Localized states and self-similar states of electrons on a two-dimensional Penrose lattice

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Eigenstates with an energy $E=2$ are analyzed for a tight-binding Schrödinger equation $-\sum_{\langle j,i\rangle}\psi_j=E\psi_i$ on a two-dimensional Penrose lattice. Two different kinds of eigenstates exist. One is strictly localized and the other is on certain strings of rhombuses with one three-edge vertex plus some additions. The latter tends to states whose support is self-similar and fractal with a dimension $\ln 2/\ln\tau$ on an infinite lattice. The fraction of eigenstates in the spectrum with $E=2$ is obtained exactly and is 6.8189%.

I. INTRODUCTION

The discovery of quasicrystals[1,2] has strongly stimulated theoretical activity. Much of the work is devoted to the structure and defects of these fascinating new phases. Several examples of quasicrystals had been known previously, such as one-dimensional (1D) Fibonacci lattices,[3] and 2D and 3D Penrose lattices.[4,5,6] Quasicrystals are rather high-symmetric systems with long-range order and self-similarity but without translational symmetry or periodicity. We expect that quasicrystals (or quasiperiodic systems) can show exotic physical properties which are quite different from those in crystal phases and amorphous phases.

Electronic and phonon eigenstates on a 1D Fibonacci lattice have been treated theoretically by several groups,[3,7–11] and the Cantor-set spectrum with zero Lebesgue measure and self-similarity of some wave functions are known. Compared with the 1D Fibonacci lattice, the 2D Penrose lattice has not been much investigated[12–18] and most of the work is based on numerical analysis. We should be very careful in discussing numerical results, because a new phenomenon is difficult to understand and often leads to some misunderstanding. In a recent work,[13] we argued that some of the eigenstates are critical and show power-law decay due to Conway's theorem. Sutherland defined a Hamiltonian for a 2D Penrose lattice whose ground-state wave function is critical and non-normalizable.[15] Recently we also defined a Hamiltonian and obtained an eigenstate which is not the ground state and either is critical or extends over all sites without decaying.[19] From these experiences, we believe that the spectrum may be very complicated and multifractal. We realize the importance of exact solutions for an unexplored field such as quasicrystals.

In this paper, we demonstrate new exact solutions of the electronic states of a standard tight-binding Hamiltonian on a 2D Penrose lattice. In Sec. II, the Hamiltonian is defined, and the existence of strictly localized states (confined states) is shown at an energy $E=2$. We analyze the supporting region of these localized states in Sec. III and introduce another state (string state) whose supporting region is a string of rhombuses with one three-edge vertex. The support of the string states is self-similar and fractal. The wave functions are explicitly shown. The fraction of these two kinds of states and the fractal dimension of the string states are evaluated in Sec. IV. Section V is a summary.

II. CONFINED STATES

We now define a tight-binding Hamiltonian on a 2D Penrose lattice (the center model). The s-type atomic orbitals sit on rhombuses both wide and narrow. Electrons can transfer from one orbital to another on adjacent rhombuses and the transfer matrix element is assumed to be constant ($t=-1$). The resulting Schrödinger equation for the energy eigenfunction $|\Psi\rangle=\sum_i\psi_i\,|\,i\,\rangle$ with an energy E is

$$-\sum_{\langle i,j\rangle}\psi_j=E\psi_i\ , \tag{2.1}$$

where the summation is over four rhombuses j adjacent to the central one i. The energy spectrum extends from $E=-4$ ($\psi_i=1$) to $E\approx 2.6865$.[13]

The Schrödinger equation of the 2D Penrose lattice gives unusual states at $E=+2$, found for the first time in a system of 440 rhombuses by Semba and Ninomiya.[12] They argued that the degeneracy of the states at $E=+2$ is proportional to the system size and estimated the fraction as $N_0/N\geq 0.004\,29$. We call these infinitely degenerate localized states at $E=+2$ "confined states" because we observed, by numerical calculation, that the extent of the states is confined in some special regions.[13] In another 2D Penrose lattice (the vertex model), where atomic orbitals sit on vertices of rhombuses, Kohmoto and Sutherland found infinitely degenerate localized states at $E=0$.[14]

In Fig. 1, we show a wave function of one of the confined states. We refer to this as an $A1$ state. Of

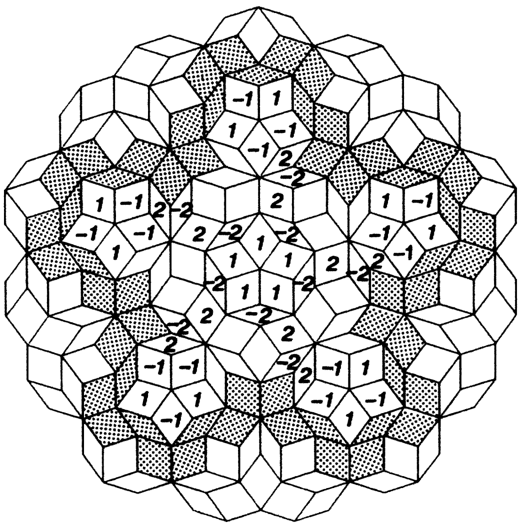

FIG. 1. One of the irreducible confined states, the $A1$ state. Amplitudes are shown on each rhombus without normalization.

course, there should be another very similar state ($A2$ state) in which rhombuses with nonzero wave-function amplitude are on the opposite side of the $S3$ vertices. There are eight different vertices (Q, D, K, S, $S5$, J, $S4$, $S3$), named by de Bruijn.[5] The $S3$ vertex is that at which seven edges (two double-arrow and five single-arrow edges) join. These $A1$ and $A2$ states have a vanishing angular momentum around the central S vertex. The vertex S is that at which five double-arrow edges join.

Another typical example is shown in Fig. 2(a) and has an angular momentum m ($m=0$, 1, 2, 3, and 4), which is just the state found by Semba and Ninomiya. The supporting region of this state is doubly connected and we call these sorts of states "reducible," because we can have some "irreducible" confined states whose supporting region is minimal and singly connected by choosing an appropriate linear combination of these states. We show an irreducible confined state, referred to as a B state, in Fig. 2(b) obtained as a linear combination of states in Fig. 2(a). These irreducible confined states can be found in another region and two of them are shown in Fig. 3. One has the same pattern of the supporting region as the B state and we also call this state a B state. The other state in Fig. 3 has a different pattern of the supporting region from the $A1$, $A2$, and B states, and we refer to this as a C state. It should be noted that appropriate linear combinations of five B states and five C states form ten reducible states folding around the inner shaded region of Fig. 3. They can be simultaneous eigenstates of the $2\pi/5$ rotation around the central S vertex.

In Fig. 4, we show two other irreducible states. One of them is a B state. The other state has a different pattern of the supporting region and we refer to this as a D state. Though its supporting region is multiply connected around an $S5$ vertex, the D state is irreducible. There are five B states and five D states in the unshaded region of Fig. 4. Some linear combinations of these states can be eigenstates of the $2\pi/5$ rotation around the central $S5$ vertex and these eigenstates are folding around the inner shaded region.

In the preceding part, we found several irreducible confined states. Once we construct simultaneous eigenstates of the $2\pi/5$ rotation and if they are reducible, they fold around a central unsupporting region. To specify in more detail the supporting region of confined

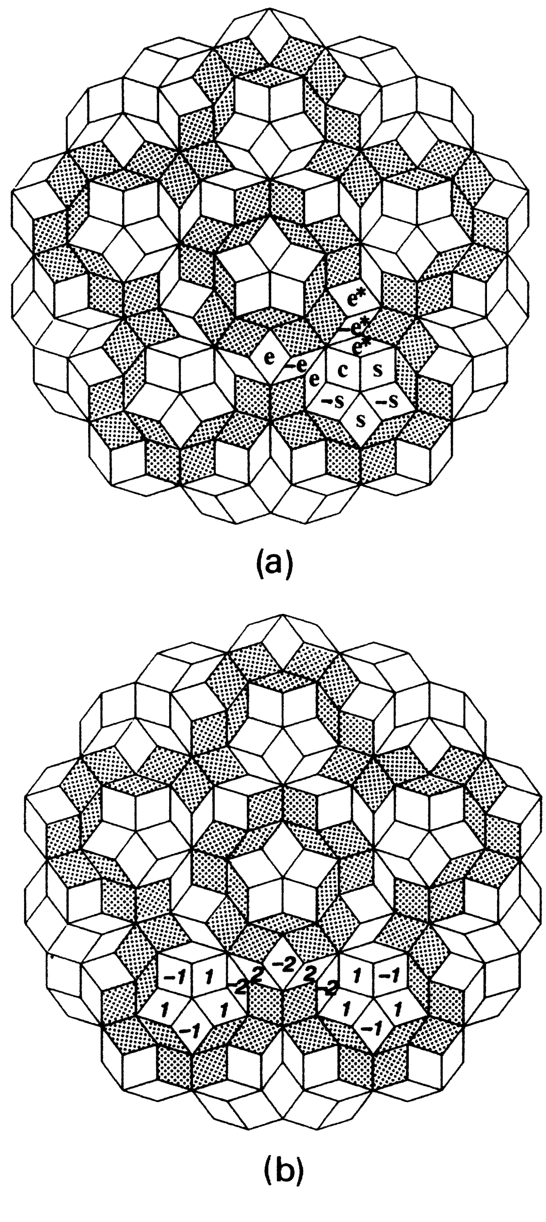

FIG. 2. (a) The reducible confined states found by Semba and Ninomiya. The amplitude shown is denoted by $s=i\sin\phi$, $c=-\cos\phi$, $e=\exp(i\phi)$, $e^*=\exp(-i\phi)$, with $\phi=m\pi/5$ ($m=0$, 1, 2, 3, and 4). Others in an inner region are not shown but can be determined in a (clockwise) cyclic way with an additional phase factor $\exp(2i\phi)$. (b) Another irreducible confined state (B state) obtained by a linear combination of the states in Fig. 2(a).

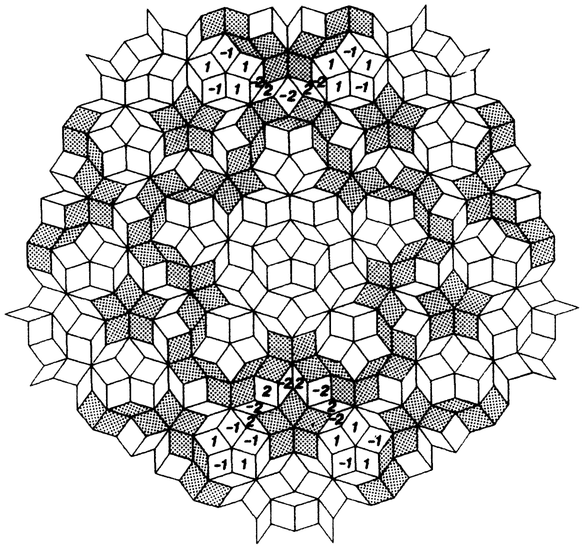

FIG. 3. Examples of other irreducible confined states around S vertices (B state and C state).

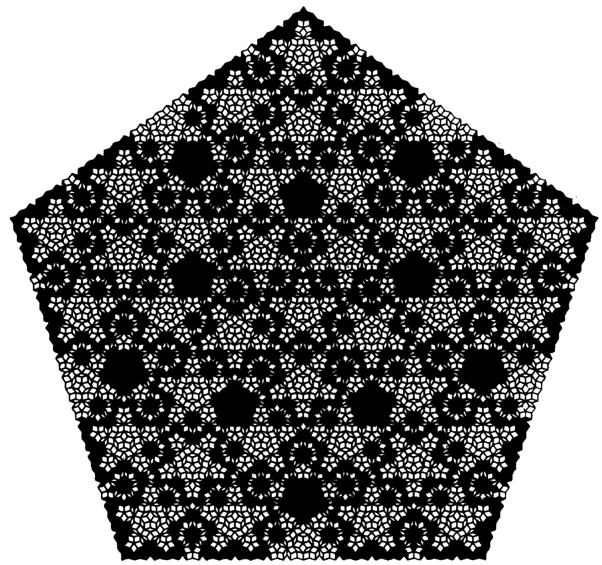

FIG. 5. The unsupporting region for confined states is shaded with black. The Dirichlet boundary condition is imposed on a system of 6545 rhombuses.

states, we calculated all eigenstates with $E=+2$ on the Penrose lattice of 6545 rhombuses under the Dirichlet boundary condition and found that there are some specific clusters and strings of rhombuses on which any confined state is forbidden (Fig. 5). There are two different kinds of such clusters, both of which are fivefold symmetric around $S5$ vertices. One is of five wide rhombuses and the other is of 30 wide and 15 narrow rhombuses. One-dimensional closed strings consisting of rhombuses with only one three-edge vertex (the D or Q vertices) are also forbidden regions with some exceptions. For example, the $A1$ and $A2$ confined states extend across one of these strings. The aforementioned forbidden clusters are also thought to be such strings. These forbidden regions in the system of 6545 rhombuses are shaded with black in the figure. We will find another exceptional case of strings which support new states with $E=2$.

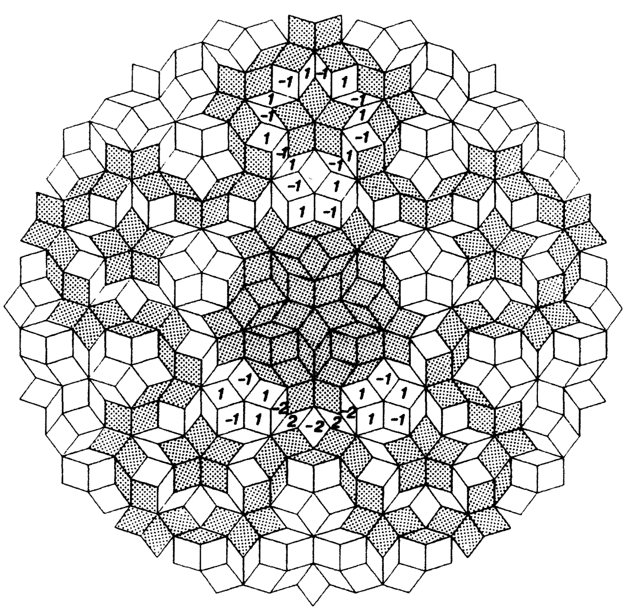

FIG. 4. Two examples of irreducible confined states around the $S5$ vertex (B state and D state).

III. IRREDUCIBLE CONFINED STATES, STRING STATES, AND THEIR SUPPORTING REGIONS

There are seven different nearest configurations for rhombuses, four for wide rhombuses and three for narrow, shown in Fig. 6. Every rhombus has one or two three-edge vertices (the Q vertex with three double-arrow edges and the D vertex with two single-arrow and one double-arrow edges). We can observe the following.

(1) There are four configurations in which a rhombus has one three-edge vertex, i.e., (b), (c), (d), and (e) in Fig. 6. (2) It is impossible that both ends of an arrow are three-edge vertices. (3) A tail end of a single arrow is never a three-edge vertex. (4) One of the 144° vertices of a narrow rhombus (i.e., the top end of the single-arrow edge of a narrow rhombus) is always the D vertex and a three-edge vertex. (5) There are only three cases in which neither side of a double arrow is a three-edge vertex [(c), (d), and (e)]. (6) There is no isolated rhombus with two three-edge vertices.

From the above considerations, we have the following theorem.

Theorem 1. A rhombus with one three-edge vertex always has two (and only two) nearest-neighboring rhombuses with one three-edge vertex. This implies that all rhombuses with one three-edge vertex are connected with strings and there are no end points in the strings.

It can be easily shown that, on a 10-rhombus cluster (five wide and five narrow) around an S vertex and on a

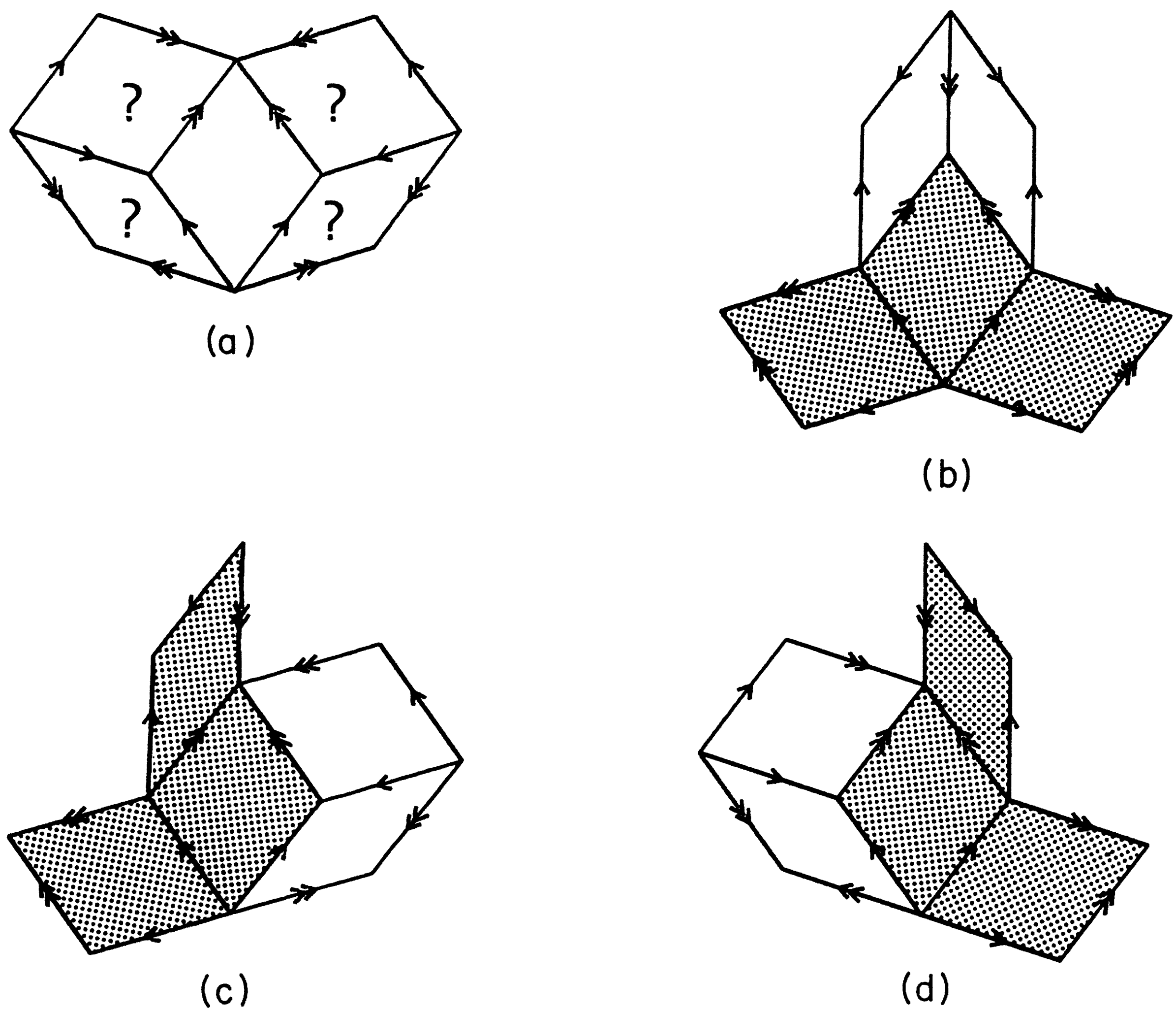

FIG. 6. Nearest-neighbor configurations of wide and narrow rhombuses. Shaded and unshaded rhombuses have one and two three-edge vertices, respectively. The number of three-edge vertices cannot be known for those with question marks.

30-rhombus cluster (twenty wide and ten narrow) around an $S5$ vertex, three adjacent rhombuses, a, b, and c, meeting at a three-edge vertex should always satisfy the equation for $E=+2$,

$$\psi_a+\psi_b+\psi_c=0 \ . \tag{3.1}$$

For a rhombus 0 with adjacent neighbors 1–4, we have from Eq. (2.1)

$$2\psi_0+\psi_1+\psi_2+\psi_3+\psi_4=0 \ . \tag{3.2}$$

Any three-edge vertex can be approached from some S or $S5$ vertex by going along rhombuses with two three-edge vertices. Furthermore, two three-edge vertices of one rhombus do not share the same edge. Therefore, from Eqs. (3.1) and (3.2), we can obtain the next theorem.

Theorem 2. Three adjacent rhombuses a, b, and c meeting at any three-edge vertex satisfy the equation for $E=+2$,

$$\psi_a+\psi_b+\psi_c=0 \ . \tag{3.3}$$

A rhombus with one three-edge vertex has two neighboring rhombuses with one three-edge vertex and no two of these three rhombuses share a three-edge vertex. (See Fig. 6.) We number the rhombuses in a string mentioned in theorem 1 in a sequential way as $1,2,\ldots,N$, and will refer to this 1D arrangement of rhombuses simply as the "string." From Eqs. (3.2) and (3.3), the wave function on a string with $E=+2$ should satisfy the equation

$$\psi_{l-1}+\psi_l+\psi_{l+1}=0 \quad (l=1,2,\ldots,N;\ \psi_{N+1}=\psi_1) \ , \tag{3.4}$$

where N is the total number of rhombuses in a string.

The solution of Eq. (3.4) can be assumed as

$$\psi_l^{(m)}=\psi^{(m)}\exp[i2\pi(m/N)l], \quad m=1,2,\ldots,N \ . \tag{3.5}$$

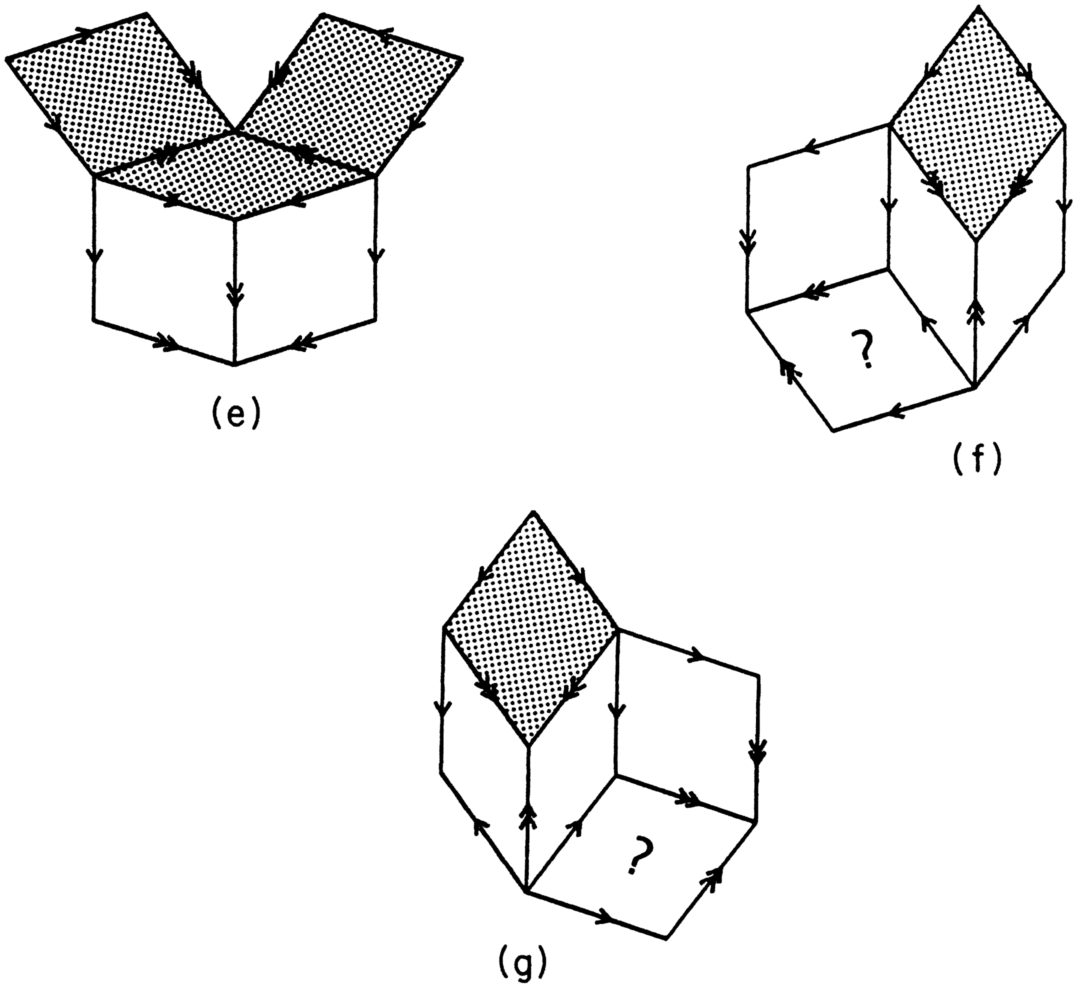

FIG. 6. (*Continued*).

Substituting Eq. (3.5) into Eq. (3.4), we can obtain

$$N=3m \text{ or } 2N=3m\ , \tag{3.6}$$

otherwise we have a trivial solution $\psi_l=0$ for all l. Then the wave function of a nontrivial solution on the string should be

$$\psi_l=\psi_0\exp\left[i\frac{2\pi}{3}l\right] \text{ or } \psi_0\exp\left[i\frac{4\pi}{3}l\right]\ , \tag{3.7}$$

which are zero-angular-momentum solutions.

The strings are generated by the deflation of the smallest strings around the S or $S5$ vertices. In the following discussion, we refer to the smallest strings around S and $S5$ vertices as the first-order S and zeroth-order $S5$ strings, respectively. It must be noticed that five wide rhombuses around an S vertex do not really constitute a string because these five rhombuses have two three-edge vertices. The n-times twofold deflations of the first-order S string and the zeroth-order $S5$ string, respectively, generate the $(n+1)$-st S and nth $S5$ strings. We denote the numbers of rhombuses of these strings by $R^{(S)}$ and $R^{(S5)}$. By successive twofold deflations, we can get the recursion relations

$$R_{n+1}^{(S)}=4R_n^{(S)}-5,\quad R_1^{(S)}=15\ , \tag{3.8a}$$

$$R_{n+1}^{(S5)}=4R_n^{(S5)}+5,\quad R_0^{(S5)}=5\ , \tag{3.8b}$$

and easily solve them as

$$R_n^{(S)}=\tfrac{5}{3}(2^{2n+1}+1)\quad (n\geq 1)\ , \tag{3.9a}$$

$$R_n^{(S5)}=\tfrac{5}{3}(2^{2n+2}-1)\quad (n\geq 0)\ . \tag{3.9b}$$

The nontrivial solutions with $E=+2$ on the strings can appear on the $(3m+1)$-st S strings and the $(3m+2)$-th $S5$ strings ($m=0,1,2,\ldots$) from Eqs. (3.6), (3.9a), and (3.9b).

By the inflation-deflation rule, we found two units, shown in Fig. 7, of which any nontrivial solution can be constructed with certain phases of wave functions. There exist two nontrivial solutions corresponding to the two cases of Eq. (3.7) and we will call them "string states."

There are several possibilities of some additions to a string for the support of string states. We can easily understand that this ambiguity is not essential, because the additions are restricted on the regions of confined states B, C, and D. It must be mentioned that the largest shaded string in Fig. 5 is just the fourth-order S string, which supports the string states. However, in the numerical calculation, this string and several rhombuses on the boundary are shaded, i.e., unsupporting regions. This is simply due to the Dirichlet boundary condition of our numerical calculation. The confined states $A1$ and $A2$ can be classified as string states on a first-order S string but cannot be constructed by units in Fig. 7.

Theorem 3. There are two and only two (nontrivial, zero-angular momentum) string states with $E=+2$ on each $(3m+1)$-st S string and $(3m+2)$-th $S5$ string. Other strings can never support any states with $E=+2$. The wave functions of string states can be constructed of two units in Fig. 7 except on the first-order S string.

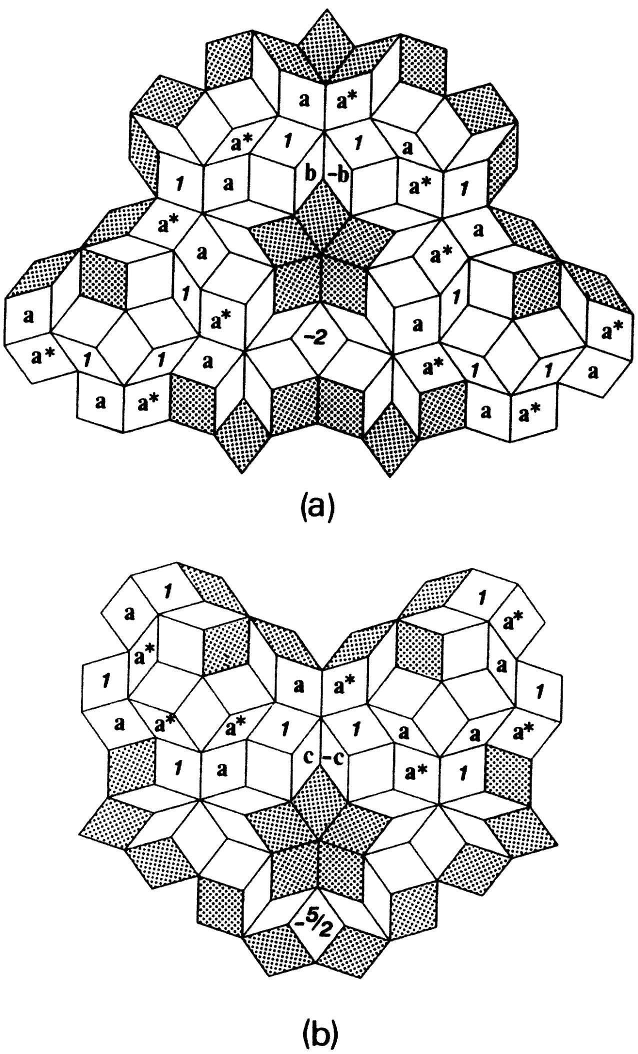

FIG. 7. Two units of string states. The factor a is either $\exp(i2\pi/3)$ or $\exp(i4\pi/3)$, $a+a^*+1=0$, $b=2(a-a^*)$, and $c=\frac{1}{2}(a-a^*)$. The shaded rhombuses have vanishing wave functions and others are shown explicitly or uniquely determined. The strings are 1D chains of rhombuses with numbers 1, a, and a^*. An appropriate common phase factor is not shown here. The string states can be constructed of these two units, overlapping S vertices with each other.

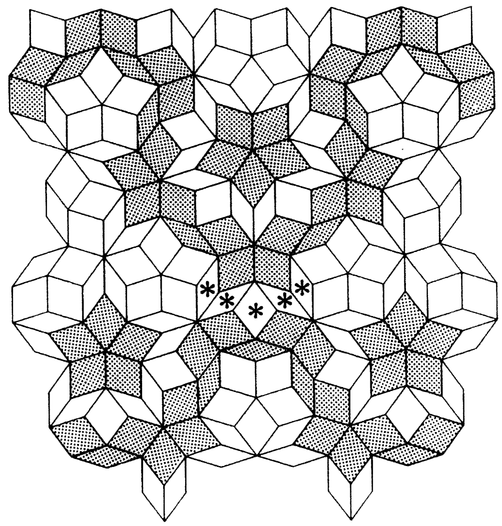

FIG. 8. An example of a bridge (marked by an asterisk) connecting two subparts, between nth and $(n+1)$-st strings. Shaded regions are strings.

Before closing this section, we present one more important theorem which will help to clarify the term "irreducible."

Theorem 4. All possible eigenstates with $E=+2$ are covered by the confined states $A1$, $A2$, B, C, and D and the string states.

This can be proved in the following way. Any region caught between successive nth and $(n+1)$-st strings can be divided into five smaller equivalent regions by cutting "bridges" of rhombuses. (See Figs. 5 and 8.) These bridges are just the central parts of the B, C, or D confined states. The smaller subparts are decorated by other bridges which are also the central parts of B, C, or D confined states. The component of a wave function on a bridge is uniquely determined except for an appropriate constant phase factor. Therefore, if we choose a wave function of confined states on a region between two successive strings to be orthogonal to those of B, C, and D confined states on the bridges, this is equivalent to imposing a condition that the wave function vanishes on the bridges. After discarding five bridges and several additions to B, C, and D confined states, five smaller subparts remain and each of them is identical with a region surrounded by an nth string. Therefore, it is not necessary to consider larger regions except for string states already examined completely.

IV. NUMBER OF $E=2$ STATES AND FRACTAL DIMENSION OF THE $N_R \to \infty$ STRING STATE

In order to understand the hierarchical structure of the lattice, the original tiling of Penrose,[4] shown in Fig. 9, is convenient. There are four prototiles in it, pentagon, five-pointed star, three-pointed half-star, and diamond. There are three different pentagonal pieces. The unit prototiles are defined as the first-order ones.

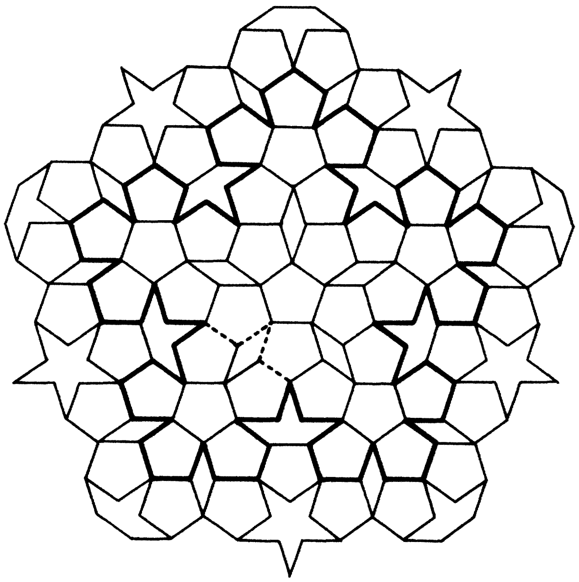

FIG. 9. Tiling of six prototiles by Penrose (Ref. 4). The unit prototiles are defined as the first-order ones. $\{S_3\}$ and B_2 are shown by bold solid and dashed lines, respectively.

A narrow rhombus in a string shares a K vertex (at which four double-arrow edges join) with two wide rhombuses in the string [(e) in Fig. 6]. There exists one other wide rhombus meeting at this K vertex, which is not in the string. Adding a cluster of five wide rhombuses around an S vertex, we can get the result that all wide rhombuses are connected by strings and that there are no end points. We will call them nth-order S rings and $S5$ rings in order to distinguish them from strings of rhombuses with one three-edge vertex. The rings are almost identical with the strings shown in Fig. 5. The numbers of wide rhombuses in rings around S and $S5$ vertices are also given by Eqs. (3.8a) and (3.8b), respectively.

It must be noticed that the hierarchical structure of rings is just the same as the structure in Fig. 9. Each edge of the nth-order pentagon (nth-order boundary) can be understood as a sum of two edges of $(n-1)$-st-order pentagons and an $(n-1)$-st-order wedge (a half-diamond). Let us introduce a new notation $\{\ \}$ such as $\{S_n\}$ or $\{S5_n\}$. The notation $\{S_n\}$ means an nth-order S ring denoted by S_n plus all rings within it. Furthermore we use notations $\{S'_n\}=\{S_n\}-S_n$, and B_n and W_n for a boundary and a wedge. The first-order pentagons and stars correspond to $\{S'_1\}$ and $\{S5'_1\}$, respectively. A wedge W_n includes rings within a half diamond but not the boundary ring, because the boundary ring is already taken account of in $\{S_n\}$ or $\{S5_n\}$. Then the hierarchical structures of rings are as follows:

$$\{S_0\}=S_0\ , \tag{4.1}$$

$$\{S_1\}=S_1+\{S_0\}\ , \tag{4.2}$$

$$\{S_n\}=S_n+\{S_{n-1}\}+5\{S'_{n-1}\}+5B_{n-1}\quad (n\geq 2)\ , \tag{4.3}$$

$$\{S5_0\}=S5_0\ , \tag{4.4}$$

$$\{S5_1\}=S5_1+\{S5_0\}\ , \tag{4.5}$$

$$\{S5_n\}=S5_n+\{S5_{n-1}\}+5W_n\quad (n\geq 2)\ , \tag{4.6}$$

$$B_n=2B_{n-1}+W_{n-1}\ , \tag{4.7}$$

$$W_n=2B_{n-1}+3W_{n-1}\ . \tag{4.8}$$

Each $(n-1)$-st-order diamond contains one $\{S'_{n-2}\}$, one $\{S5_{n-3}\}$, and one $\{S5_{n-2}\}$. The first two are in the $(n-1)$-st-order wedge (a half diamond). The last one uniquely corresponds to B_n. Therefore, we have additional generating rules,

$$W_n\rightarrow\{S'_{n-1}\}+\{S5_{n-2}\}\ , \tag{4.9}$$

$$B_n\rightarrow\{S5_{n-2}\}\ . \tag{4.10}$$

Narrow rhombuses are always isolated or paired, and are denoted by T_1 and T_2, respectively. Every S_0 (five wide rhombuses around an S vertex) is surrounded by five isolated narrow rhombuses; every W_1 contains one pair of narrow rhombuses and every B_1 converts two isolated narrow rhombuses (around an S_0) to two pairs of narrow rhombuses. Therefore, we have the following generating rule for narrow rhombuses:

$$S_0\rightarrow 5T_1\ , \tag{4.11}$$

$$W_1\rightarrow 1T_2\ , \tag{4.12}$$

$$B_1\rightarrow -2T_1+2T_2\ . \tag{4.13}$$

The fraction of rings can be evaluated by the above recursion relations but here we use the deflation procedure. We denote a number of X's by $N(X)$ and the total number of rhombuses by N_R. The generating rule of eight vertices can be seen by deflation and the fractions of $\{S_0\}$ and $\{S5_0\}$ are calculated as

$$N(\{S_0\})/N_R=\tau^{-4}/(1+\tau^2)\ , \tag{4.14}$$

$$N(\{S5_0\})/N_R=\tau^{-6}/(1+\tau^2)\ , \tag{4.15}$$

where $\tau=(\sqrt{5}+1)/2$. The nth-order rings can be generated from the $(n-1)$-st-order ones by a twofold deflation and, therefore, we get

$$N(\{S_n\})=\tau^{-4n}N(\{S_0\})\ , \tag{4.16}$$

$$N(\{S5_n\})=\tau^{-4n}N(\{S5_0\})\ . \tag{4.17}$$

One $A1$, one $A2$, and five B confined states appear in every $\{S'_2\}$ [Figs. 1 and 2(b)]. An $\{S5_0\}$ in an $\{S5_1\}$ (30 wide and 15 narrow rhombuses) does not carry any confined state. From Figs. 3 and 4, we can see that every other $\{S5_0\}$ carries one B confined state and one C or D confined state. Therefore, the total number of $A1$, $A2$, B, C, and D confined states can be evaluated as

$$7N(\{S'_2\})+2[N(\{S5_0\})-N(\{S5_1\})]\ . \tag{4.18}$$

Furthermore, the number of string states other than $A1$ and $A2$ confined states is

$$2\sum_{\substack{m\\m\geq 1}} N(\{S_{3m+1}\})+2\sum_{\substack{m\\m\geq 0}} N(\{S5_{3m+2}\}) . \quad (4.19)$$

By using the additional relation

$$N(\{S_2'\})=N(\{S_1\}) , \quad (4.20)$$

we obtain the following results. The fraction of confined states ($A1$, $A2$, B, C, and D) is equal to

$$\frac{7}{\tau^8(1+\tau^2)}+2\left[\frac{1}{\tau^6(1+\tau^2)}-\frac{1}{\tau^{10}(1+\tau^2)}\right]$$
$$=6.749\,483\,200\,4\times 10^{-2} , \quad (4.21)$$

and the fraction of string states (other than $A1,A2$) is equal to

$$2\left[\frac{1}{\tau^4(1+\tau^2)}\frac{\tau^{-16}}{1-\tau^{-12}}+\frac{1}{\tau^6(1+\tau^2)}\frac{\tau^{-8}}{1-\tau^{-12}}\right]$$
$$=6.944\,377\,661\,4\times 10^{-4} . \quad (4.22)$$

The fraction of degenerate states at $E=2$ is 6.818 927% in total.

Confined states and string states are localized and normalizable except for infinite-length string states. In the thermodymamic limit $N_R\to\infty$, this limiting operation generates two string states on the largest (infinite length) string. The wave function is not normalizable and extends from one end to the other. The support of this $N_R\to\infty$ string state is really self-similar and fractal. It is easily seen that the similarity ratios of one twofold deflation are τ^4 and 2^2 for the total number of rhombuses in a system and on its boundary. Therefore, the fractal dimension of the ($N_R\to\infty$) string states (a support of states) is

$$\ln(2^2)/\ln(\tau^2)=\ln 2/\ln\tau\cong 1.4404 . \quad (4.23)$$

The finite width of the support does not change the result.

V. SUMMARY

The eigenstates with $E=+2$ of the 2D Penrose center model are studied. Five irreducible confined states and string states are defined. We have proven that these states cover all eigenstates with $E=+2$. Two string states appear on every $(3m+1)$-st S string and $(3m+2)$-th $S5$ string. Any other state with $E=+2$ cannot be supported on strings of rhombuses with one three-edge vertex. The fractions of confined states and string states are exactly evaluated. The fractal dimension of the support of ($N_R\to\infty$) self-similar string states is $\ln 2/\ln\tau$.

ACKNOWLEDGMENT

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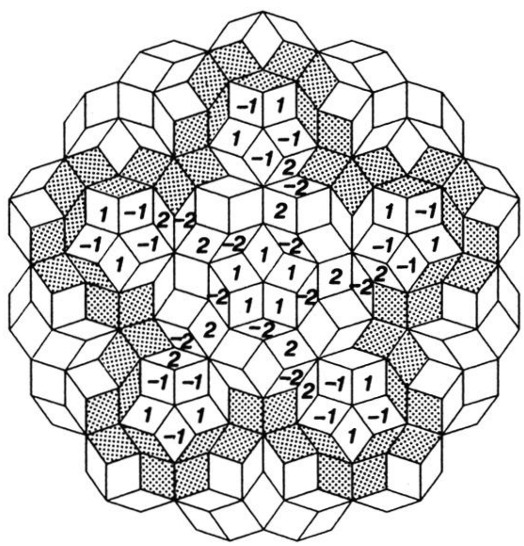

FIG. 1. One of the irreducible confined states, the $A1$ state. Amplitudes are shown on each rhombus without normalization.

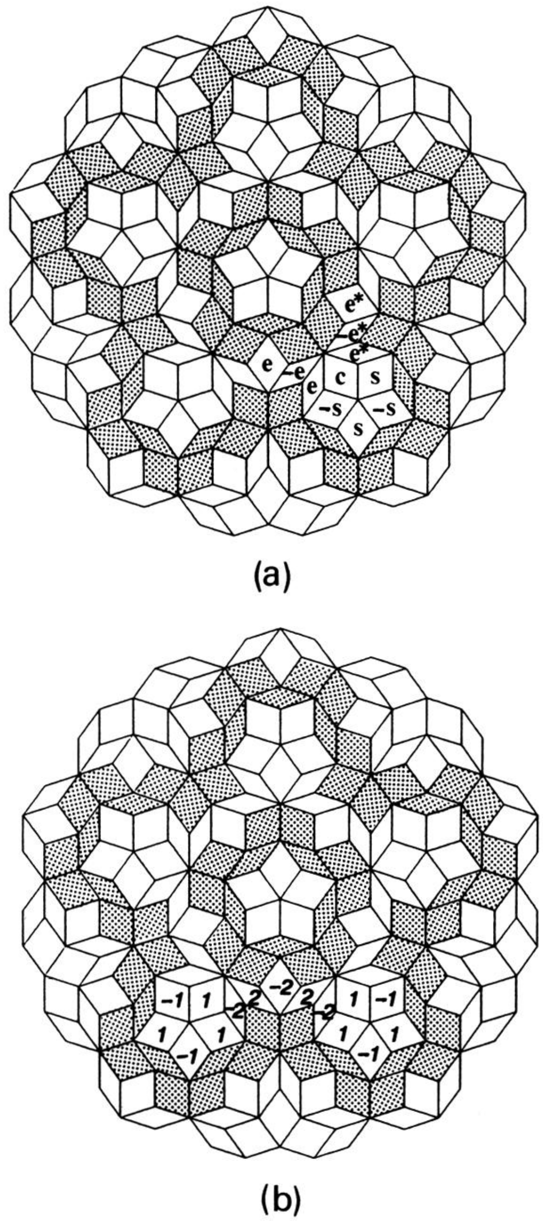

FIG. 2. (a) The reducible confined states found by Semba and Ninomiya. The amplitude shown is denoted by $s = i\sin\phi$, $c = -\cos\phi$, $e = \exp(i\phi)$, $e^* = \exp(-i\phi)$, with $\phi = m\pi/5$ ($m = 0$, 1, 2, 3, and 4). Others in an inner region are not shown but can be determined in a (clockwise) cyclic way with an additional phase factor $\exp(2i\phi)$. (b) Another irreducible confined state (*B* state) obtained by a linear combination of the states in Fig. 2(a).

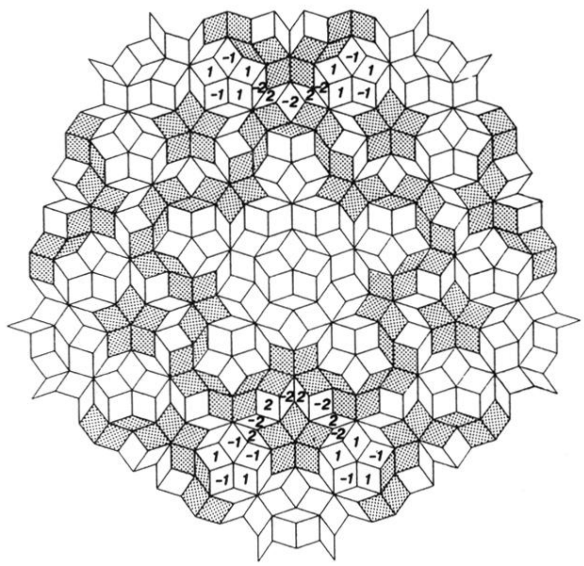

FIG. 3. Examples of other irreducible confined states around S vertices (B state and C state).

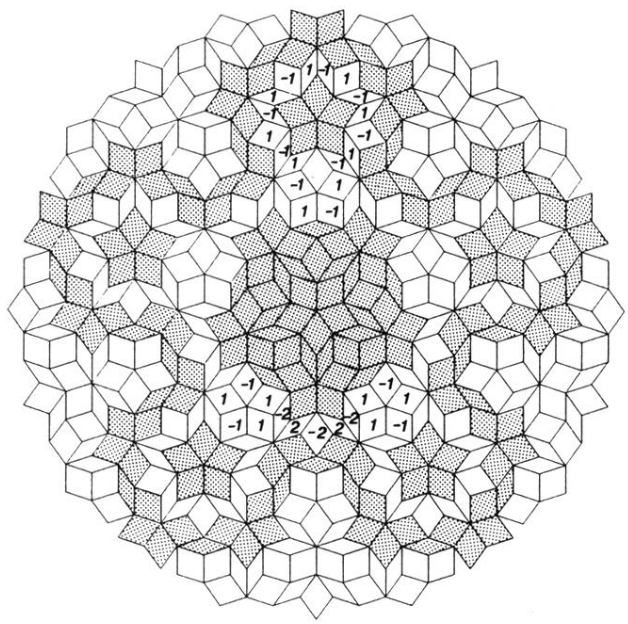

FIG. 4. Two examples of irreducible confined states around the $S5$ vertex (B state and D state).

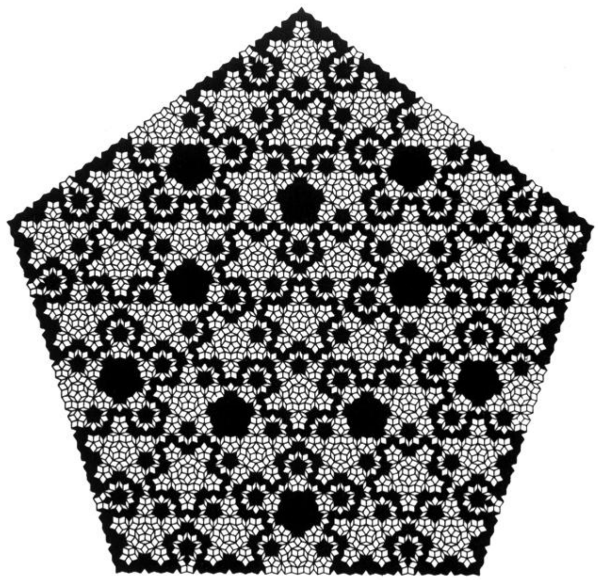

FIG. 5. The unsupporting region for confined states is shaded with black. The Dirichlet boundary condition is imposed on a system of 6545 rhombuses.

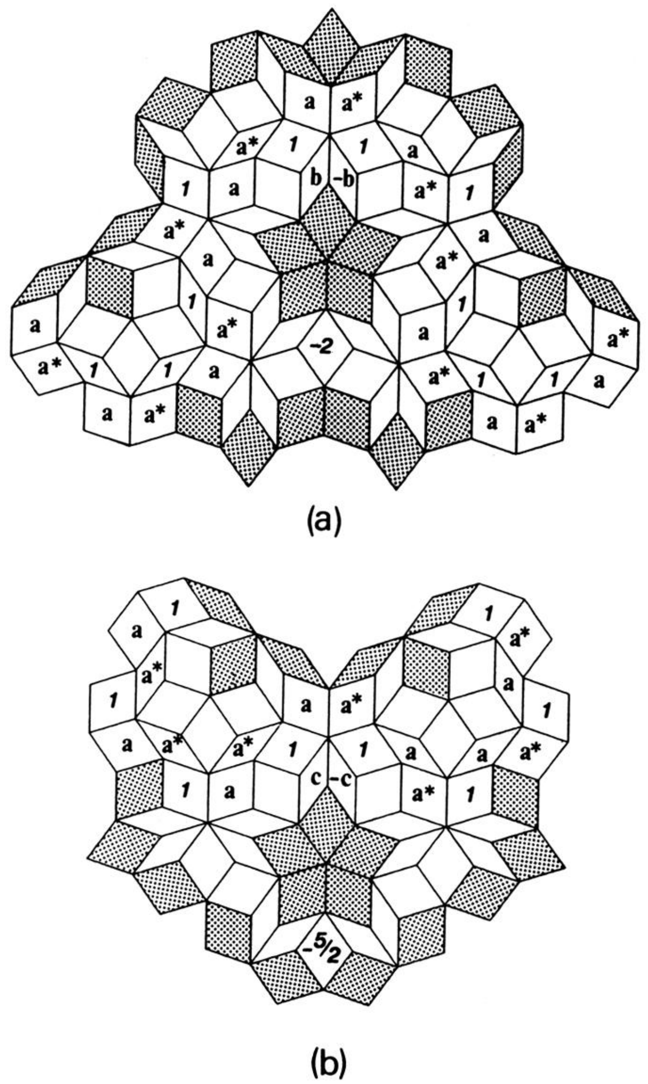

FIG. 7. Two units of string states. The factor a is either $\exp(i2\pi/3)$ or $\exp(i4\pi/3)$, $a+a^*+1=0$, $b=2(a-a^*)$, and $c=\frac{1}{2}(a-a^*)$. The shaded rhombuses have vanishing wave functions and others are shown explicitly or uniquely determined. The strings are 1D chains of rhombuses with numbers 1, a, and a^*. An appropriate common phase factor is not shown here. The string states can be constructed of these two units, overlapping S vertices with each other.

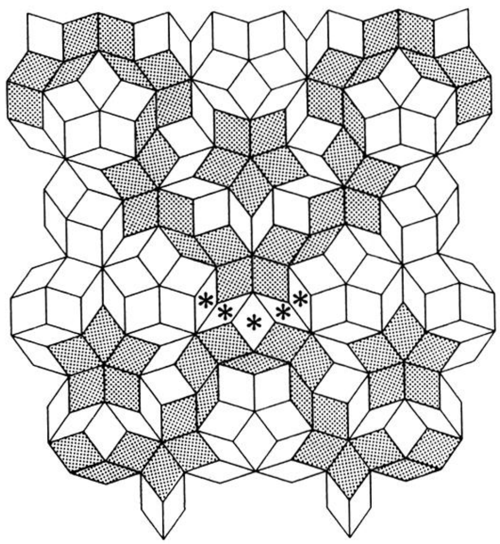

FIG. 8. An example of a bridge (marked by an asterisk) connecting two subparts, between nth and $(n+1)$-st strings. Shaded regions are strings.