phase a next-order (at least) development is necessary—a point to be discussed in greater detail in Sec. VI.

In the delocalized regime, due to the presence of disorder, the singlet superconducting phase grows at the expense of the triplet phase which is in agreement with the fact that triplet pairing is more sensitive to disorder than singlet pairing.[23] However, the triplet phase is more stable against localization than the singlet one, i.e., it extends further to the left in the phase diagram (Fig. 3). This can be qualitatively explained by the fact that the phase in competition with singlet superconductivity is a CDW which is easily stabilized by the impurities. This was also pointed out by Suzumura.[24]

IV. LOCALIZATION LENGTH AND FINITE-TEMPERATURE EFFECTS

The renormalization equations allow us to obtain the asymptotic form of the localization length. These equations describe the modification of the interaction constants under a change of the short-range cutoff, due to coherence effects. Starting from the initial values of the

interaction constants, the renormalization equations will be iterated up to a point where the system is dominated by the localization effects. Practically, the iteration stops when $\mathcal{D}$ has reached a given value $\mathcal{D}_x$. The precise point does not affect the asymptotic behavior of the quantities studied. We have

$$L_{\rm loc}(\mathcal{D})=L_{\rm loc}(\mathcal{D}_x)e^{[l(\mathcal{D}_x)-l(\mathcal{D})]} . \tag{4.1}$$

$L_{\rm loc}$ can be easily obtained in the $\mathcal{D}\to 0$ limit. Then we can neglect the renormalization of K_ρ and K_σ by the disorder and obtain from (3.4)

$$\ln(\mathcal{D}_x/\mathcal{D})=[l(\mathcal{D}_x)-l(\mathcal{D})](3-K_\rho-K_\sigma^*) , \tag{4.2}$$

where K_σ^* is the fixed-point value in the absence of disorder. The asymptotic behavior of $L_{\rm loc}$ is given by

$$L_{\rm loc}(\mathcal{D})\propto\left[\frac{1}{\mathcal{D}}\right]^{1/(3-K_\rho-K_\sigma^*)} \tag{4.3}$$

If we take into account the renormalization of the interactions by the disorder this power-law behavior is changed. The solution in the general case can be obtained numerically. Close to the critical surface an analytic solution can be obtained (see Appendix B for details). Let us write $\eta=K_\rho-2$. For $y/\eta\ll 1$ and $\mathcal{D}/\eta^2\ll 1$ the critical surface is $\mathcal{D}=\eta y$. The asymptotic behavior of the localization length is

$$L_{\rm loc}\propto\exp\left[\frac{\eta}{\mathcal{D}-\eta y}\right] . \tag{4.4}$$

In fact, on the boundary between the localized and delocalized phases we always renormalize to $y\to 0$, $\mathcal{D}\to 0$. There is always a length scale l_0 for which the conditions $y(l_0)/\eta(l_0)\ll 1$ and $\mathcal{D}(l_0)/\eta^2(l_0)\ll 1$ are fulfilled. Equation (4.4) can thus still be used by replacing the initial values $\mathcal{D},y$ by the renormalized ones $\mathcal{D}(l_0),y(l_0)$, which are continuous functions of $\mathcal{D},y$.

If the spin degrees of freedom are frozen the equations simplify and reduce to (3.14). Writing $3+\eta_\rho=K_\rho$ the equations become

$$\begin{aligned}\frac{d\eta_\rho}{dl}&=-\tfrac{9}{2}\mathcal{D} ,\\ \frac{d\mathcal{D}}{dl}&=-\mathcal{D}\eta_\rho .\end{aligned} \tag{4.5}$$

In terms of the new variables $2\chi^2=\frac{9}{2}\mathcal{D}$ and $\gamma=\eta_\rho/2$ we recover the equations of a Kosterlitz-Thouless[25] transition. In this case the localization length is given in terms of a crossover function f,[26]

$$L_{\rm loc}=Ce^{f(\gamma/\chi)/\chi} , \tag{4.6}$$

where C is a constant which cannot be given by the renormalization equations. f is given by

$$f(x)=\begin{cases}\phi/\sin(\phi)\ \ \cos(\phi)=-x\ \ (-1<x<1) ,\\ \phi/\sinh(\phi)\ \ \cosh(\phi)=-x\ \ (x<-1) .\end{cases} \tag{4.7}$$

We have the two following limiting behaviors of the localization length.

(1) Deep in the localized region (or in the limit $\mathcal{D}\to 0$) we have $|\gamma|\gg\chi$ and

$$L_{\rm loc}\propto\left[\frac{1}{\mathcal{D}}\right]^{1/(3-K_\rho)} . \tag{4.8}$$

(2) Close to transition ($\gamma\simeq\chi$),

$$L_{\rm loc}\propto\exp[2\pi/(9\mathcal{D}-\eta_\rho^2)^{1/2}] . \tag{4.9}$$

As our renormalization equations are equivalent in the $\mathcal{D}\to 0$ limit to the SCHA method, we thus recover in (4.8) the localization length computed by Suzumura and Fukuyama.[8] In their method the disorder effects seem to be dependent both on the impurity concentration n_i and on the disorder potential V. One can check that in the physical quantities (such as the localization length and the energy) *only* the product $n_i V^2$ (i.e., our $\mathcal{D}$) occurs.

One can remark that, in the case of spinless fermions, the fermion Hamiltonian in the presence of impurities is equivalent to that of a charge-density wave in the presence of impurities.[21,27] The main difference comes from the fact that in the case of fermions (at least for reasonably strong interactions) the quantum effects are important: The quantum term in the Hamiltonian $[(uK)\Pi^2]$ has a prefactor close to v_F. In the case of a charge-density wave one often neglects the quantum terms due to the large effective mass of the charge-density wave. The delocalization transition found in this study corresponds to a depinning of the charge-density wave due to quantum effects. The renormalization equations are obtained from (3.4), setting $K_\rho=K_\sigma=K$, $u_\rho=u_\sigma=u$, and $y=0$. In the localized region we can use (4.3). The classical limit of the Hamiltonian (2.5) is obtained by writing $K=0$. The "localization length" becomes

$$L_{\rm loc}=\left[\frac{1}{\mathcal{D}}\right]^{1/3} . \tag{4.10}$$

In this case we simply recover the Lee-Fukuyama pinning length (in the weak-pinning case) for the charge-density-wave system.[21]

The renormalization equations also provide some insight into the influence of finite temperature on the delocalized phase. We will neglect here all the phonon processes taking only into account the inelastic processes due to the electron-electron interactions. At zero temperature we can iterate the renormalization equations up to an infinite length (as long as the coupling remains weak, which is the case in the delocalized phase). At finite temperature, the coherence effects disappear for any length greater than $\xi_T\approx v_F/T$ due to inelastic processes, and the renormalization stops at this length scale. Under the assumption that temperature does not drastically modify the renormalization equations of the system until the thermal length is reached, the renormalization stops at $e^{l^*}\approx\xi_T/\alpha$. At this length scale the disorder can be treated in the Born approximation. As the conductivity is a physical quantity it is not changed under renormalization and we have

$$\sigma[n(0),\mathcal{D}(0),0]=\sigma(n(l),\mathcal{D}(l),l)$$

$$=\sigma_0\frac{n(l)\mathcal{D}(0)}{n(0)\mathcal{D}(l)}=\sigma_0\frac{e^l\mathcal{D}(0)}{\mathcal{D}(l)}\ , \tag{4.11}$$

where $\sigma[n(l),\mathcal{D}(l),l]=\sigma(l)$ and $n(l)$ are, respectively, the conductivity and the electronic density at the scale l. $\sigma_0=e^2v_F^2/2\pi\hbar D_\xi$ is the conductivity in the Born approximation, expressed with the initial parameters. When the renormalization stops at e^{l^*} we get at low temperatures $\mathcal{D}\sim T^\gamma$, with $\gamma=K_\rho^*-2$ in the TS region and $\gamma=K_\rho^*-3$ in the SS region. In both cases one has $\gamma\geq 0$ in the delocalized phase. The conductivity then behaves as

$$\sigma(T)\sim T^{-1-\gamma}\ . \tag{4.12}$$

The exponent γ decreases with increasing disorder as can be seen from Fig. 2. Thus in the limit $\mathcal{D}\to 0$ we recover the result given in Refs. 28 and 29.

The renormalization can also give the crossover toward the localized regime. At a high temperature the properties of the system are correctly obtained from the Born approximation. If the temperature is lowered the quantum (interference) effects will become increasingly important, and at low temperature they will dominate the behavior of the system. The crossover can be described by the renormalization equations: Even if the solution flows toward the localized regime (i.e., strong coupling), if the temperature is high enough the renormalization will stop when the renormalization equations are still valid (i.e., for small-coupling constants). What we are left with is a system where the disorder can be treated perturbatively and there will be no localization. We can thus introduce a temperature $T_{\rm loc}$ which will roughly indicate when the quantum-interference effects will dominate the behavior of the system. This will happen when the thermal length ξ_T is greater than the localization length, and we thus have

$$T_{\rm loc}\sim v_F/L_{\rm loc}\ . \tag{4.13}$$

The complete temperature dependence of the conductivity, as obtained from a numerical integration of Eqs. (3.12), is shown in Fig. 4 for different values of the electron-electron coupling and the strength of the disorder. It is remarkable that, provided that initially $2-K_\rho-3y/2<1$ (i.e., superconducting fluctuations would dominate in the pure system), the resistivity first decreases with temperature, and increase occurs only at rather low temperature, below (approximately) $T_{\rm loc}$. This behavior can be understood, noting that for $T>T_{\rm loc}$ quantum-interference effects from the disorder are absent, and the description in terms of a modified Born approximation (so as to take into account the interactions) of Ref. 28 is correct. Only below $T_{\rm loc}$ does one have $\xi_T>L_{\rm loc}$, and the localization effects become important, leading to a sharp increase in resistivity. Thus, in the presence of interactions, a diffusive regime exists at short-length scales, when interference effects are unimportant. This regime is of course especially large close to the localization transition where $L_{\rm loc}$ becomes very large.

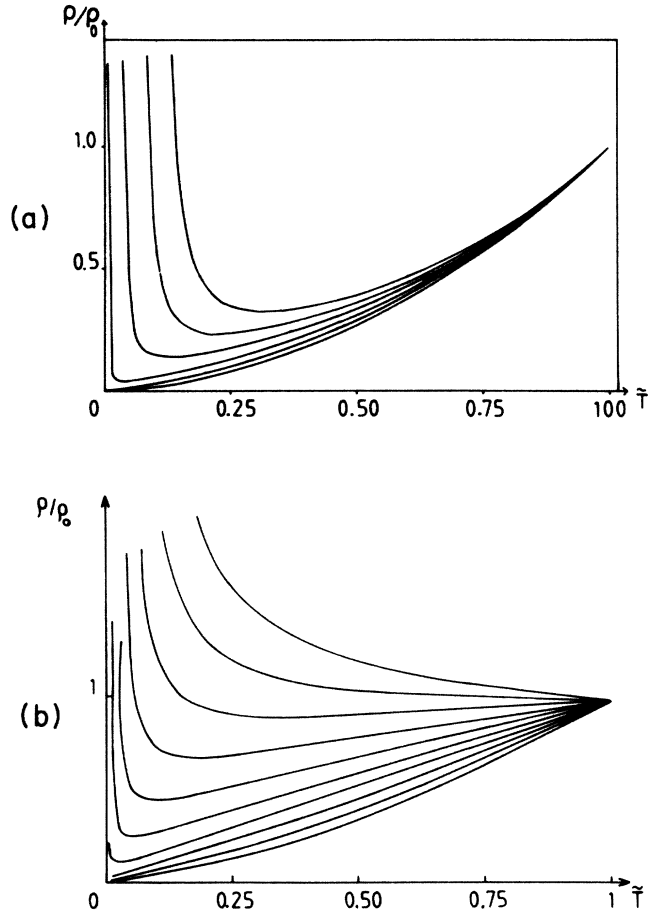

FIG. 4. Temperature dependence of the resistivity. ρ_0 is the value of the Born approximation, and $\tilde{T}=T/E_F$. (a) Curves are for $y=0.3$, $K=2.5$, and $\mathcal{D}=0.02$, 0.10, 0.18, 0.26, 0.34, 0.42 (ρ increasing). (b) Curves are for $y=0.3$, $\mathcal{D}=0.05$, and $K=2.7$, 2.5, 2.3, 2.1, 1.9, 1.7, 1.5, 1.3, 1.1 (ρ increasing). Except for strongly repulsive interactions, a diffusive regime exists at finite temperature.

If Ref. 29 the behavior in the delocalized region is interpreted as complete screening of the impurity potential, the effective scattering potential being $V_{\rm eff}(q)=V_{\rm imp}(q)\Pi(q,0)$, where $\Pi(q,\omega)$ is the density-density response function. One should, however, notice that even if $\alpha_{\rm CDW}<0$, so that $\Pi(q,\omega)$ does not diverge for $(q,\omega)\to(2k_F,0)$, $\Pi(2k_F,0)$ is finite and does not vanish. Thus, there is no complete screening of the impurity potential. The increasing conductivity can be imputed to the *superconductive* behavior of the system. In this context, it should be noted that the long-range order of the superconducting correlation function is not absolutely necessary to have a perfectly superconducting response behavior. A sufficiently slow power-law decay is actually sufficient.[30] In this sense the zero-temperature state is a real superconductor.

V. LOCALIZATION TRANSITION IN A ONE-DIMENSIONAL BOSON SYSTEM

The method developed here can be applied to a one-dimensional system of interacting bosons in a random potential, and specifically to the localized-superfluid transition in such a system. We use a representation of boson operators in terms of phase fields introduced by Haldane.[18] The single-boson creation operator is written as

$$\Psi_b^\dagger(x)=[\rho(x)]^{1/2}e^{i\phi(x)} , \tag{5.1}$$

where $\rho(x)$ is the particle-density operator and $\phi(x)$ the phase of the boson field. Taking the discrete nature of the particle density into account, the density operator is

$$\rho(x)=\frac{1}{\pi}\frac{\partial\theta(x)}{\partial x}\sum_{m=-\infty}^{+\infty}\exp[2im\theta(x)] , \tag{5.2}$$

where $\partial\theta(x)/\partial x=\pi[\rho_0+\Pi(x)]$, ρ_0 is the average density, and $\Pi(x)$ obeys the canonical commutation relations,

$$[\phi(x),\Pi(x')]=i\delta(x-x') . \tag{5.3}$$

The long-wavelength–low-energy properties are described by the Hamiltonian,[18]

$$H=\frac{1}{2\pi}\int dx\left[(vK)(\pi\Pi)^2+\left[\frac{v}{K}\right](\partial_x\phi)^2\right] , \tag{5.4}$$

where from Galilean invariance one has $v/(\pi K)=\rho_0/m$, and $\pi vK=\kappa/(\pi^2\rho_0^2)$, where κ is the compressibility. Clearly, the excited states of H are sound waves with phase velocity v, which from Eq. (5.1) are the phonon modes typical of a Bose superfluid. As already pointed out in the preceding section, the existence of such modes is sufficient for true superfluidity to exist.[31] The coefficient K determines the asymptotic behavior of the correlation functions,[18]

$$\langle\Psi_B^\dagger(r)\Psi_B(0)\rangle=B\rho_0(\rho_0 r)^{-K/2} , \tag{5.5}$$

$$\langle\rho(r)\rho(0)\rangle=\frac{2}{K}(2\pi\rho_0 r)^{-2}+A\rho_0^2(\rho_0 r)^{-2/K}\cos(2\pi\rho_0 r) , \tag{5.6}$$

with some numerical constants A and B.

We now introduce a random potential V, described by an additional term

$$H_r=\int dx\, V(x)\rho(x) \tag{5.7}$$

in the Hamiltonian. Inserting from (5.2) and retaining only the most important terms ($m=0$, $m=1$), this becomes

$$H_r=\int dx\{\eta(x)\Pi(x)+[\xi(x)\rho_0 e^{2i\tilde{\theta}(x)}+\text{H.c.}]\} , \tag{5.8}$$

where in $\tilde{\theta}$ the linearly increasing part of θ is subtracted,

$$\frac{\partial\tilde{\theta}(x)}{\partial x}=\pi\Pi(x) , \tag{5.9}$$

and η and ξ are the parts of the random potential with Fourier components around $q\simeq 0$ and $q\simeq\pm 2\pi\rho_0$, respectively. We will take for η and ξ the same probability distributions as in (2.10). A unitary transformation which exchanges ϕ and θ [$\rho_-(p)\to-\rho_-(p)$ in Eq. (2.3)] turns the complete problem [Eqs. (5.4) and (5.8)], into the one discussed in Sec. II, Eqs. (2.4) to (2.11) (or more precisely its spinless analogue). Following the same route as before, we find the scaling equations

$$\frac{dK}{dl}=\tfrac{1}{2}\mathcal{D} ,$$
$$\frac{d\mathcal{D}}{dl}=(3-2K^{-1})\mathcal{D} , \tag{5.10}$$

where $\mathcal{D}=\mathcal{D}_\xi/\pi^2u^2\rho_0$. Thus for $K<\frac{2}{3}$ the disorder scales to zero, indicating a delocalized, superfluid phase with renormalized exponents due to the renormalization of K. For $K>\frac{2}{3}$, the disorder grows under scaling, indicating that the properties of the system are qualitatively different from the region $K<\frac{2}{3}$. In analogy with the previous chapter, we interpret this as the localized region. The phase diagram in the disorder K plane is shown in Fig. 5. Along the superfluid-localized transition line, the fixed-point value of K is $K^*=\frac{2}{3}$, and consequently the single-boson correlation function, Eq. (5.5) decays with a *universal power* $\frac{1}{3}$ along this line. This is the same exponent as found along the SS-PCDW limit before. This equivalence certainly is not unexpected, as in the SS region fermions are bound into singlet pairs, with binding energy Δ_σ, and these pairs behave like bosons, at least at large length scales.[32]

The coefficient K increases with increasing repulsion between bosons ($K\sim\kappa^{1/2}$), and consequently the transition discussed above occurs with increasing repulsion. In this context we may note than an external potential with period $1/\rho_0$ (1 boson per site) is a relevant perturbation, i.e., leads to a long-range ordered state, for $K>\frac{1}{2}$, i.e., within the stable (against disorder) superfluid region. On the other hand, a potential with period $\frac{1}{2}\rho_0$ (one boson per two sites) needs $K>2$ to lead to an ordered state, and this can only be achieved for a very strong repulsion with a finite range [δ-function repulsion leads to $K<1$ (Ref. 18)].

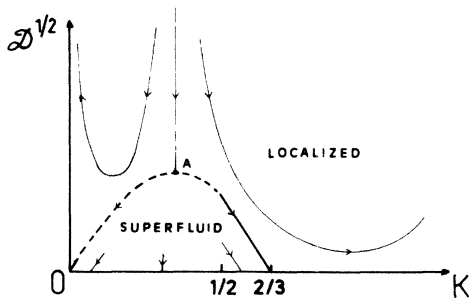

FIG. 5. Phase diagrams for a one-dimensional boson system. The thin lines indicate the qualitative shape of scaling trajectories, as discussed in the text. The dashed lines are for parts of the diagram that cannot be obtained by the present method. The multicritical point can also be located at $K=0$.

Clearly, noninteracting bosons are localized for arbitrarily weak disorder. Consequently, the superfluid-localized boundary is bent down for small values of K, as shown schematically by the dashed line in Fig. 5. According to Eq. (5.10), on the phase boundary, scaling is toward the end of the fixed line $\mathcal{D}=0$, $K=\frac{2}{3}$. Such a behavior is questionable for points of the phase boundary close to $K=0$. It seems more reasonable to assume the existence of a multicritical point A on the boundary. This point will separate the scaling to $K=\frac{2}{3}$ from the scaling to $K=0$. It also implies the existence of *two localized phases*. If K scales to zero all the particles condense into the single-particle ground state of the random potential, giving a highly inhomogeneous state. On the contrary, if the repulsion between the bosons dominates (scaling to large K) we have a localized state with a homogeneous density similar to the case of fermions. We have indicated on Fig. 5 by thin lines the qualitative shape of the scaling trajectories corresponding to this problem. It is also possible that the noninteracting line is singular. If that is the case the multicritical point is at $K=0$. However, such a picture gives very peculiar renormalization trajectories. The location of the multicritical point, the detailed shape of the phase boundary, and the scaling behavior close to $K=0$ cannot be determined by the present approach.

VI. DISCUSSION AND CONCLUSION

In this paper we have developed a renormalization-group method to study the combined effects of disorder and interactions in one-dimensional quantum fluids. In the case of fermions we find a localized-delocalized transition for sufficiently attractive interactions. The delocalized phase is dominated by superconducting fluctuations, whether singlet or triplet, depending on the interactions, whereas the localized phase corresponds to a RAF or a CDW pinned by impurities. The singlet phase is found to extend over the triplet one. It is less stable against disorder because of competition with a CDW phase which is easily pinned by impurities.

We calculate the localization length. In the localized regime we find a power-law dependence on the disorder. In the limit where quantum effects can be neglected in the dynamics of the CDW, the localization length calculated here becomes identical to the Lee-Fukuyama pinning length. Close to the localized-delocalized transitions, the localization length diverges exponentially.

Our calculation also allows us to discuss the temperature dependence of the conductivity and the temperature below which localization effects become important. Specifically, we have shown that in the localized region, but not too far from the localized-delocalized transition, the resistivity initially decreases with decreasing temperature, and localization effects set in only at very low temperatures. This can be explained, noting that in the presence of attractive electron-electron interactions there is, at short-length scales, a diffusive regime where quantum interference effects are absent. Such a regime does not exist for noninteracting particles in one dimension.

Finally, we have shown the existence of a localized-superfluid transition in a one-dimensional Bose fluid. The problem is formally similar to fermions with a short-range (g_1) interaction. We find universal power laws for correlation functions on the localized-superfluid transition line.

As we have seen in Sec. III, the fact that we use a first-order development leads to an incorrect description of the magnetic properties of localized phase. But there is another limitation specific to our study. As we have used a first-order expansion in $\mathcal{D}$, there is no difference between annealed and quenched disorder (annealed disorder could also be seen as a classical phonon field, and in this case similar equations are found[33]). Close to the transition line there is thus no difference between a localized-delocalized transition, which occurs for quenched disorder, and a disorder-induced density-fluctuation-superconducting transition for annealed disorder. The transition for annealed disorder is similar to a Peierls transition.[34] The difference between quenched and annealed disorder has also to appear in the renormalization equations. One can check in a second-order development in $\mathcal{D}$ that diagrams with different replica indices, i.e., those responsible for differences between quenched and annealed disorder, become relevant when the couplings flow into the localized region. Thus a correct description of the localized region and the vicinity of the multicritical point would require higher-order renormalization equations and taking into account the diagrams with different replica indices.

We try now to give some possible application of the effects discussed to real compounds. We find a localized-delocalized transition which occurs at $K_\rho=2$ which means $g_2=-(3\pi v_F)/5$. Thus the delocalization transition occurs for strongly attractive (comparable to the bandwidth) interaction only. Even for compounds with a superconducting ground state it is reasonable to think that the interaction will not be attractive enough to be in the delocalized region. The absence of localization effects in these compounds is likely to be due to three-dimensional coupling between chains.

This model predicts at low enough temperature a transition from a superconducting state to a random antiferromagnetic state if disorder is put into the system. This could be a possible explanation of the transition observed in (TMTSF)ClO_4 compounds. (TMTSF)ClO_4 is superconducting at ambient pressure. A small amount of disorder [either by anion disorder or in (TMTSF)$(ClO_4)_{1-x}(ReO_4)_x$ alloys] induces a transition toward a SDW-type state.[11] Under pressure this SDW state is suppressed and one recovers the metallic state. This could be explained in our model by saying that the pressure increases the interactions and the system enters into the delocalized phase where the conductivity behaves like $T^{-\gamma}$ at finite temperature. But this explanation has to be considered with a critical eye: At the temperatures involved in such transitions (5 K) three-dimensional effects due to the interchain coupling $t_\perp$ are expected to be important. A complete explanation of the low-temperature effects would require a study of the influence of the interchain coupling.

ACKNOWLEDGMENTS

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APPENDIX A

In order to derive renormalization-group equations, following José *et al.*,[35] we consider the two correlation functions:

$$R_\rho(x_1-x_2,\tau_1-\tau_2)=\langle T_\tau e^{i\sqrt{2}\phi_\rho^i(x_i,\tau_1)}e^{-i\sqrt{2}\phi_\rho^i(x_2,\tau_2)}\rangle ,$$
$$R_\sigma(x_1-x_2,\tau_1-\tau_2)=\langle T_\tau e^{i\sqrt{2}\phi_\sigma^i(x_i,\tau_1)}e^{-i\sqrt{2}\phi_\alpha^i(x_2,\tau_2)}\rangle . \tag{A1}$$

We will not indicate in the following the replica indices because we will make a first-order development in $\mathcal{D}$. To this order the expansion can be carried out without using replicas, and the calculation is similar to that for the electron-phonon problem.[33]

We develop R_σ in powers of D_ξ and $g_{1\perp}$ we find

$$R_\sigma(r_1-r_2)=e^{-F_\sigma(r_1-r_2)}+\frac{1}{2}\left[\frac{g_{1\perp}}{(2\pi\alpha)^2}\right]^2\sum_{\epsilon_i=\pm1}\int dx_3d\tau_3dx_4d\tau_4\langle T_\tau e^{i\sqrt{2}\phi_\sigma(r_1)}e^{-i\sqrt{2}\phi_\sigma(r_2)}\times e^{i\epsilon_3\sqrt{8}\phi_\sigma(r_3)}e^{-i\epsilon_4\sqrt{8}\phi_\sigma(r_4)}\rangle$$
$$+\frac{1}{8}\frac{D_\xi}{(\pi\alpha)^2}\sum_{\epsilon_i=\pm1}\int dx_3d\tau_3dx_4d\tau_4\delta(x_3-x_4)\times\langle T_\tau e^{i\sqrt{2}\phi_\sigma(r_1)}e^{-i\sqrt{2}\phi_\sigma(r_2)}e^{i\epsilon_3\sqrt{2}\phi_\sigma(r_3)}e^{-i\epsilon_4\sqrt{2}\phi_\sigma(r_4)}e^{i\epsilon_5[\sqrt{2}\phi_\rho(r_3)-\sqrt{2}\phi_\rho(r_4)]}\rangle , \tag{A2}$$

where r_i denotes (x_i,τ_i) and F_ν is the correlation function in the absence of interactions. We have

$$F_\nu(r_1-r_2)=\frac{K_\nu}{2}\ln\left[\frac{(x-x')^2+[u_\nu(\tau-\tau')]^2}{\alpha^2}\right]+d_\nu\cos(2\theta_{\nu,r_1-r_2}) , \tag{A3}$$

where $\theta_{\nu,r}$ is the angle between the vector $(x,u_\nu\tau)$ and the x axis, and d_ν parametrizes the anisotropy between the space and time directions. In the original Hamiltonian d_ν is zero but will be generated during the renormalization. After a straightforward calculation we find

$$R_\sigma(r_1-r_2)=e^{-F_\sigma(r_1-r_2)}\left[1+2\pi I_+(r_1-r_2)\left[\frac{g_{1\perp}}{(2\pi)^2u_\sigma}\right]^2K_\sigma^2\int_\alpha^{+\infty}dr\frac{1}{\alpha}\left[\frac{r}{\alpha}\right]^{3-4K_\sigma}+\left[\frac{u_\sigma}{u_\rho}\right]^{K_\rho}\left[\frac{D_\xi\alpha}{8\pi^2}\right]K_\sigma^2[I_+(r_1-r_2)-I_-(r_1-r_2)]J\right] \tag{A4}$$

with

$$I_\pm(r_1-r_2)=\int d^2R\,\ln|r_1-R|\,(\partial_X^2\pm\partial_Y^2)\ln|R-r_2| ,$$
$$J=\frac{2}{u_\sigma^2}\int_\alpha^{+\infty}dr\frac{1}{\alpha}\left[\frac{r}{\alpha}\right]^{2-K_\sigma-K_\rho} . \tag{A5}$$

The integrals in (A5) are

$$I_+(r_1-r_2)=2\pi\ln|r_1-r_2| ,$$
$$I_-(r_1-r_2)=\pi\cos(2\theta_{r_1-r_2}) . \tag{A6}$$

Equation (A4) is a first-order cumulant expansion of the correlation function (A1) (Ref. 35), and if one reexponentiates Eq. (A4), one recovers the original functional form, but with the effective quantities

$$K_\sigma^{\rm eff}=K_\sigma-\tfrac{1}{2}\mathcal{D}K_\sigma^2\int_\alpha^{+\infty}dr\frac{1}{\alpha}\left[\frac{r}{\alpha}\right]^{2-K_\sigma-K_\rho}-\tfrac{1}{2}y^2K_\sigma^2\int_\alpha^{+\infty}dr\frac{1}{\alpha}\left[\frac{r}{\alpha}\right]^{3-4K_\sigma} ,$$
$$d_\sigma^{\rm eff}=d_\sigma+\tfrac{1}{4}\mathcal{D}K_\sigma^2\int_\alpha^{+\infty}dr\frac{1}{\alpha}\left[\frac{r}{\alpha}\right]^{2-K_\sigma-K_\rho} . \tag{A7}$$

We have introduced

$$\mathcal{D}=\frac{2D_\xi\alpha}{\pi u_\sigma^2}\left[\frac{u_\sigma}{u_\rho}\right]^{K_\rho},$$
$$y=g_{1\perp}/(\pi u_\sigma). \tag{A8}$$

The correlation function R_ρ can be expanded in the same way,

$$R_\rho(r_1-r_2)=e^{-F_\rho(r_1-r_2)}+\frac{1}{8}\frac{D_\xi}{(\pi\alpha)^2}\sum_{\epsilon_i=\pm1}\int\int\int\int dx_3 d\tau_3 dx_4 d\tau_4\,\delta(x_3-x_4)$$
$$\times\langle T_\tau e^{i\sqrt{2}\phi_\rho(r_1)}e^{-i\sqrt{2}\phi_\rho(r_2)}e^{i\epsilon_3\sqrt{2}\phi_\sigma(r_3)}e^{-i\epsilon_4\sqrt{2}\phi_\sigma(r_4)}$$
$$\times e^{i\epsilon_5[\sqrt{2}\phi_\rho(r_3)-\sqrt{2}\phi_\rho(r_4)]}\rangle, \tag{A9}$$

and from this effective constants follow as

$$K_\rho^{\text{eff}}=K_\rho-\frac{u_\rho}{2u_\sigma}\mathcal{D}K_\rho^2\int_\alpha^{+\infty}dr\frac{1}{\alpha}\left[\frac{r}{\alpha}\right]^{2-K_\sigma-K_\rho},$$
$$d_\rho^{\text{eff}}=d_\rho+\frac{u_\rho}{4u_\sigma}\mathcal{D}K_\rho^2\int_\alpha^{+\infty}dr\frac{1}{\alpha}\left[\frac{r}{\alpha}\right]^{2-K_\sigma-K_\rho}. \tag{A10}$$

We have used a short-range cutoff so that $x^2+u_\sigma^2\tau^2\geq\alpha^2$. Under a change of the cutoff $\alpha\to e^l\alpha=\alpha+d\alpha$ we get the following renormalization equations:

$$\frac{dK_\rho}{dl}=-\frac{u_\rho}{2u_\sigma}K_\rho^2\mathcal{D}(l),\quad \frac{dK_\sigma}{dl}=-\left[\frac{\mathcal{D}(l)}{2}+\frac{y^2(l)}{2}\right]K_\sigma^2,$$
$$\frac{dy}{dl}=[2-2K_\sigma(l)]y(l),\quad \frac{d\mathcal{D}}{dl}=[3-K_\rho(l)-K_\sigma(l)]\mathcal{D}(l), \tag{A11}$$
$$\frac{dd_\rho}{dl}=\frac{u_\rho}{4u_\sigma}K_\rho^2\mathcal{D}(l),\quad \frac{dd_\sigma}{dl}=\tfrac{1}{4}K_\sigma^2\mathcal{D}(l).$$

The renormalization of d_ν is equivalent to a renormalization of the velocities u_ν [cf. Eq. (A3)],

$$\frac{1}{u_\nu}\frac{du_\nu}{dl}=-\frac{2}{K_\nu}\frac{dd_\nu}{dl}. \tag{A12}$$

As the anisotropy parameter is of the first order in $\mathcal{D}$, the corrections to the u_ν can be neglected in the first of Eqs. (A11).

These equations do not preserve the symmetries of the original system: Obviously if we start the renormalization from a Hamiltonian invariant under spin rotation ($g_{2\parallel}=g_{2\perp}$ and $g_{1\parallel}=g_{1\perp}$) this symmetry has to be preserved during the renormalization, which is not the case for Eqs. (A11). The problem arises from the necessity to take into account renormalizations coming from a third-order term in the expansion of R_ν which is

$$R_\nu^{(\text{III})}(r_1-r_2)=-\frac{1}{8}\frac{g_{1\perp}}{(2\pi\alpha)^2}\frac{D_\xi}{(\pi\alpha)^2}\sum_{\epsilon_3,\epsilon_4,\epsilon_5,\epsilon_6}\int\int\int\int\int\int dx_3 d\tau_3 dx_4 d\tau_4 dx_5 d\tau_5\,\delta(x_4-x_5)$$
$$\times\langle T_\tau e^{i\sqrt{2}\phi_\nu(r_1)}e^{-i\sqrt{2}\phi_\nu(r_2)}e^{i\epsilon_3\sqrt{8}\phi_\sigma(r_3)}$$
$$\times e^{i\epsilon_4\sqrt{2}\phi_\sigma(r_4)}e^{-i\epsilon_5\sqrt{2}\phi_\sigma(r_5)}e^{i\epsilon_6[\sqrt{2}\phi_\rho(r_4)-\sqrt{2}\phi_\rho(r_5)]}\rangle. \tag{A13}$$

Following the method of Nelson and Halperin,[36] only the case where two internal points of this term are close to a distance of α needs to be taken into account. Under renormalization this will give terms similar to those previously examined (we will call them "contractions"). Only two contractions are possible (see Fig. 6).

(1) If the points $(x_4,u_\sigma\tau_4)$ and $(x_5,u_\sigma\tau_5)$ are close together in a ring of inner radius α and width $d\alpha$ the element of integration becomes

$$\int\int\int\int\int\int dx_3 d\tau_3 dx_4 d\tau_4 dx_5 d\tau_5\,\delta(x_4-x_5)\to\frac{2d\alpha}{u_\sigma}\int\int\int\int dx_3 d\tau_3 dx_4 d\tau_4. \tag{A14}$$

When contracted and summed over ϵ_6 the

$$\langle e^{i\epsilon_6[\sqrt{2}\phi_\rho(r_4)-\sqrt{2}\phi_\rho(r_5)]}\rangle = \exp\left[-F_\rho\left[0,\frac{\alpha}{u_\sigma}\right]\right]$$

part gives a factor $2(u_\sigma/u_\rho)^{K_\rho}$, and from the eliminated degrees of freedom we get a contribution,

$$-\frac{d\alpha}{2u_\sigma}\left[\frac{u_\sigma}{u_\rho}\right]^{K_\rho}\frac{g_{1\perp}}{(2\pi\alpha)^2}\frac{D_\xi}{(\pi\alpha)^2}\sum_{\epsilon_i=\pm1}\int\int\int\int dx_3 d\tau_3 dx_4 d\tau_4\langle T_\tau e^{i\sqrt{2}\phi_\nu(r_1)}e^{-i\sqrt{2}\phi_\nu(r_2)}e^{i\epsilon_3\sqrt{8}\phi_\sigma(r_3)}e^{-i\epsilon_4\sqrt{8}\phi_\sigma(r_4)}\rangle\ , \tag{A15}$$

which is similar to a $g_{1\perp}^2$ term and contributes only to R_σ.

(2) If the points (x_3,τ_3) and (x_4,τ_4) [or (x_5,τ_5)] are close together the element of integration becomes

$$\int\int\int\int\int\int dx_3 d\tau_3 dx_4 d\tau_4 dx_5 d\tau_5\delta(x_4-x_5)\to\frac{2\pi\alpha d\alpha}{u_\sigma}\int\int\int\int dx_4 d\tau_4 dx_5 d\tau_5\delta(x_4-x_5)\ . \tag{A16}$$

As there are two contractions $[(x_3,\tau_3)\to(x_4,\tau_4)$ and $(x_3,\tau_3)\to(x_5,\tau_5)]$ we have a factor 2 and we get a contribution

$$-\frac{\pi\alpha d\alpha}{2u_\sigma}\frac{g_{1\perp}}{(2\pi\alpha)^2}\frac{D_\xi}{(\pi\alpha)^2}\sum_{\epsilon_i=\pm1}\int\int\int\int dx_4 d\tau_4 dx_5 d\tau_5\delta(x_4-x_5)$$

$$\times\langle T_\tau e^{i\sqrt{2}\phi_\nu(r_1)}e^{-i\sqrt{2}\phi_\nu(r_2)}e^{i\epsilon_4\sqrt{2}\phi_\sigma(r_4)}e^{-i\epsilon_5\sqrt{2}\phi_\sigma(r_5)}e^{i\epsilon_6[\sqrt{2}\phi_\rho(r_4)-\sqrt{2}\phi_\rho(r_5)]}\rangle\ , \tag{A17}$$

which is similar to a D_ξ term and contributes both to R_σ and to R_ρ. If one takes into account these terms in the renormalization equations, one finds that the last two equations in (A11) become

$$\frac{dy}{dl}=(2-2K_\sigma)y-\mathcal{D}\ ,$$
$$\frac{d\mathcal{D}}{dl}=(3-K_\rho-K_\sigma-y)\mathcal{D}\ , \tag{A18}$$

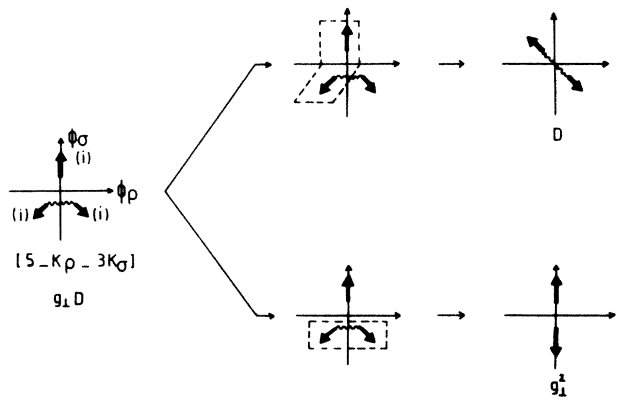

FIG. 6. We represent the correlation functions by analogy with a gas of vectorial charges. Only "neutral" ($\sum \mathbf{q}$) terms are not zero. The wiggly line indicates that the two operators are at the same time. Nonlinked arrows are at different positions (space and time). The box denotes the contraction of the two operators.

which are now spin-rotation invariant.

For $g_1<0$ the spin degrees of freedom are frozen. We will use the same expansion of R_ρ [cf. Eq. (A9)]. But if $g_1<0$ we have for large r_1-r_2,[15]

$$\langle T_\tau e^{i\sqrt{2}\phi_\sigma(r_1)}e^{-i\sqrt{2}\phi_\sigma(r_2)}\rangle\simeq C_\sigma\ . \tag{A19}$$

If we use the cutoff $[x^2+(u_\rho\tau)^2]\geq\alpha$ the effective constants are

$$K_\rho^{\text{eff}}=K_\rho-\frac{\mathcal{D}}{2}K_\rho^2\int_\alpha^{+\infty}dr\frac{1}{\alpha}\left[\frac{r}{\alpha}\right]^{2-K_\rho}\ ,$$
$$d_\rho^{\text{eff}}=d_\rho+\frac{\mathcal{D}}{4}K_\rho^2\int_\alpha^{+\infty}dr\frac{1}{\alpha}\left[\frac{r}{\alpha}\right]^{2-K_\rho}\ , \tag{A20}$$

where we have introduced

$$\mathcal{D}=\frac{2C_\sigma D_\xi\alpha}{\pi u_\rho^2}\ . \tag{A21}$$

The renormalization equations then are

$$\frac{dK_\rho}{dl}=-\frac{K_\rho^2}{2}\mathcal{D}\ ,$$
$$\frac{d\mathcal{D}}{dl}=(3-K_\rho)\mathcal{D}\ , \tag{A22}$$
$$\frac{dd_\rho}{dl}=\frac{K_\rho^2}{4}\mathcal{D}\ .$$

APPENDIX B

We want to obtain the asymptotic behavior of the correlation length close to the transition. The renormalization equations are [cf. (3.4)]

$$\begin{aligned}\frac{d\eta}{dl}&=-C\mathcal{D}\ ,\\ \frac{dy}{dl}&=-y^2-\mathcal{D}\ ,\\ \frac{d\mathcal{D}}{dl}&=-(\eta+\tfrac{3}{2}y)\mathcal{D}\ ,\end{aligned}\tag{B1}$$

with $\eta=K_\rho-2$ and $C=K_\rho^2 u_\rho/2u_\sigma$. On the fixed line one has $\eta^*>0$, $y=0$; thus close to the critical surface we write $y=\eta u$, $\mathcal{D}=\eta^2 v$, and $\eta=e^t$. Equation (B1) becomes

$$\begin{aligned}\frac{du}{dt}&=\frac{1}{C}\left[\frac{u^2}{v}+1-Cu\right]\ ,\\ \frac{dv}{dt}&=\frac{1}{C}\left[1+\frac{3u}{2}-2Cv\right]\ .\end{aligned}\tag{B2}$$

We will solve these equations in the limit $u\ll 1$ and $v\ll 1$. We obtain from (B2)

$$\frac{dv}{dt}=\frac{1}{C}\ ,\tag{B3}$$

which comes to (the index 0 indicates initial values)

$$\eta=\eta_0 e^{C(v-v_0)}\ .\tag{B4}$$

Moreover we have from (B2),

$$\frac{du}{dv}=1+\frac{u^2}{v}\ .\tag{B5}$$

It is a Ricatti equation, the solution of which is

$$u(v)=\frac{z}{2}\left[\frac{AJ_1(z)+BN_1(z)}{AJ_0(z)+BN_0(z)}\right]\ ,\tag{B6}$$

where J and N are the Bessel functions, and $z=2\sqrt{v}$. In the delocalized regime the trajectories will flow to $u=v=0$, whereas in the localized regime the trajectories will cross the line $u=0$ at finite v. This corresponds to a zero of the numerator in (B6) for a value of z which is given by

$$AJ_1(z_c)+BN_1(z_c)=0\ .\tag{B7}$$

Using the expansion of the Bessel functions for small values of the argument we obtain $z_c^2\simeq 4\epsilon/\pi$ (where $\epsilon=B/A$). The value ϵ is imposed by the initial conditions, and using (B6) we find

$$\epsilon=\frac{\pi}{4}\left[\frac{z_0^2-4u_0}{1+2u_0\ln(z_0/2)}\right]\ .\tag{B8}$$

The phase boundary between the delocalized and localized regimes is given by $\epsilon=0$ ($\epsilon>0$ is for the delocalized phase, $\epsilon<0$ for the localized one). This gives $z_0^2-4u_0=0$. When expressed in terms of $\mathcal{D}$ and η the boundary is given by

$$\mathcal{D}=\eta y\ .\tag{B9}$$

The localization length is given by the length to which we have to renormalize until the parameters reach a given point in the localized phase (the precise point introduces only finite corrections and does not affect the asymptotic behavior close to the critical surface). Suppose we stop the renormalization at a point $u=u_f<0$. We have, using (B6) and the development of the Bessel functions

$$2u_f=\frac{z_f^2-4\epsilon/\pi}{2+(4\epsilon/\pi)\ln(z_f/2)}\ .\tag{B10}$$

If we are close to the critical surface $\epsilon\simeq 0$, and (B10) gives

$$v_f\propto e^{-\pi/\epsilon}\ ,\tag{B11}$$

then from (B4) and $d\eta/dl=-C\mathcal{D}$ we deduce

$$\frac{dv}{v}=-\eta_0 dl\ .\tag{B12}$$

As $L_{\rm loc}=e^l$, using (B11), we have

$$L_{\rm loc}=\exp\left[\frac{\eta_0}{\mathcal{D}_0-y_0\eta_0}\right]\ .\tag{B13}$$

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