Quasiparticle spectra and specific heat of a normal Fermi liquid in a spin-fluctuation model

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We calculate statistical and dynamical quasiparticle energies of a normal Fermi liquid in a spin-fluctuation model. The properties of spin fluctuations in the model are the same as those given by Landau Fermi-liquid theory at long wavelengths, low frequencies, and low temperatures if all Landau parameters except F_0^a are neglected. The statistical quasiparticle spectrum is found to be significantly less dependent on momentum and temperature than the dynamical quasiparticle one. The specific heat is calculated both from the statistical quasiparticle spectrum and from the thermodynamic potential, and for parameters appropriate for liquid ^{3}He, the corrections to the leading low-temperature result ($\propto T$) are well characterized by $T^3\ln T$ behavior up to temperatures of order 100 mK, which is in qualitative agreement with what Greywall finds experimentally. If the Landau parameter F_1^a is included in the calculation in addition to F_0^a, we find rather good agreement between theory and experiment. A further conclusion of our work is that the finite-temperature contributions to the specific heat are much less sensitive to variations of the cutoff momentum in our calculations than is the spin-fluctuation contribution to the effective mass. Spin fluctuations are therefore able to account for finite-temperature effects, even though they provide only a modest contribution to the effective mass. We also investigate the reason for earlier calculations in the paramagnon model giving very low estimates of the temperature below which the specific heat should exhibit $T^3\ln T$ behavior, and find that this is due to (i) the fact that the dynamical quasiparticle contribution to the specific heat was calculated and (ii) the use of the paramagnon model, rather than Landau theory.

I. INTRODUCTION

At low temperatures the specific heat at constant volume of normal liquid ^{3}He is much enhanced over that of a free Fermi gas. With increasing temperature the enhancement decreases rapidly, which indicates a strongly temperature-dependent effective mass. Doniach and Engelsberg[1] and Berk and Schrieffer[2] showed that this rapid temperature dependence can be accounted for by the coupling of quasiparticles to spin fluctuations, which gives rise to a contribution to the specific heat varying as $T^3\ln T$. In addition they argued that this mechanism could provide a plausible explanation for the value of the effective mass at zero temperature. The early calculations were done assuming that the thermodynamic properties could be determined using the ordinary quasiparticle expression and taking for the quasiparticle energies the poles of the single-particle propagator. Amit, Kane, and Wagner[3] showed that the nonanalytic terms in the quasiparticle energies defined by the poles of the single-particle propagator, the so-called dynamical quasiparticle spectrum, may be expressed in terms of Landau parameters. They then calculated the entropy by putting the dynamical quasiparticle spectrum into the quasiparticle expression for the entropy,

$$S=-k_B\sum_{\mathbf{p},\sigma}[n_{\mathbf{p}}\ln n_{\mathbf{p}}+(1-n_{\mathbf{p}})\ln(1-n_{\mathbf{p}})]\ , \tag{1}$$

where $n_{\mathbf{p}}$ is the equilibrium distribution function for quasiparticles. It was shown by Brenig *et al.*,[4] Riedel,[5] and Brinkman and Engelsberg[6] that in models closely related to the random-phase approximation, thermodynamic properties cannot be described by the dynamical quasiparticle spectrum alone, but that extra Bose terms must be considered. Carneiro and Pethick[7] generalized these results and showed that, as long as there are no real collective modes, it is unnecessary to introduce the Bose terms, and thermodynamic properties can be described in terms of another spectrum, the statistical quasiparticle spectrum, which had previously been considered by Balian and De Dominicis,[8] Luttinger and Liu,[9] and Luttinger.[10] They also showed that the entropy can be expressed in another way, as a sum of two terms. One is the expression for the entropy of a noninteracting Fermi gas evaluated using the dynamical quasiparticle spectrum and the other term comes from the on-shell scattering of quasiparticle-quasihole pairs, and includes the Bose term discussed in Refs. 4, 5, and 6. Carneiro and Pethick[7] justified this procedure for calculating the leading finite-temperature contributions to the specific heat using microscopic theory, but the problem of implementing the procedure of Balian and De Dominicis and of Luttinger at higher temperatures has not yet been solved.

The statistical quasiparticle spectrum is defined by

$$\varepsilon_{\mathbf{p}}^{\mathrm{st}}=\frac{\delta E[n_{\mathbf{p}}]}{\delta n_{\mathbf{p}}}\ , \tag{2}$$

where $E[n_{\mathbf{p}}]$, the energy of the system, is a functional of the distribution function. $\varepsilon_{\mathbf{p}}^{\mathrm{st}}$ is therefore the quasiparticle energy introduced originally by Landau.[11] In equilibrium at finite temperature the distribution function is determined by the Fermi-Dirac function in which the argument is the statistical quasiparticle spectrum, and therefore one has a set of self-consistency conditions for the spectrum and the distribution function. The specific heat is calculated by putting the statistical quasiparticle spectrum into the expression for the entropy of a free Fermi gas and differentiating. We may, in a similar way, define a dynamical quasiparticle contribution to the specific heat which is calculated using the dynamical quasiparticle spectrum.

Pethick and Carneiro[12] gave a phenomenological model from which they deduced the behavior of these spectra near the Fermi surface. The effective interaction between quasiparticles and quasiholes was expressed in terms of Landau parameters since only interactions with low momentum transfers are important in determining the dominant contributions to the $T^3\ln T$ term in the specific heat. This calculation emphasized that the difference between the two spectra arises from the different ways in which on-shell scattering of quasiparticle-quasihole pairs is treated in the calculation of the self-energy and the contribution to the energy.

More recently theoretical work on the quasiparticle spectra of Fermi liquids has been stimulated by interest in the quasiparticle spectra in nuclear matter and finite nuclei,[13] and by new measurements of the heat capacity of liquid ^{3}He.[14–19] One of the striking features of Greywall's data is a well-defined $T^3\ln T$ term in the specific heat, which persists to temperatures of order 200 mK at low pressure and 100 mK near the melting line. This result is in disagreement with paramagnon theory calculations of Brinkman and Engelsberg, which indicated that the $T^3\ln T$ behavior should be observable only at very low temperatures. Brown, Pethick, and Zaringhalam[20] (BPZ) investigated over what region of excitation energies the effective-mass enhancement due to spin fluctuations went away. Their approach was to calculate first of all the dynamical quasiparticle energy at higher energies. From this they estimated the statistical quasiparticle energy by using the relationship between the spectra derived by Pethick and Carneiro[12] for low temperatures and low excitation energies. Mishra, Brown, and Pethick[21] showed that the model of BPZ could be fitted to the recent experimental results of Greywall[19] quite well at several pressures.

The purpose of this paper is to make estimates of properties of ^{3}He at temperatures and energies where the low-energy, low-temperature results of Landau theory and no longer valid. Our general philosophy in this paper is not to produce a model which we regard as being a realistic quantitative model for liquid ^{3}He, but rather to adopt a simplified model and to investigate qualitative features such as the difference between the two quasiparticle energies and the various contributions to thermodynamic quantities. First we explore the difference between the two sorts of quasiparticle energy, and show that it is significant. We then calculate the entropy and specific heat, and determine the temperature below which the $T^3\ln T$ behavior gives a good approximation to the results of the full calculation. Our results show that the $T^3\ln T$ behavior should persist to temperatures well above those indicated by Brinkman and Engelsberg, and we discuss the reasons for the difference.

The plan of the paper is as follows. In Sec. II we describe the basic model, and derive expressions for the statistical and dynamical quasiparticle spectra, the entropy, and the specific heat. Results of numerical calculations are described in Sec. III, and in Sec. IV we compare results for our model with those based on the paramagnon model.[6] Section V is a brief conclusion.

II. THE MODEL, QUASIPARTICLE SPECTRA, AND THERMODYNAMIC PROPERTIES

A. The spin-fluctuation energy

The model we employ is a spin-fluctuation model which at long wavelengths and low temperatures reduces to Landau theory, in the sense that spin fluctuations are made up of quasiparticle-quasihole pairs with the true effective mass m^*, and the magnetic susceptibility agrees with that given by Landau theory. It is essentially a generalization to finite wave numbers of the model of Ref. 12, in which the energy is obtained by evaluating the contribution from diagrams consisting of rings of quasiparticle-quasihole pairs. Since pairs with spin one provide the dominant contribution to the leading finite-temperature properties, we shall neglect pairs with spin zero. Also, since the largest contributions to the spin-one channels are due to the large enhancement of the magnetic susceptibility, we shall for the most part take the interaction in these channels to be simply V_q, where q is the total momentum of the pair, and shall neglect the dependence on other momentum variables. At long wavelengths we shall take V_q to tend to $f_0^a \equiv F_0^a/N(0)$, where F_0^a is the angular averaged spin-antisymmetric Landau parameter, and $N(0)$ is the density of quasiparticle states at the Fermi surface. In addition we shall take the quasiparticles making up the spin fluctuation to have an energy spectrum $p^2/2m^*$, to within an additive constant, where m^* is the effective mass at the Fermi surface. This ensures that at long wavelengths the properties of spin fluctuations are the same as we would obtain in Landau theory, provided all Landau parameters except F_0^a are neglected. Our main purpose in this paper is to explore qualitative features of spectra and thermodynamic properties, and these are influenced little by the inclusion of further Landau parameters. Higher Landau parameters have important quantitative effects, and these will be considered briefly.

The contribution to the energy from pairs with spin one has been discussed by Baym and Pethick[22] and is given by

$$\Delta E = -\tfrac{1}{2} \sum_{\substack{\mathbf{q},\mathbf{p} \\ \sigma,\sigma'}} n_{\mathbf{p}\sigma}(1-n_{\mathbf{p}+\mathbf{q}\sigma'})\tilde{f}^{\,\mathrm{st}}(q,\omega_{\mathbf{pq}}) , \tag{3}$$

where

$$\tilde{f}^{\,\mathrm{st}}(q,\omega_{\mathbf{pq}}) = f^{\mathrm{st}}(q,\omega_{\mathbf{pq}}) - \tfrac{3}{4}V_q^2 \mathrm{Re}\chi(q,\omega_{\mathbf{pq}}) ,$$

$$f^{\mathrm{st}}(q,\omega_{\mathbf{pq}}) = \frac{3}{2}\frac{1}{\mathrm{Im}\chi(q,\omega_{\mathbf{pq}})}\tan^{-1}\left[\frac{-V_q \mathrm{Im}\chi(q,\omega_{\mathbf{pq}})}{1-V_q\mathrm{Re}\chi(q,\omega_{\mathbf{pq}})}\right] ,$$

$$\chi(q,\omega) = 2\sum_{\mathbf{p}'} \frac{n_{\mathbf{p}'}-n_{\mathbf{p}'+\mathbf{q}}}{\omega-\omega_{\mathbf{p}'\mathbf{q}}} ,$$

$$\omega_{\mathbf{pq}} = \frac{(\mathbf{p}+\mathbf{q})^2-p^2}{2m^*} , \tag{4}$$

and $n_{\mathbf{p}}$ is the Fermi-Dirac distribution function. The second term in the statistical effective interaction, $\tilde{f}^{\,\mathrm{st}}$, is included to avoid counting the second-order diagram twice.

B. Quasiparticle spectra

To get the spin-fluctuation contribution to the statistical quasiparticle spectrum we functionally differentiate the expression for the energy with respect to the distribution function, and find

$$\Delta\varepsilon_p^{\mathrm{st}} = -\sum_{\mathbf{q}}(1-2n_{\mathbf{p}+\mathbf{q}})f^{\mathrm{st}}(q,\omega_{\mathbf{pq}}) - \sum_{\mathbf{q},\mathbf{p}'} n_{\mathbf{p}'}(1-n_{\mathbf{p}'+\mathbf{q}})\frac{\delta f^{\mathrm{st}}}{\delta n_{\mathbf{p}}}(q,\omega_{\mathbf{p}'\mathbf{q}}) . \tag{5}$$

For small q the last term in Eq. (5), the so-called rearrangement term, has been shown by Carneiro and Pethick[23] to give negligible contributions to any quantity which is independent of spin. In line with our aim of exploring differences among various calculations we shall generally adopt a policy of neglecting contributions like the rearrangement term, which are unimportant at long wavelengths, and shall introduce a cutoff wave number q_c for the spin fluctuations. We therefore lump our lack of knowledge of the finite momentum effects and other features into the single parameter q_c. Note that it is f^{st}, not $\tilde{f}^{\,\mathrm{st}}$, that appears in (4). This is because in the second-order diagram for ΔE there are two ways of selecting the momentum of the particle-hole pair and, consequently, two ways of selecting the momentum of the particle-hole pair in $\Delta\varepsilon_p^{\mathrm{st}}$. Since we assume that the momenta q of pairs of interest are small enough that the phase spaces for the two ways of choosing the pair momentum do not overlap significantly, these two ways of assigning the momentum correspond to physically different processes. The factor of $\frac{1}{2}$ in the second-order contribution to ΔE is therefore compensated in $\Delta\varepsilon_p^{\mathrm{st}}$ by the factor 2 coming from the two ways of assigning the quasiparticle-quasihole momentum. We shall evaluate the effective interaction at zero temperature, since it is a weak function of temperature.

As we shall explain below, not all of the effective-mass enhancement in liquid ^{3}He comes from spin fluctuations, and to take this effect into account we shall assume that the quasiparticle spectrum in the absence of spin fluctuations is given by $p^2/2m_0$, where m_0 is generally different from the bare mass m.

The expression for the statistical quasiparticle spectrum is then

$$\varepsilon_p^{\mathrm{st}} = p^2/2m_0 - p_F^2/2m_0 - \sum_{\mathbf{q}}(1-2n_{\mathbf{p}+\mathbf{q}})f^{\mathrm{st}}(q,\omega_{\mathbf{pq}}) , \tag{6}$$

where $f^{\mathrm{st}}(q,\omega)$ is defined as above. For convenience we have subtracted $p_F^2/2m_0$, where p_F is the Fermi momentum, from the quasiparticle energies, so that ε_p vanishes at the Fermi surface. This calculation will be consistent with the Landau theory calculations of the contributions to ε_p beyond those linear in $\xi_p = (p-p_F)p_F/m^*$ if the long-wavelength low-frequency interaction between the quasiparticles is given by the Landau theory expression.

In evaluating Eq. (6) we shall make a number of long-wavelength approximations. We replace the argument of the Fermi-Dirac distribution $n_{\mathbf{p}+\mathbf{q}}$ by

$$\varepsilon_{\mathbf{p}+\mathbf{q}} - \varepsilon_{p_F} = (p - p_F + q\mu)v_F , \tag{7}$$

where μ is the cosine of the angle between $\mathbf{q}$ and $\mathbf{p}$, and we use the following long-wavelength expressions for $\chi(q,\omega)$:

$$\mathrm{Re}\chi(s) = -\left[1-\frac{s}{2}\ln\left|\frac{1+s}{1-s}\right|\right] ,$$

$$\mathrm{Im}\chi(s) = -\frac{\pi}{2}s\Theta(1-|s|) , \tag{8}$$

where $s=\omega/(qv_F)$ and Θ is the step function. $v_F = p_F/m^*$ is the Fermi velocity. In addition, rather than modeling V_q in detail, we shall take it to be equal to $F_0^a/N(0)$ for all momenta out to some cutoff q_c. Here F_0^a is the usual spin-antisymmetric Landau parameter which is related to the magnetic susceptibility χ_m by

$$\chi_m = \chi_0\frac{m^*/m}{1+F_0^a} , \tag{9}$$

where χ_0 is the magnetic susceptibility of a free Fermi gas with bare mass m, and $N(0)=m^*p_F/\pi^2\hbar^3$ is the density of states. These long-wavelength approximations should be reasonable provided $q/2p_F$ is small, and they will automatically ensure that our results for the leading corrections to the quasiparticle spectrum at finite values of $|p-p_F|$ and finite temperature are consistent with the results of Landau theory with all the Landau parameters except F_0^a neglected.

Our model is a simplified one, since it does not incorporate all the physics one could wish for. In particular, we have not allowed for the fact that the quasiparticle spectra and the interactions should depend on energy and temperature and should be calculated self-consistently. One might argue that one should work with quasiparticles with an effective mass m^* and interaction F_0^a at zero temperature, while at temperatures so high that the effective mass is no longer enhanced by spin-fluctuation effects, one should work with quasiparticles having a mass closer to the bare mass, and an interaction closer to that in the paramagnon model. We shall not take such effects into account explicitly, since we expect that, in the region of energies and frequencies of interest here, neglect of these effects will not seriously affect our conclusions about finite-frequency and finite-temperature corrections.

We now consider how to choose q_c. If one evaluates the effective mass from Eq. (6), and makes the long-wavelength approximations described above, one finds

$$\frac{1}{m^*}=\frac{1}{p_F}\left[\frac{\partial\varepsilon_p^{\rm st}}{\partial p}\right]_{p=p_F}=\frac{1}{m_0}+\frac{3}{4m^*}\frac{1}{p_F^2}\int_0^{q_c} q\,dq\frac{F_0^a}{1-F_0^a\chi(q,0)} . \quad (10)$$

The effective-mass enhancement $\delta m^*_{\rm SF}$ due to spin fluctuations is given by

$$\delta m^*_{\rm SF}=m^*-m_0=-\tfrac{3}{8}m_0\int_0^{q_c}\frac{q\,dq}{p_F^2}\frac{F_0^a}{1-F_0^a\chi(q,0)} , \quad (11)$$

and if one neglects the momentum dependence of $\chi(q,0)$, one finds

$$\delta m^*_{\rm SF}=-\tfrac{3}{2}m_0\frac{F_0^a}{1+F_0^a}\left[\frac{q_c}{2p_F}\right]^2 . \quad (12)$$

Since q_c cannot exceed $2p_F$, the maximum mass enhancement from spin fluctuations is $3.5m$ at zero pressure, and $4.7m$ at 27 bars, if mass enhancements due to other effects are neglected ($m_0=m$), which might suggest that spin fluctuations alone could account for the experimentally observed enhancement, $1.76m$ at zero pressure and $4.17m$ at 27 bars. However, as we now show, these theoretical values are gross overestimates for a number of reasons. To see this we take the standard expression

$$\frac{m^*}{m}=1+\frac{F_1^s}{3} \quad (13)$$

for the effective mass in terms of the Landau parameter

$$F_1^s=3\int_{-1}^{1}\frac{d\mu}{2}F^s(\mu)\mu , \quad (14)$$

where $F^s(\cos\theta)$ is the spin-symmetric quasiparticle interaction (multiplied by the density of states at the Fermi surface) as a function of the angle θ between the two quasiparticle momenta. $F^s(\cos\theta)$ may be evaluated by using crossing symmetry, as is commonly done in induced-interaction models,[24] and for small angles θ it may be expressed in terms of Landau parameters

$$F^s(\theta\simeq 0)=F^s_{\rm dir}(\theta=0)-\sum_p\left[\frac{3}{2}\frac{{F_l^a}^2}{1+\dfrac{F_l^a}{2l+1}}+\frac{1}{2}\frac{{F_l^s}^2}{1+\dfrac{F_l^s}{2l+1}}\right] . \quad (15)$$

Here $F^s_{\rm dir}$ is the so-called direct interaction, which consists of all diagrams irreducible in both the particle-hole channels. One may recover the result (10) for m^* from Eqs. (14) and (15) by (i) neglecting the F_l^s terms in Eq. (15), (ii) replacing $F^s_{\rm dir}$ by $\frac{3}{2}\sum_l F_l^a$, and (iii) setting $\mu=1-q^2/(2p_F^2)$ in Eq. (14) equal to unity, which is equivalent to assuming that $F^s(\mu)$ is appreciable only for small values of θ (i.e., that only long-wavelength spin fluctuations are important). In Eq. (10) the fluctuation-induced contributions are contained in the second term. When we include only the induced-interaction terms in evaluating the effective mass, neglect all Landau parameters except F_0^a and again replace μ by 1, we find

$$\delta m^*_{\rm SF}=\frac{3m_0}{2}\frac{(F_0^a)^2}{(1+F_0^a)}\left[\frac{q_c}{2p_F}\right]^2 , \quad (16)$$

which is less than 1.45 m at $P=0$ and 2.59 m at $P=27$ bars if we neglect enhancements from other sources, and m_0 is chosen so that $m_0+\delta m^*_{\rm SF}$ is the experimentally measured effective mass. Because of the approximations we have made, even these estimates are too large, and there must be significant contributions to m^* from either the direct interaction or other parts of the induced interaction. These conclusions are in accord with the calculations of Ainsworth, Bedell, Brown, and Quader.[24] In their induced-interaction model for ^{3}He more than half of the enhancement of m^*/m comes from the direct term, and less than 20% from spin fluctuations. In view of the uncertainty as to the size of the mass enhancement from spin fluctuations we shall consider two choices of q_c. One value is determined from Eq. (10) where m_0 is the bare mass. Since this expression is not realistic for $q_c/p_F \gtrsim 1$, the actual mass enhancement due to spin fluctuations will account for only part of the total. The second value we take is one-half the first one and m_0 is chosen so that the value of m^* given by Eq. (10) is the experimentally determined value. With this latter choice, the spin-fluctuation contribution to the effective mass evaluated from Eq. (16) is about 50% of the total, m^*-m, but if finite wave number effects are included, it is of order 40%. By making calculations for two values of q_c we shall be able to see how dependent our results are on q_c.

The dynamical quasiparticle spectrum is given by the poles of the single-particle propagator. The self-energy due to spin fluctuations is obtained by summing self-energy diagrams corresponding to those summed in the calculation of the energy, and we find the dynamical quasiparticle spectrum to be given by

$$\varepsilon_p^{\rm dy}=p^2/2m_0-p_F^2/2m_0+\Sigma(p,\varepsilon_p^{\rm dy}) , \quad (17)$$

where $\Sigma(p,\varepsilon_p^{\rm dy})$ is the self-energy. This definition of the dynamical quasiparticle spectrum introduces renormalization effects which were not included in the calculation of the statistical quasiparticle spectrum. In order to compare the two spectra, we replace $\varepsilon_p^{\rm dy}$ in Σ by its zero-temperature value, which we shall approximate by the leading contribution, $\xi_p=(p-p_F)v_F$, and we shall employ the same long-wavelength approximation as we used in evaluating Eq. (6). The dynamical quasiparticle spectrum is therefore given by

$$\begin{aligned}\varepsilon_p^{\rm dy} &= \frac{p^2}{2m_0} - \frac{p_F^2}{2m_0} + \Sigma(p,\xi_p) \\ &= \frac{p^2}{2m_0} - \frac{p_F^2}{2m_0} + \sum_{\mathbf{q}} (1-2n_{\mathbf{pq}}) f^{\rm dy}(q,\omega_{\mathbf{pq}}) \\ &\quad + \sum_{\mathbf{q}} [1-2n_B(\omega_{\mathbf{pq}})] g(q,\omega_{\mathbf{pq}}) , \end{aligned} \tag{18}$$

where

$$\begin{aligned} f^{\rm dy}(q,\omega_{\mathbf{pq}}) &= \tfrac{3}{2}\,\mathrm{Re}\left[\frac{V_q}{1-V_q\chi(q,\omega_{\mathbf{pq}})}\right] , \\ g(q,\omega_{\mathbf{pq}}) &= \tfrac{3}{2}\,\mathrm{Im}\left[\frac{V_q}{1-V_q\chi(q,\omega_{\mathbf{pq}})}\right] , \end{aligned} \tag{19}$$

and $n_B(\omega_{\mathbf{pq}})$ is the Bose-Einstein distribution function. The term involving $g(q,\omega_{\mathbf{pq}})$ is essentially the analogue of the rearrangement term in Eq. (5). It vanishes for long-wavelength fluctuations and we shall neglect it. In the calculations of $\varepsilon_p^{\rm dy}$, the spectrum which appears in the distribution function is taken to be the linear part of the zero-temperature spectrum so that, in the distribution function, $(\varepsilon_p-\mu)/T$ is replaced by ξ_p/T and the chemical potential is not allowed to vary with temperature. Neglecting the temperature dependence of the chemical potential should be a good approximation since the characteristic temperature for variations of the chemical potential is of order the Fermi temperature whereas the quasiparticle spectrum varies on a scale of temperatures typical of spin-fluctuation energies, which are much smaller than the Fermi temperature. In the calculations of the two quasiparticle energies we have made completely equivalent approximations, and consequently meaningful comparisons can be made between them.

C. Entropy and specific heat

In order to evaluate the entropy and the dynamical contribution to it we expand the expression in Eq. (1) to linear order in $(\varepsilon_p-\xi_p)$ and integrate numerically. To include contributions from higher powers of $(\varepsilon_p-\xi_p)$ in the entropy and its dynamical part would be to include higher-order effects which are not present in our calculation of the quasiparticle spectra. However, since we have not calculated the spectrum correctly to this order we shall drop these contributions here. The leading terms at low temperatures are

$$\frac{S}{nk_B} = \frac{S^0}{nk_B} + \frac{\Gamma}{3} T^3 \ln(T/T_s^{\rm st}) , \tag{20}$$

where

$$\frac{S^0}{nk_B} = \frac{\pi^2}{2}\frac{T}{T_F^0}\frac{m^*}{m} ,$$

$$T_F^0 = \frac{p_F^2}{2mk_B} ,$$

and $T_s^{\rm st}$ is a characteristic temperature not given in terms of Landau parameters alone. The quantity Γ is defined by

$$\Gamma = \frac{3\pi^4}{20} B^{\rm st} \frac{1}{T_F^3} , \tag{21}$$

where

$$B^{\rm st} = \frac{3}{2}\left[\frac{F_0^a}{1+F_0^a}\right]^2 \left[1-\frac{\pi^2}{12}\frac{F_0^a}{1+F_0^a}\right]$$

and $T_F = p_F^2/2m^*k_B$. The corresponding result for the specific heat is

$$\frac{C_v}{nk_B} = \frac{T}{nk_B}\left[\frac{\partial S}{\partial T}\right]_v = \frac{C_v^0}{nk_B} + \Gamma T^3 \ln(T/T_c^{\rm st}) , \tag{22}$$

where

$$\frac{C_v^0}{nk_B} = \frac{\pi^2}{2}\frac{T}{T_F^0}\frac{m^*}{m} ,$$

and $T_c^{\rm st} = e^{-1/3} T_s^{\rm st}$. The dynamical quasiparticle contribution to the entropy is obtained by evaluating Eq. (1) using the dynamical quasiparticle spectrum and the results are similar to Eqs. (20) and (22), except that Γ is replaced by $\Gamma^{\rm dy}$:

$$\frac{S^{\rm dy}}{nk_B} = \frac{S^0}{nk_B} + \frac{\Gamma^{\rm dy}}{3} T^3 \ln(T/T_s^{\rm dy}) , \tag{23}$$

and

$$\frac{C_v^{\rm dy}}{nk_B} = \frac{C_v^0}{nk_B} + \Gamma^{\rm dy} T^3 \ln(T/T_c^{\rm dy}) . \tag{24}$$

Here

$$\begin{aligned} T_c^{\rm dy} &= e^{-1/3} T_s^{\rm dy} , \\ \Gamma^{\rm dy} &= \frac{3\pi^4}{20} B^{\rm dy} \frac{1}{T_F^3} , \end{aligned} \tag{25}$$

and

$$B^{\rm dy} = \frac{3}{2}\left[\frac{F_0^a}{1+F_0^a}\right]^2 \left[1-\frac{\pi^2}{4}\frac{F_0^a}{1+F_0^a}\right] . \tag{26}$$

Note that the $T^{\rm dy}$ are not equal to the $T^{\rm st}$.

The calculation of thermodynamic quantities from the quasiparticle spectrum is somewhat laborious, since one has to perform, first, a double integral to evaluate the quasiparticle energies, and, second, a single integral to evaluate the entropy. The labor can be reduced significantly if one works directly with the thermodynamic potential, in which case one has to perform one double integral. To calculate the thermodynamic potential we adopt a phenomenological approach. We consider quasiparticles with effective mass m^* interacting via an effective interaction V_q. The contribution to the thermodynamic potential coming from ring diagrams consisting of spin-one fluctuations is

$$\begin{aligned}\Omega_{\rm SF} = \tfrac{3}{2} k_B T \sum_{q,\nu_n} \{ &\ln[1-V_q\chi(q,\nu_n)] \\ &+ \tfrac{1}{4} V_q^2 \chi^2(q,\nu_n) + V_q\chi(q,\nu_n) \} . \end{aligned} \tag{27}$$

The χ^2 term in (27) is to avoid overcounting the second-order term, as we discussed in connection with the calculation of the energy, Eq. (3). The terms of first order in V, which correspond to Hartree-Fock-like contributions are not evaluated properly by treating them as a single bubble, so we have subtracted them too. Since they are not enhanced by spin-fluctuation effects they are generally small compared with the other terms. We have not attempted to carry out a self-consistent calculation starting from the expression for the thermodynamic potential as a functional of the single-particle Green function, since this is too ambitious a project at this stage. From Eq. (27) the contribution to the entropy from spin fluctuations is calculated as follows:

$$S_{\rm SF}=-\left.\frac{\partial\Omega_{\rm SF}}{\partial T}\right|_{\mu}=-2\sum_{\mathbf{q}}\int_0^\infty\frac{d\omega}{\pi}\left[\frac{\partial n_B(\omega)}{\partial T}\mathrm{Im}\chi(q,\omega)f^{\rm st}(q,\omega)+n_B(\omega)\frac{\partial}{\partial T}[\mathrm{Im}\chi(q,\omega)f^{\rm st}(q,\omega)]\right]. \tag{28}$$

In our earlier calculations starting from the energy we neglected temperature dependence of $\chi(q,\omega)$ and we shall do the same here. We therefore neglect the second term in Eq. (28) and write simply

$$S_{\rm SF}=-2\sum_{\mathbf{q}}\int_0^\infty\frac{d\omega}{\pi}\frac{\partial n_B(\omega)}{\partial T}\mathrm{Im}\chi(q,\omega)f^{\rm st}(q,\omega)\ . \tag{29}$$

We note in passing that if we were to neglect only the part of the second term in (28) proportional to $\partial f^{\rm st}/\partial T$, we would obtain a result for the entropy, beyond the term linear in T, that agrees with the entropy calculated from the statistical quasiparticle spectrum retaining only terms linear in $\varepsilon_p-\xi_p$, as described earlier in this subsection. At low temperatures Eq. (29) has a term proportional to T, whose magnitude, which is determined by expanding $\mathrm{Im}\chi(q,\omega)f^{\rm st}(q,\omega)$ in powers of ω, is

$$S_1=-\sum_{\mathbf{q}}\int\frac{d\omega}{\pi}\frac{\partial n_B(\omega)}{\partial T}\left[\frac{\pi}{2}\frac{\omega}{qv_F}\right]\frac{3V_q}{1-V_q\chi(q,0)}$$
$$=\frac{\pi^2}{2}nk_B\frac{T}{T_F^0}\frac{3m^*}{4m}\frac{1}{p_F^2}\int_0^{q_c}q\,dq\frac{-V_q}{1-V_q\chi(q,0)}\ . \tag{30}$$

This corresponds to a correction to the quasiparticle effective mass but since the full mass renormalization is already included in the basic quasiparticles from which we built the spin fluctuations, to include it again would be double counting, and we shall drop it. This procedure can be justified by starting from the expression for the thermodynamic potential as a functional of the fully renormalized single-particle propagator.

The next term in the expansion of $\mathrm{Im}\chi(q,\omega)f^{\rm st}(q,\omega)$ in powers of ω gives the $T^3\ln T$ contribution to the entropy. Using the long-wavelength form of $\chi(q,\omega)$ [Eq. (8)], so that $\mathrm{Im}\chi(q,\omega)f^{\rm st}(q,\omega)$ is a function only of the Landau variable $s=\omega/qv_F$, we obtain for the logarithmic contribution

$$S_2=B^{\rm st}\int_0^\infty d\omega\frac{\beta^2\omega e^{\beta\omega}}{(e^{\beta\omega}-1)^2}\sum_{\substack{\mathbf{q}\\(\omega/v_F<q<q_c)}}\left[\frac{\omega}{qv_F}\right]^3, \tag{31}$$

with

$$B^{\rm st}=\tfrac{3}{2}A_0^{a\,2}\left[1-\frac{\pi^2}{12}A_0^a\right],$$

where

$$A_0^a=\frac{N(0)V_{q=0}}{1+N(0)V_{q=0}}\ .$$

This is the same as $B^{\rm st}$ defined in Eq. (21) if F_0^a in (21) is replaced by $N(0)V_{q=0}$. One finds

$$S_2=n\left[\frac{T}{T_F}\right]^3\frac{3}{16}B^{\rm st}\left[\frac{4\pi^4}{15}\ln\left[\frac{T}{q_cv_F}\right]+\int_0^\infty dx\frac{x^4e^x}{(e^x-1)^2}\ln x\right]$$
$$=nk_B\frac{\Gamma}{3}T^3\ln\left[\frac{T}{\zeta q_cv_F}\right], \tag{32}$$

where

$$\zeta=\exp\left[-\frac{15}{4\pi^4}\int_0^\infty dx\frac{x^4\ln x\,e^x}{(e^x-1)^2}\right]\approx 0.236\ .$$

Higher-order terms in s give a contribution

$$S_3=-2\int_0^\infty\frac{d\omega}{\pi}\frac{\beta^2\omega e^{\beta\omega}}{(e^{\beta\omega}-1)^2}\times\sum_{\substack{\mathbf{q}\\(\omega/v_F\le q\le q_c)}}\mathrm{Im}\chi(s)[f^{\rm st}(s)-f^{\rm st}(0)-B^{\rm st}s^2] \tag{33}$$

whose leading contribution is equal to

$$-nk_B\Upsilon(\Gamma/3)T^3\ ,$$

where

$$\Upsilon=\int_0^1\frac{ds}{s^3}\frac{[f^{\rm st}(s)-f^{\rm st}(0)-R^{\rm st}s^2]}{B^{\rm st}}\ .$$

The entropy to order T^3 is given by

$$\Delta S/nk_B=\Gamma/3T^3\ln(T/T_s^{\rm st})\ ,$$

where

$$T_s^{\rm st}=\zeta q_cv_F\exp(\Upsilon)\ . \tag{34}$$

This correction to the linear part of the entropy is of the same form as in Eq. (20). However, because of the different approximations that have been made in calculating the entropy from the statistical quasiparticle spectrum and from the thermodynamic potential, T_s^{st} will be different. The specific heat is evaluated by differentiating the entropy. As we shall see, the differences, which are a result of different approximations for finite q fluctuations, are rather modest.

III. RESULTS

In this section we describe the results of numerical calculations of the quasiparticle spectra and thermodynamic functions.

A. Quasiparticle energies

The statistical quasiparticle spectrum is calculated from Eq. (6) and the dynamical quasiparticle spectrum from Eq. (18), with g put equal to zero. With the parameters appropriate to normal liquid ^{3}He at zero pressure and all Landau parameters except F_0^a equal to zero, the values of q_c used are $1.418p_F$, for which one finds that all the effective mass can be accounted for by spin fluctuations if m^* is given by Eq. (10) with $m_0=m$, and $q_c=0.709p_F$, for which m_0 determined by Eq. (10) is $1.92m$, and hence in this approximation spin fluctuations contribute roughly half the mass enhancement. However, as we discussed earlier, these estimates of the mass enhancement are unrealistically high, due to the approximations made in deriving Eq. (10). In Fig. 1 we show the zero-temperature statistical and dynamical quasiparticle spectra for $q_c=0.709p_F$ calculated with the long-wavelength form of $\chi(q,\omega)$. For $|p-p_F|>q_c$, which is outside the range displayed in Fig. 1, the two spectra differ from the free-particle spectrum by constant amounts, the deviation from the free-particle spectrum being greater for the statistical spectrum than for the dynamical one. In Fig. 2 we show the spin-fluctuation contributions to the statistical and dynamical quasiparticle spectra at zero temperature for $q_c=0.709p_F$ and for $q_c=1.418p_F$. We have also carried out calculations of the spectra at finite temperature, and find that, far from the Fermi surface, they do not change very much with temperature. However, the deviations of the $q_c=0.709p_F$ spectra from the free-particle spectrum are less than those of the $q_c=1.418p_F$ spectra for all temperatures. Close to the Fermi surface the analytic calculation of Pethick and Carneiro[12] predicts logarithmic corrections to the quasiparticle spectra of the form

$$\varepsilon_p^{st}=\xi_p+\frac{B^{st}}{24}\frac{\xi_p^3}{T_F^2}\ln\left[\frac{|\xi_p|}{\xi_c^{st}}\right],$$

and (35)

$$\varepsilon_p^{dy}=\xi_p+\frac{B^{dy}}{24}\frac{\xi_p^3}{T_F^2}\ln\left[\frac{|\xi_p|}{\xi_c^{dy}}\right].$$

B^{st} and B^{dy} are the same as B^s and B_D^s, respectively, in Ref. 12, and are given by

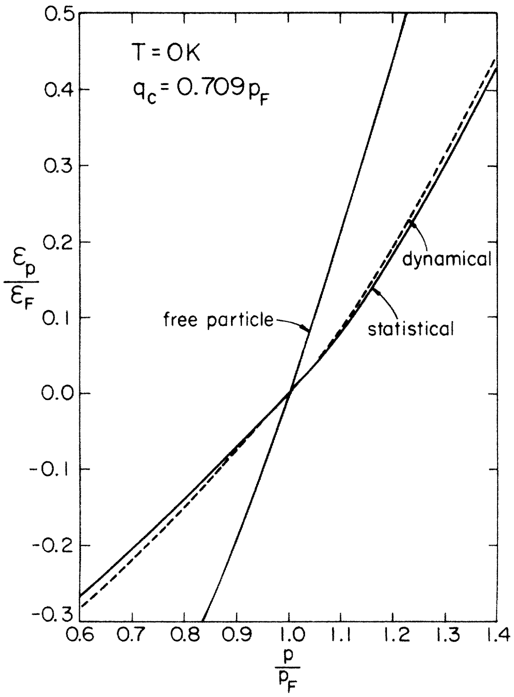

FIG. 1. Statistical and dynamical quasiparticle spectra at zero temperature. The free-particle curve is given by $p^2/2m$, where m is the bare mass. Energies are measured with respect to their values at p_F and are scaled by $\varepsilon_F=p_F^2/2m$. The parameters employed are $F_0^a=-0.70$, $q_c=0.709p_F$, and $m^*=2.76m$.

$$B^{st}=\tfrac{3}{2}(A_0^a)^2\left[1-\frac{\pi^2}{12}A_0^a\right],$$

$$B^{dy}=\tfrac{3}{2}(A_0^a)^2\left[1-\frac{\pi^2}{4}A_0^a\right], \tag{36}$$

and

$$A_0^a=F_0^a/(1+F_0^a)$$

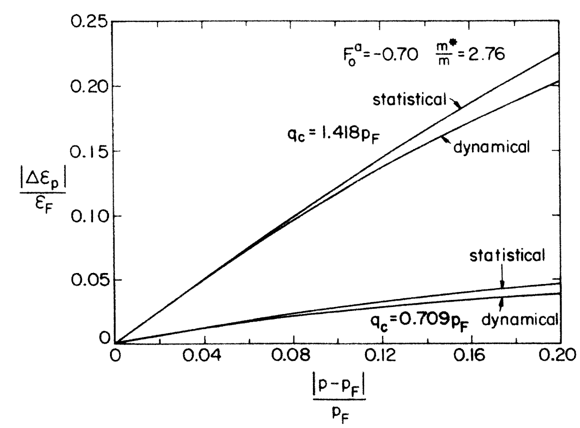

FIG. 2. Spin-fluctuation contributions to the statistical and dynamical quasiparticle spectra at zero temperature for $q_c=1.418p_F$ and $q_c=0.709p_F$. The other parameters are the same as in Fig. 1.

if only the Landau parameter F_0^a is included. The quantities ξ_c^{st} and ξ_c^{dy} are characteristic energies which cannot be calculated in Landau theory.

To investigate up to what energies the analytic results (35) hold, in Fig. 3 we plot $(\xi_p - \varepsilon_p)/\xi_p^3$ versus $\ln\xi_p$ for the statistical and dynamical spectra for $q_c = 0.709p_F$ and for $q_c = 1.418p_F$. The straight lines drawn in Fig. 3 have slopes calculated from Eq. (35), and are given by

$$\varepsilon_p^{st} = \xi_p + (0.303\ \mathrm{K}^{-2})\xi_p^3 \ln(|\xi_p|/\xi_c^{st}) \tag{37a}$$

and

$$\varepsilon_p^{dy} = \xi_p + (0.701\ \mathrm{K}^{-2})\xi_p^3 \ln(|\xi_p|/\xi_c^{dy}) \ . \tag{37b}$$

For $q_c = 0.709p_F$ we find $\xi_c^{st} = 1200$ mK and $\xi_c^{dy} = 860$ mK, while for $q_c = 1.418p_F$ we find $\xi_c^{st} = 2400$ mK and $\xi_c^{dy} = 1710$ mK. The analytic expression (37a) represents deviations of the calculated statistical quasiparticle spectra from the form linear in ξ_p to within 10% for energies up to 650 mK for $q_c = 0.709p_F$ and up to 800 mK for $q_c = 1.418p_F$. For the dynamical quasiparticle spectra Eq. (37b) represents deviations less well, and it is in error by 10% at 330 mK for $q_c = 0.709p_F$ and at 650 mK for $q_c = 1.418p_F$.

In order to see how sensitive the values of ξ_c^{st} and ξ_c^{dy} are to a wave number dependence in the effective interaction, we used a more general expression for χ that takes into account its q dependence. We took the full Lindhard function $\chi(q,\omega)$ evaluated for quasiparticles with a spectrum $p^2/2m^*$, but in evaluating the frequency variable we approximated ω_{pq} by $p_F q\mu/m^*$, where $\mu = \hat{\mathbf{p}}\cdot\hat{\mathbf{q}}$. This replacement is correct for small q, and it amounts to neglecting the curvature of the Fermi surface and putting the velocity of all quasiparticles equal to p_F/m^*. It also maintains the particle-hole symmetry of the spin-fluctuation contributions to the quasiparticle energies. With this form of $\chi(q,\omega)$, the value of q_c required for spin fluctuations to account for all the mass enhancement is $1.585p_F$, if one evaluates the effective mass from Eq. (10) with $m_0 = m$. The values of ξ_c^{st} and ξ_c^{dy} we find for this case are 1600 mK and 1170 mK, respectively, which are to be compared with the values 2400 mK and 1710 mK obtained if the momentum dependence of $\chi(q,\omega)$ is neglected. Considering the large value of q_c used, it is encouraging that the effect of the q dependence is not larger.

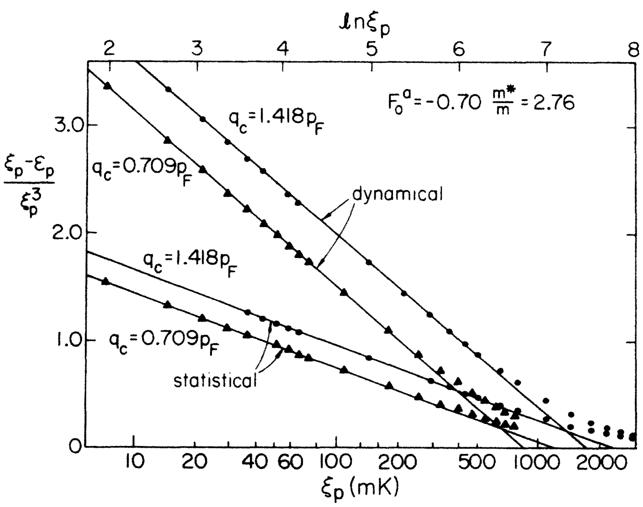

FIG. 3. Plot of $(\xi_p - \varepsilon_p)/\xi_p^3$ for the statistical and dynamical quasiparticle spectra vs $\ln\xi_p$ for $q_c = 1.418p_F$ (dots) and for $q_c = 0.709p_F$ (triangles). The straight lines are the $\xi_p^3\ln\xi_p$ fits in Eqs. (37a) and (37b). For $q_c = 1.418p_F$, $\xi_c^{st} = 2400$ mK and $\xi_c^{dy} = 1710$ mK and for $q_c = 0.709p_F$, $\xi_c^{st} = 1200$ mK and $\xi_c^{dy} = 860$ mK. The other parameters are the same as in Fig. 1.

We now consider calculations using parameters appropriate for liquid ^{3}He at 27 bars ($m^* = 5.17m$ and $F_0^a = -0.759$) and the long-wavelength form of $\chi(q,\omega)$ in the effective interaction. With these values of m^* and F_0^a, the value of q_c determined from Eq. (10) is $1.879p_F$ when $m_0 = m$, and $0.940p_F$ when $m_0 = 2.53m$. In the latter case spin fluctuations are responsible for about 40% of the enhancement of the effective mass. As we mentioned previously, there are overestimates of the mass enhancement due to spin fluctuations because of the approximations made in deriving Eq. (10). The analytic expressions for the spectra for small ξ_p, given by Eq. (35), are

$$\varepsilon_p^{st} = \xi_p + (1.540\ \mathrm{K}^{-2})\xi_p^3 \ln(|\xi_p|/\xi_c^{st}) \ ,$$

and (38)

$$\varepsilon_p^{dy} = \xi_p + (3.762\ \mathrm{K}^{-2})\xi_p^3 \ln(|\xi_p|/\xi_c^{dy}) \ .$$

The coefficient of $\xi_p^3\ln|\xi_p|$ is much larger for the high-pressure case than for zero pressure while the ξ_c's are smaller at the higher pressure. We find that $\xi_c^{st} = 812$ mK and $\xi_c^{dy} = 572$ mK for $q_c = 0.940p_F$ and $\xi_c^{st} = 1600$ mK and $\xi_c^{dy} = 1140$ mK for $q_c = 1.879p_F$. Thus, for values of q_c that give comparable fractions of the total mass enhancement, the ξ_c at high pressure are typically $\sim 30\%$ less than those at low pressure. We have also calculated the statistical quasiparticle spectrum for the case where both F_0^a and F_1^a are nonzero which is discussed in more detail in Sec. III C. In Table I we show the calculated cutoff energies ξ_c^{st} for the cases where one and two Landau parameters are nonzero at high and low pressure.

Our calculations enable us to investigate one of the assumptions made by BPZ.[20] In order to derive the statistical quasiparticle spectrum from the dynamical quasiparticle spectrum at finite values of $|p - p_F|$ and finite temperatures, BPZ assumed that the ratio of the devia-

TABLE I. Energy cutoffs, ξ_c^{st}, found by fitting the calculated statistical quasiparticle spectrum to the analytically determined $\xi_p^3\ln\xi_p$ dependence.

q_c/p_F	F_0^a	F_1^a	m^*/m	ξ_c^{st} (mK)
1.4182	−0.70	0.0	2.76	2400
1.879	−0.759	0.0	5.17	1600
0.709	−0.70	0.0	2.76	1200
0.940	−0.759	0.0	5.17	812
1.249	−0.7	−0.55	2.76	3300
1.550	−0.759	−0.99	5.17	3100

tions from the linear behavior in ξ_p of the two spectra was given by its value for small $p-p_F$, $B^{\rm dy}/B^{\rm st}$ [see Eqs. (21) and (26)]. In Fig. 4 we plot the ratio of the deviations of the spectra from ξ_p as functions of $|p-p_F|$ for zero pressure with $q_c=0.709p_F$ and $q_c=1.418p_F$. Since the numerical calculation of the ratio was not accurate for p close to p_F at zero temperature, we used the analytical result obtained from the fits (37) here

$$\frac{\xi_p-\varepsilon_p^{\rm dy}}{\xi_p-\varepsilon_p^{\rm st}}=\frac{B^{\rm dy}}{B^{\rm st}}\frac{\ln\left[\dfrac{|\xi_p|}{\xi_c^{\rm dy}}\right]}{\ln\left[\dfrac{|\xi_p|}{\xi_c^{\rm dy}}\right]}=\frac{B^{\rm dy}}{B^{\rm st}}\left[1-\frac{\ln\left[\dfrac{\xi_c^{\rm st}}{\xi_c^{\rm dy}}\right]}{\ln\left[\dfrac{|\xi_p|}{\xi_c^{\rm st}}\right]}\right]. \tag{39}$$

The ratio has a term varying as $1/\ln|\xi_p|$, which causes it to fall rapidly as p becomes different from p_F. The ratio falls faster as one goes further from the Fermi surface for the smaller value of q_c. At zero temperature and for $q_c=0.709p_F$, the ratio has fallen by about 16% from its value at the Fermi surface, 2.3148, for $\xi_p=150$ mK, which corresponds to a momentum $0.04p_F$ from the Fermi surface. For $q_c=1.418p_F$ at the same ξ_p the ratio has fallen by about 10%. At a temperature of 100 mK the numerically determined value of $(\xi_p-\varepsilon_p^{\rm dy})/(\xi_p-\varepsilon_p^{\rm st})$ at $p=p_F$ is about 1.80 for $q_c=0.709p_F$ and about 1.94 for $q_c=1.418p_F$, so we see that the temperature dependence of the ratio is also important. The dependence of the ratio on $|p-p_F|$ is not as strong at finite temperatures as at zero temperature and is much the same for the two values of q_c. With the values of parameters appropriate to normal liquid ^{3}He at 27 bars, the zero-temperature value of the ratio at the Fermi surface is 2.24 and again, because $\xi_c^{\rm st}$ and $\xi_c^{\rm dy}$ are different, there is a sharp drop in the value of the ratio at small $|p-p_F|$.

B. Effective masses

We now turn to the quasiparticle effective masses, which are defined, for arbitrary p and arbitrary temperature, by

$$\frac{1}{m_p^*}=\frac{1}{p}\frac{d\varepsilon_p}{dp}\ . \tag{40}$$

Our approximation for ε_p is odd in ξ_p, and to preserve the particle-hole symmetry we shall replace p^{-1} in the interaction term by p_F^{-1}. Thus

$$\frac{1}{m_p^*}=\frac{1}{m_0}+\frac{1}{p_F}\frac{\partial\Delta\varepsilon_p}{\partial p}\ . \tag{41}$$

The statistical and the dynamical effective masses at zero temperature are shown in Fig. 5 as functions of $|p-p_F|$ for $q_c=0.709p_F$ and $q_c=1.418p_F$. With $q_c=0.709p_F$ the effective mass due to other sources, m_0, is $1.917m$. As a consequence of this small value of q_c, m_p^* approaches its limiting value more rapidly than when $q_c=1.418p_F$ and $m_0=m$. The dynamical effective mass falls more rapidly than the corresponding statistical effective mass. For $q_c=1.418p_F$, the enhancement due to spin fluctuations at $|p-p_F|=0.1p_F$ is 30% less for the statistical spectrum and 45% for the dynamical spectrum than the enhancement at the Fermi surface, $1.76m$. For $q_c=0.709p_F$, the enhancement at $|p-p_F|=0.1p_F$ has fallen by 40% for the statistical

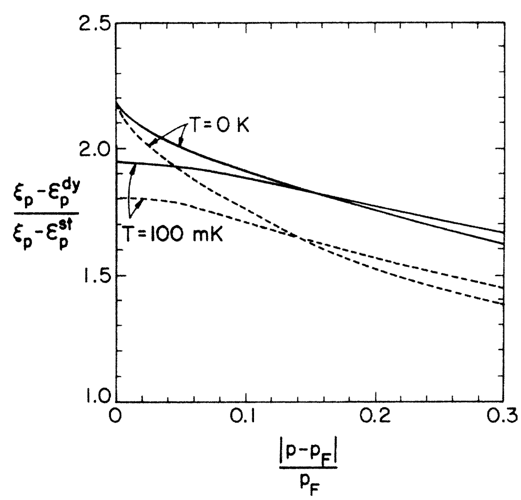

FIG. 4. Ratio of the deviations of the quasiparticle spectra from ξ_p at zero temperature and at 100 mK as a function of $|p-p_F|$ for $q_c=1.418p_F$ (solid lines) and for $q_c=0.709p_F$ (dashed lines). At zero temperature and $p\to p_F$ this ratio is given by the analytic result $B^{\rm dy}/B^{\rm st}=2.3148$, where $B^{\rm st}$ and $B^{\rm dy}$ are defined in Eqs. (21) and (26). The other parameters used are the same as in Fig. 1.

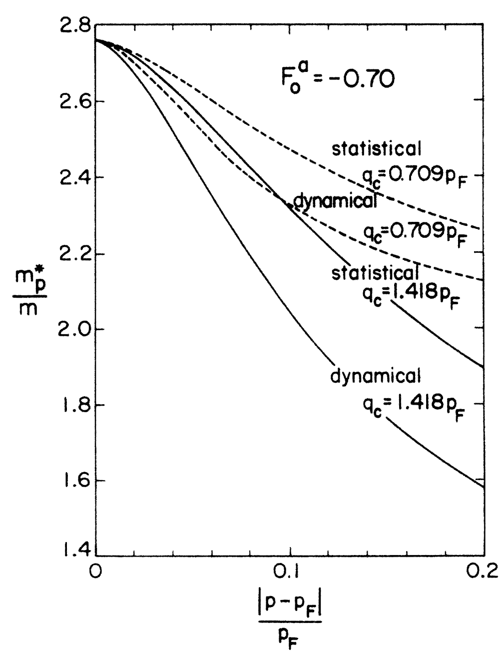

FIG. 5. Statistical and dynamical quasiparticle effective masses at zero temperature as functions of $|p-p_F|$ for $q_c=1.418p_F$ (solid lines) and for $q_c=0.709p_F$ (dashed lines). The other parameters are the same as in Fig. 1.

spectrum and by 50% for the dynamical spectrum. The statistical and dynamical effective masses are compared at different temperatures for $q_c = 1.418 p_F$ in Fig. 6. One sees that the variation of m_p^*/m with $|p - p_F|$ decreases as the temperature increases. In Fig. 7 we show the variation of the statistical and dynamical effective masses at the Fermi surface as a function of temperature for the two values of q_c. The temperature-dependent contribution to the effective masses for $q_c = 0.709 p_F$ is much smaller than for $q_c = 1.418 p_F$ because the effective mass due to other sources, m_0, is comparable with m^*, whereas for $q_c = 1.418 p_F$ all of the enhancement comes from spin fluctuations.

C. Thermodynamic properties

As we described above, we have used two ways to calculate thermodynamic properties, the first based on the statistical quasiparticle spectrum, and the second on a direct evaluation of the thermodynamic potential. We begin by describing the calculations starting from the spectrum.

1. Calculations using the quasiparticle spectrum

The entropy and the dynamical quasiparticle contribution to the entropy are evaluated by putting the statistical quasiparticle spectrum and the dynamical quasiparticle spectrum into the quasiparticle expression for the entropy [Eq. (1)]. The analytic results for the coefficients of the $T^3 \ln T$ term in the entropy and the dynamical contribution to it are given by Eqs. (20) and (23). For parameters appropriate to ^{3}He at zero pressure,

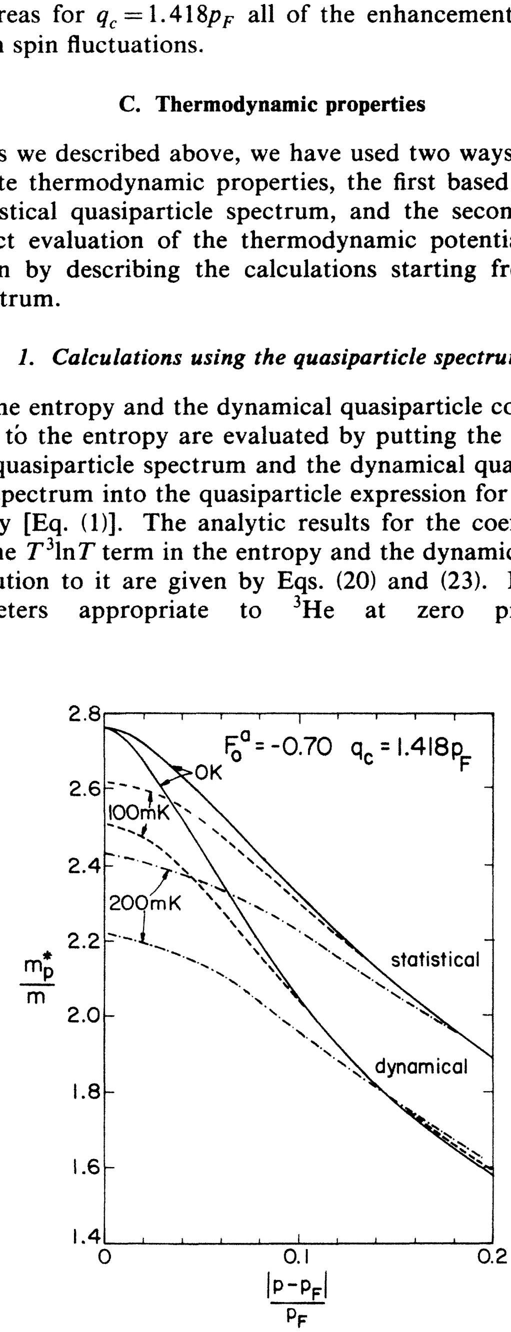

FIG. 6. Statistical and dynamical effective masses as functions of $|p - p_F|$ at zero temperature, 100 mK, and 200 mK for $q_c = 1.418 p_F$. The other parameters are the same as in Fig. 1.

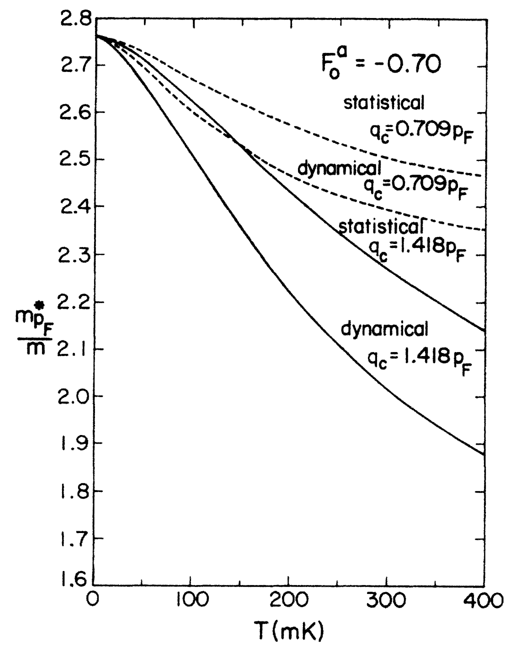

FIG. 7. Statistical and dynamical quasiparticle effective masses on the Fermi surface as functions of temperature for $q_c = 1.418 p_F$ (solid lines) and for $q_c = 0.709 p_F$ (dashed lines). The other parameters are the same as in Fig. 1.

$A_0^a = -2.333$ and $m^*/m = 2.76$, we find

$$S/nk_B = S^0/nk_B + (19.53 \text{ K}^{-3}) T^3 \ln(T/T_s^{\text{st}}) ,$$

and (42)

$$S^{\text{dy}}/nk_B = S^0/nk_B + (45.21 \text{ K}^{-3}) T^3 \ln(T/T_s^{\text{dy}}) .$$

To find the entropy and the specific heat at higher temperatures we evaluated the expression for the entropy, Eq. (1), using the two quasiparticle spectra which were calculated with the long-wavelength form of $\chi(q,\omega)$. To determine the temperatures T_s^{st} and T_s^{dy} and to investigate how good the low-temperature expansions are, we display in Fig. 8 plots of $(S^0 - S)/nk_B T^3$ versus $\ln T$ for the two values of q_c. The straight lines in Fig. 8 are drawn with the theoretical slopes given by Eq. (42). The characteristic temperatures obtained from the fits for $q_c = 0.709 p_F$ are $T_s^{\text{st}} = 182$ mK and $T_s^{\text{dy}} = 136$ mK, and, for $q_c = 1.418 p_F$, $T_s^{\text{st}} = 359$ mK and $T_s^{\text{dy}} = 268$ mK. In Eq. (34) we showed that the characteristic temperature T_s is proportional to q_c when the entropy is calculated from the thermodynamic potential with the long-wavelength form of $\chi(q,\omega)$. One sees from the fits here that T_s is approximately proportional to q_c and that the deviation from exact proportionality, which is a measure of the error introduced in carrying out the triple integral, is quite small. These results exhibit a feature we have found in all calculations, namely, that characteristic temperature for calculations using dynamical quasiparticle energies are significantly lower than those for calculations using statistical quasiparticle energies. Also we find that the $T^3 \ln T$ behavior holds to higher temperatures in the statistical case than in the dynamical one:

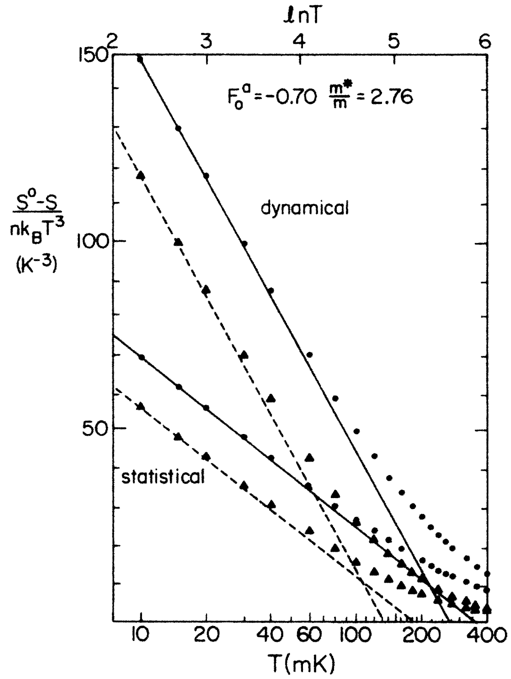

FIG. 8. Entropy and the dynamical contribution to the entropy for $q_c=1.418p_F$ (dots) and $q_c=0.709p_F$ (triangles). Also plotted are the $T^3\ln T$ fits, Eq. (42), for $q_c=1.418p_F$ (solid lines) and for $q_c=0.709p_F$ (dashed lines). For $q_c=1.418p_F$, $T_s^{\rm st}=359$ mK and $T_s^{\rm dy}=268$ mK and for $q_c=0.709p_F$, $T_s^{\rm st}=182$ mK and $T_s^{\rm dy}=136$ mK. The other parameters are as in Fig. 1.

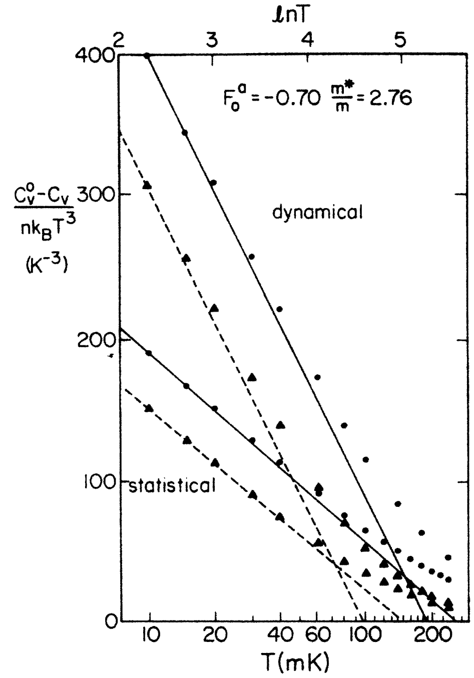

FIG. 9. Specific heat and the dynamical contribution to the specific heat for $q_c=1.418p_F$ (dots) and $q_c=0.709p_F$ (triangles). Also plotted are the $T^3\ln T$ fits, Eq. (43), for $q_c=1.418p_F$ (solid line) and $q_c=0.709p_F$ (dashed line). For $q_c=1.418p_F$, $T_c^{\rm st}=257$ mK and $T_c^{\rm dy}=192$ mK and, for $q_c=0.709p_F$, $T_c^{\rm st}=130$ mK and $T_c^{\rm dy}=97$ mK. The other parameters are the same as in Fig. 1.

for $q_c=0.709p_F$ the deviations of the entropy and of the dynamical contribution to the entropy are given to better than 10% by the low-temperature expansions in (42) up to 60 mK for the entropy and up to 46 mK for the dynamical contribution to the entropy. Similarly, for $q_c=1.418p_F$, the corresponding temperatures are 110 mK and 90 mK, respectively, which shows that the T_s roughly scale with q_c, as the ξ_c do.

Now we turn to the specific heat. The low-temperature expressions calculated from Eqs. (42) are

$$\frac{C_v}{nk_B}=\frac{C_v^0}{nk_B}+(58.59\ {\rm K}^{-3})T^3\ln(T/T_c^{\rm st})$$

and (43)

$$\frac{C_v^{\rm dy}}{nk_B}=\frac{C_v^0}{nk_B}+(135.62{\rm K}^{-3})T^3\ln(T/T_c^{\rm dy}) \ .$$

The calculations of the specific heat were made by numerically differentiating the entropy calculated above. In Fig. 9 we show a plot of $(C_v^0-C_v)/nk_BT^3$ versus $\ln T$. The straight lines have the theoretical slope given by Eqs. (43), and the values of $T_c^{\rm st}$ and $T_c^{\rm dy}$ determined from the plots are 130 mK and 97 mK for $q_c=0.709p_F$, and 257 mK and 192 mK for $q_c=1.418p_F$, which are in satisfactory agreement with the values obtained from the expressions $T_c^{\rm st}=e^{-1/3}T_s^{\rm st}$ and $T_c^{\rm dy}=e^{-1/3}T_s^{\rm dy}$, which follow from differentiation of the low-temperature expansion for the entropy. The deviation of the specific heat and of the dynamical contribution to the specific are given to better than 10% by the low-temperature expansions (43) up to 67 mK and 33 mK, respectively, for $q_c=0.709p_F$ and up to 86 mK and 60 mK, respectively, for $q_c=1.418p_F$. The temperature at which this 10% deviation occurs is roughly proportional to q_c for the dynamical contribution to the specific heat but decreases less with decreasing q_c for the specific heat.

Terms in the specific heat beyond the $T^3\ln T$ contribution cannot be evaluated in terms of Landau theory, and it is therefore of interest to know the extent to which results depend on uncertain features of the model at finite wave numbers. We performed calculations similar to those above, except that we replaced the long-wavelength form of $\chi(q,\omega)$ in Eqs. (4), (5), and (19) by its q-dependent form, as described in Sec. III A. As we mentioned above, for a cutoff $q_c=1.585p_F$, this model gives the measured effective mass. The characteristic temperature in the logarithmic fits to these results is somewhat lower than for the earlier calculations using the long-wavelength form of $\chi(q,\omega)$ with $q_c=1.418p_F$; $T_s^{\rm st}$ is 270 mK and $T_s^{\rm dy}$ is 189 mK. Also the deviations of the entropy from the linear temperature dependence are given by the logarithmic fit to better than 10% up to 69 mK, while for the dynamical contribution to the entropy the fit is good to 10% up to only 52 mK. The specific heat and the dynamical contribution to the specific heat are calculated from the entropy and the dynamical contribution to it. In this case $T_c^{\rm st}$ is 193 mK and $T_c^{\rm dy}$ is 135 mK. The deviation of the specific heat from linear temperature dependence is fitted by the logarithmic term in Eq. (22) to within 10% up to a temperature of 56 mK. For the dynamical case the fit is good to 10% up to 46 mK.

These results show that characteristic temperatures can be altered by about 25% by treating the finite momentum response in different ways. We find it reassuring that the effects are no larger, considering how large the values of q_c are, but we should stress that the differences between the two calculations are rather modest, and larger changes in characteristic temperatures could result from, for example, allowing the interaction, V_q, to be momentum dependent.

In the earlier calculations, we included only the single Landau parameter F_0^a. However, the experimental value of the coefficient Γ of the $T^3\ln T$ term in the specific heat at zero pressure is 36.8 K^{-3}, which is considerably smaller than the theoretical value 58.6 K^{-3} [see Eq. (43)] which one obtains in this approximation. The difference may be attributed to the effects of Landau parameters other than F_0^a which will modify the spin-fluctuation propagator and give rise to interactions in other channels. To investigate these effects we have carried out a calculation in which we include a second Landau parameter, F_1^a. As in the earlier calculations, we consider the scattering of a quasiparticle-quasihole pair with small total momentum $\mathbf{q}$. The K matrix for scattering a pair with a quasiparticle of momentum $\mathbf{p}$ and a quasihole in the state $\mathbf{p}-\mathbf{q}$ to a state where the quasiparticle and quasihole are in the states $\mathbf{p}'$ and $\mathbf{p}'-\mathbf{q}$ is a function of θ, the angle between $\mathbf{p}$ and $\mathbf{q}$, θ', the angle between $\mathbf{p}'$ and $\mathbf{q}$, ϕ, the angle between the plane containing $\mathbf{p}$ and $\mathbf{q}$ and the plane containing $\mathbf{p}'$ and $\mathbf{q}$, and the Landau variable s. The equations for K may be solved by expanding K in associated Legendre polynomials of θ and θ', and powers of $e^{i\phi}$. We note that for on-shell processes, which are the ones of interest here, $\cos\theta$ and $\cos\theta'$ are both equal to s. For further details we refer to Ref. 12. The effective interaction between quasiparticles is related to K in essentially the same way as for the case of one Landau parameter, and one finds [cf. Eq. (4)]

$$f^{\mathrm{st}}(s)=\frac{3}{2}\left[\frac{\tan^{-1}[\mathrm{Im}\chi(s)K_0^a(s)]}{\mathrm{Im}\chi(s)}+\frac{2\tan^{-1}[\mathrm{Im}\chi(s)K_1^a(s)]}{\mathrm{Im}\chi(s)}\right], \tag{44}$$

where

$$K_0^a(s)=\frac{F_0^a+A_1^a s^2}{1-(F_0^a+A_1^a s^2)\mathrm{Re}\chi(s)},$$

$$K_1^a(s)=\frac{F_1^a(1-s^2)}{2[1-F_1^a\mathrm{Re}\Omega_{11}^1(s)]},$$

$$\mathrm{Re}\Omega_{11}^1(s)=-\tfrac{1}{2}[\tfrac{1}{3}+(1-s^2)\mathrm{Re}\chi(s)],$$

and

$$A_1^a=F_1^a/(1+F_1^a/3).$$

χ is given in Eqs. (8). The generalizations of Eqs. (21) and (26) for B^{st} and B^{dy} are

$$B^{\mathrm{st}}=\frac{3}{2}\left[(A_0^a)^2(1+A_1^a)-\frac{\pi^2}{12}(A_0^a)^3-2A_0^aA_1^a+(A_1^a)^2-\frac{\pi^2}{48}(A_1^a)^3\right]$$

and (45)

$$B^{\mathrm{dy}}=\frac{3}{2}\left[(A_0^a)^2(1+A_1^a)-\frac{\pi^2}{4}(A_0^a)^3-2A_0^aA_1^a+(A_1^a)^2-\frac{\pi^2}{16}(A_1^a)^3\right].$$

The parameters A_0^a, A_0^s, and A_1^s may be determined from bulk thermodynamic measurements, but these do not determine F_1^a. A number of estimates of F_1^a are available, but they are rather imprecise. We shall adopt values Greywall estimated from Γ, the coefficient of the $T^3\ln T$ term in the specific heat, by using the theoretical expressions (21) and the generalization of (45) which includes contributions from the spin-symmetric Landau parameters. The values of F_0^a, F_0^s, and F_1^s he used were obtained from bulk thermodynamic measurements in the low-temperature limit. At low pressure he finds $F_1^a=-0.55$. If one now estimates Γ using Eqs. (21) and (45), one finds $\Gamma=35.39\ \mathrm{K}^{-3}$, compared with the experimental value of 36.8 K^{-3}. The difference is due to our neglect of contributions from spin-symmetric (density) channels in (45). The fact that the two values are so close shows how relatively unimportant the density channels are. We turn now to the numerical calculations. We need values for the cutoff parameters associated with the spin modes, and we shall take these to be the same for all spin modes. If we make the same long-wavelength approximations as we did in the derivation of Eq. (12), we find

$$\delta m_{\mathrm{SF}}^{*}=-\tfrac{3}{2}m_0\left[\frac{F_0^a}{1+F_0^a}+\frac{F_1^a}{1+\frac{1}{3}F_1^a}\right]\left[\frac{q_c}{2p_F}\right]^2. \tag{46}$$

This overestimates the contribution to the enhancement from long-wavelength fluctuations for the same reason as Eq. (12) did. When no other source of enhancement is allowed for, so that $m_0=m$, q_c is $1.249p_F$. Taking q_c equal to half this value, $0.625p_F$, the effective mass due to other sources, m_0, is $1.92m$. The value of q_c for $m_0=m$ is lower than that when only F_0^a is nonzero because transverse spin-current fluctuations make a positive contribution to the enhancement. The fit to the numerical calculations at low temperature is

$$C_v/nk_B=C_v^0/nk_B+(35.39\ \mathrm{K}^{-3})T^3\ln(T/T_c^{\mathrm{st}}), \tag{47}$$

where $T_c^{\mathrm{st}}=192$ mK. We find that Eq. (47) fits the deviation of the calculated specific heat from linear temperature dependence to better than about 10% up to a temperature of 70 mK. When all of the mass enhancement comes from spin-density and spin-current fluctuations, q_c is $1.249p_F$ and T_c^{st} is 381 mK. In this case we find that Eq. (47) fits the deviation of the calculated specific heat from linear temperature dependence to about 10% up to a temperature of 100 mK.

At high pressure the magnitude of the finite-temperature contribution to the specific heat is much larger than at zero pressure. For example, the experimental value of Γ is about 5.2 times larger at 27 bars than at $P=0$. It is therefore of interest to carry out calculations using parameters appropriate for high pressures. At 27 bars, m^* is $5.17m$, F_0^a is -0.759, and F_1^a is -0.99, where the latter value is again that estimated by Greywall from the measured value of Γ, and q_c is $1.550p_F$ if we take m_0 to be the bare mass. With this value of q_c, the fit to the numerical calculations at low temperature is

$$C_v/nk_B = C_v^0/nk_B + (182.82\ \mathrm{K}^{-3})T^3\ln(T/T_c^{\mathrm{st}}) , \qquad (48)$$

with $T_c^{\mathrm{st}} = 322$ mk. If one includes the spin-symmetric Landau parameters, one finds that Γ is 190 K^{-3}, which again shows that, in studying the $T^3\ln T$ contributions to the specific heat, the error incurred by neglecting the spin-symmetric channels is small. In this case the deviation of the calculated correction to the linear part of the specific heat from the logarithmic term in Eq. (40) is 10% at 100 mK.

As we have remarked earlier, calculations of thermodynamic properties starting from the quasiparticle spectrum are rather cumbersome numerically, and we have carried out a number of calculations starting from the thermodynamic potential, which is easier to evaluate. We now describe these.

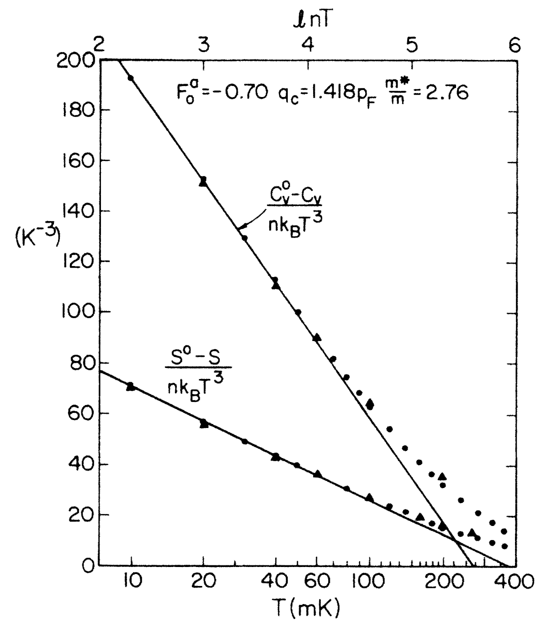

FIG. 10. Comparison of the entropy and specific heat at zero pressure calculated from the thermodynamic potential (dots) and from the statistical quasiparticle spectrum (triangles). The parameters are as in Fig. 1 except that $q_c = 1.418p_F$. The straight lines through the entropy and specific-heat points at low temperatures are given by Eqs. (42) and (43). T_s is 378 mK and T_c is 270 mK.

2. Calculations starting from the thermodynamic potential

We begin by considering parameters appropriate for the case of liquid ^{3}He at low pressure, and include only the Landau parameter F_0^a. This is important for making comparison with our earlier calculations starting from the spectrum. We find that the characteristic cutoff temperature T_s in the $T^3\ln T$ contribution to the entropy, which is defined by an equation analogous to Eq. (34), is given by $T_s = 267(q_c/p_F)$ mK if we take the long-wavelength form of the Lindhard function. When $q_c = 1.418p_F$, T_s is found to be 378 mK, compared with 359 mK for the quasiparticle calculation. This shows that the different approximations made in the two calculations have small effects. In Fig. 10 we show the entropy and specific heat calculated from the thermodynamic potential with $F_0^a = -0.70$ and all other Landau parameters set equal to zero, and with $q_c = 1.418p_F$. The triangles in Fig. 10 are the results of the corresponding calculation from the quasiparticle spectrum. We find that the $T^3\ln T$ behavior fits the numerical calculations based on the thermodynamic potential up to a somewhat higher temperature than it does for the calculations based on the quasiparticle spectrum.

As in the quasiparticle calculations we have investigated how the deviations of the specific heat from the $T^3\ln T$ behavior depend on q_c. Calculations were carried out for $q_c = 0.5p_F$, p_F, and $1.418p_F$ using the long-wavelength form of $\chi(q,\omega)$. For these values of q_c the effective mass enhancements from spin fluctuations in our model, calculated from Eq. (12), are $0.495m$, $1.288m$, and $1.76m$, respectively. Just as in the case of the quasiparticle calculations, the temperature range over which the calculations are well described by a $T^3\ln T$ behavior decreases as q_c decreases. With a value of q_c equal to $1.418p_F$ the deviation of the specific heat from linear temperature dependence fits the form in Eq. (43) to better than 10% up to 100 mK. When q_c is equal to p_F this temperature is 77 mK and when q_c is $0.5p_F$ this temperature falls further to 40 mK. To a first approximation, this temperature is proportional to q_c. Although the enhancement of the zero-temperature effective mass falls very rapidly with q_c, the region over which the $T^3\ln T$ behavior persists falls quite slowly. This demonstrates that the $T^3\ln T$ behavior can persist to reasonably high temperatures, even though spin fluctuations contribute only a modest amount to the effective-mass enhancement.

To investigate what one might expect at higher pressure, we carried out calculations for $m^* = 5.17m$ and $F_0^a = -0.759$, parameters appropriate for a pressure of 27 bars. For Γ one finds 449 K^{-3}, compared with 59 K^{-3} for the analogous calculation at zero pressure, and for $q_c = 1.879p_F$, the characteristic temperatures in the logarithms are $T_s = 260$ mK and $T_c = 188$ mK, about 30% lower than the zero-pressure values. The $T^3\ln T$ fit (20) fit to the deviation of the entropy from a linear temperature dependence is good to better than 10% up to a temperature of 100 mK, and the corresponding temperature for the $T^3\ln T$ fit to the specific heat is 70 mK. With increasing pressure, $T_c^{\rm st}$ and $T_s^{\rm st}$ decrease and deviations from logarithmic temperature dependence start at lower temperatures.

The entropy and specific heat were also calculated for the case when both F_0^a and F_1^a are taken to be nonzero, and equal to the values given earlier. For the case of zero pressure and $q_c=0.625p_F$, the deviation of the specific heat from a linear temperature dependence is fitted by the logarithmic term in Eq. (47) to better than 10% up to a temperature of 72 mK. T_c is found to be 170 mK, which is smaller than the corresponding value, 192 mK, when the specific heat is calculated from the statistical quasiparticle spectrum. However, the $T^3\ln T$ behavior of the specific heat calculated from the thermodynamic potential persists to higher temperatures than in the quasiparticle calculation. We have also calculated the specific heat for $q_c=1.249p_F$, and find $T_c=339$ mK. In this case the deviation of the specific heat from a linear dependence on temperature is fitted by the logarithmic term in Eq. (47) to 10% up to 140 mK. We have also carried out calculations for a pressure of 27 bars for $q_c=1.550p_F$, $q_c=p_F$, and $q_c=0.775p_F$, where again we took the value of the Landau parameter F_1^a from Greywall's fit, which gives $F_1^a=-0.99$. The $T^3\ln T$ fit to the specific heat is

$$\frac{\Delta C}{nk_B}=(C_v-C_v^0)/nk_B=182.82\ \mathrm{K}^{-3}T^3\ln\left[\frac{T}{T_c}\right], \quad (49)$$

with $T_c=210(q_c/p_F)$ mK. When $q_c=0.775p_F$, this form fits the deviation of the calculated specific heat from linear temperature dependence to better than 10% up to a temperature of 76 mK. For $q_c=p_F$, this temperature is 100 mK and with $q_c=1.550p_F$, the corresponding temperature is 150 mK. As in the low-pressure case, the specific heat calculated from the thermodynamic potential is described by the logarithmic term to higher temperatures than is the specific heat calculated from the quasiparticle spectrum. In Table II we show the calculated cutoffs in the entropy and specific heat for the various values of F_0^a and F_1^a.

We now investigate the extent to which our calculations with F_0^a and F_1^a included resemble the experimental data. The specific heat calculated with parameters appropriate to zero pressure and with $q_c=1.418p_F$ is compared with the experimental results of Greywall in Fig. 11. For this value of q_c, T_c is 339 mK, which is smaller than the value 450 mK, obtained by Greywall from his fit. At a given temperature, the differences between the calculated specific heat and the experimental data are larger for smaller q_c. One also sees from Fig. 11 that beyond about 160 mK the full calculation is much closer to experiment than the $T^3\ln T$ expression (47). At higher pressures the values of T_c found by Greywall by fitting to his data are much lower than at zero pressure, 238 mK at 22.22 bars and 226 mK at 29.30 bars. With parameters appropriate to 27 bars and $q_c=p_F$, the calculated value of T_c is 210 mK, which is quite close to the experimental value. In Fig. 12 we compare the specific heat calculated with these parameters with the experimental specific heat. Greywall was able to fit his data at 27 bars with a $T^3\ln T$ term alone up to 100 mK. As we can see from Fig. 12, the $T^3\ln T$ expression for the specific heat fits the full calculation very well up to about 100 mK. In addition the calculated specific heat is quite close to experiment up to 200 mK after which the difference between the two grows rapidly. What these calculations show is that a spin-fluctuation model which includes both spin-density and transverse spin-current fluctuations can give a reasonable account of the experimental specific heat for ^{3}He up to a temperature of almost 150 mK. For the high-pressure case the model can account for the experimentally determined T_c if q_c is about p_F, but with such a cutoff one cannot account for the high value of T_c observed at low pressure.

IV. THE PARAMAGNON MODEL

Calculations of contributions to the specific heat and entropy of ^{3}He beyond the $T^3\ln T$ term have previously been made by Brinkman and Engelsberg,[6] who used the paramagnon model. On the basis of their calculation they concluded that the $T^3\ln T$ behavior should be observable only at very low temperatures, less than 20 mK at high pressure, which would make it difficult to detect experimentally. In the calculations described above we found that the $T^3\ln T$ behavior persists to considerably higher temperatures, typically more than 100 mK, which is in qualitative agreement with what is observed experimentally.[21] In this section we isolate the reasons for the paramagnon model calculations giving such a low estimate of the temperature to which the $T^3\ln T$ behavior should persist.

The first point to notice is that in Ref. 6 the entropy due to spin fluctuations, apart from any contributions to

TABLE II. Characteristic temperatures T_s and T_c obtained from calculations using the thermodynamic potential and coefficients of the $T^3\ln T$ terms in the specific heat for the various values of F_0^a, F_1^a and q_c.

q_c/p_F	F_0^a	F_1^a	m^*/m	Γ (K^{-3})	T_s (mK)	T_c (mK)
1.4182	−0.70	0.0	2.76	38.59	380	270
1.879	−0.759	0.0	5.17	449.1	260	188
1.2494	−0.70	−0.55	2.76	35.39	464	337
1.550	−0.759	−0.99	5.17	182.82	455	326

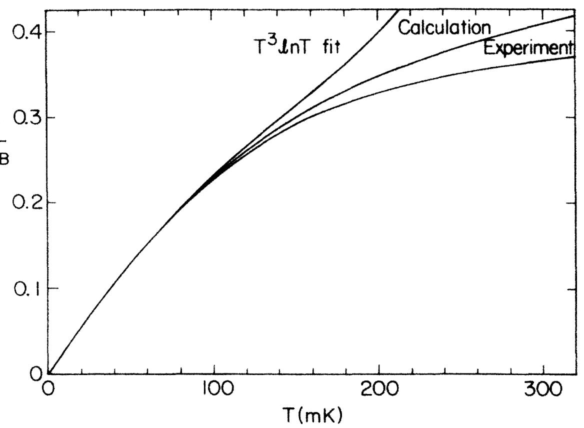

FIG. 11. Comparison of the experimental specific heat at zero pressure and the theoretical one calculated from the thermodynamic potential with two Landau parameters, F_0^a and F_1^a, and $q_c=1.249p_F$. Also plotted in the $T^3\ln T$ fit, Eq. (47), where T_c is 339 mK.

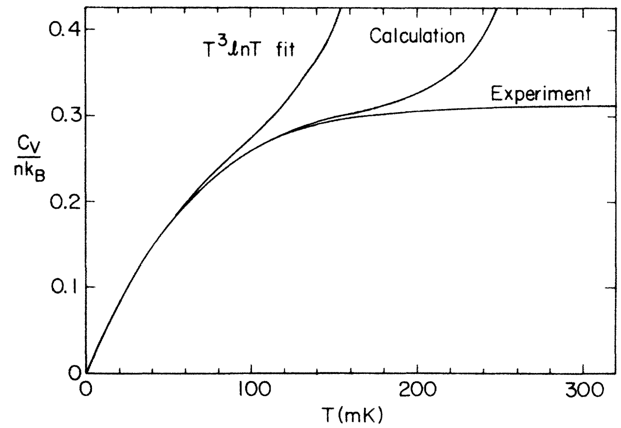

FIG. 12. Comparison of the experimental specific heat at 27 bars and the theoretical one calculated from the thermodynamic potential with two Landau parameters, F_0^a and F_1^a, and $q_c=p_F$. Also plotted is the $T^3\ln T$ fit, Eq. (49), where $T_c=210$ mK.

the $T=0$ effective mass at the Fermi surface, was approximated by the expression

$$\Delta S(T)=2\sum_{\mathbf{p}}\frac{\partial n_{\mathbf{p}}}{\partial T}\Sigma(\mathbf{p},\varepsilon_p^0)\ , \tag{50}$$

where $\Sigma(\mathbf{p},\varepsilon_p^0)$ is the self-energy due to spin fluctuations, $\varepsilon_p^0=p^2/2m$ is the free-particle energy,and $n_{\mathbf{p}}$ is the Fermi-Dirac distribution function evaluated at ε_p^0. This expression is precisely what one finds if one evaluates the dynamical quasiparticle contribution to the entropy, by inserting the dynamical quasiparticle energy (17) into the quasiparticle expression (1) for the entropy, and retains only terms of first order in Σ. The fact that the entropy and specific heat were approximated by their dynamical contributions implies that the $T^3\ln T$ contributions were overestimated by a factor $B^{\text{dy}}/B^{\text{st}}$, typically 2–3, as one can see from the calculations in Sec. III C.

To make contact with the calculations based on the thermodynamic potential we observe that (50) may be rewritten as

$$\Delta S(T)=-2\sum_{\mathbf{q}}\int\frac{d\omega}{\pi}\left[\frac{\partial n_B(\omega)}{\partial T}\right]\text{Im}\chi(q,\omega)f^{\text{dy}}(q,\omega)$$
$$+2\sum_{\mathbf{q},\mathbf{p}}f^{\text{dy}}(q,\varepsilon^0_{\mathbf{p}-\mathbf{q}}-\varepsilon_p^0)\left[\frac{\partial n_{\mathbf{p}}}{\partial T}[n_B(\varepsilon^0_{|\mathbf{p}-\mathbf{q}|}-\varepsilon_p^0)-n_B(\varepsilon_p^0-\varepsilon^0_{|\mathbf{p}-\mathbf{q}|})]+n_B(\varepsilon^0_{|\mathbf{p}-\mathbf{q}|}-\varepsilon_p^0)\left[\frac{\partial n_{\mathbf{p}}}{\partial T}-\frac{\partial n_{\mathbf{p}-\mathbf{q}}}{\partial T}\right]\right]\ . \tag{51}$$

If we neglect in (51) all but the first term, which is consistent with the approximations we have made throughout our calculations, we find

$$\Delta S(T)=-2\sum_{\mathbf{q}}\int_0^\infty\frac{d\omega}{\pi}\frac{\partial n_B(\omega)}{\partial T}\text{Im}\chi(q,\omega)f^{\text{dy}}(q,\omega)\ , \tag{52}$$

which is identical with Eq. (29), except that $f^{\text{st}}(q,\omega)$ is replaced by $f^{\text{dy}}(q,\omega)$.

We now turn to the main difference between Landau theory and the paramagnon model. This is that in the paramagnon model, the quasiparticles that make up the spin fluctuations are assumed to have the *bare* ^{3}He mass, rather than the effective mass, as they do in Landau theory. In the paramagnon model the interaction is approximated by a contact interaction whose strength $\bar{I}$ is determined by fitting the calculated static spin susceptibility to the experimental value. In Landau theory the Landau parameter F_0^a is also determined from K, the ratio of the static spin susceptibility for a normal Fermi liquid to that of a free Fermi gas, and from this it follows that the relationship between the Landau and paramagnon parameters is[25]

$$K=\frac{m^*/m}{1+F_0^a}=\frac{1}{1+\bar{I}}\ . \tag{53}$$

Using Greywall's values for m^* together with the measurements of magnetic susceptibility of Ramm *et al.*[26] one finds that $\bar{I}$ is -0.891 at zero pressure, and -0.943 at 27 bars.

To investigate quantitatively the difference between Landau theory and the paramagnon model we have carried out calculations of the specific heat at high pressure with the paramagnon model and also with Landau theory for the case when all Landau parameters except

F_0^a are zero. We have also calculated the dynamical contributions to the specific heat in both cases.

First of all we compare estimates of Γ and Γ^{dy}, the coefficients of the $T^3\ln T$ term in the specific heat and its dynamical quasiparticle contribution, in Table III. Using the values of F_0^a and $\bar{I}$ shown in Table III one finds that at 27 bars

$$\Gamma_{par}=1.51\Gamma_{Landau} . \tag{54}$$

This clearly shows that differences between Landau theory and paramagnon theory are numerically very significant and that at low temperatures one should use Landau theory, which is known to be the correct theory. For Γ^{dy} one finds

$$\Gamma_{par}^{dy}=1.79\Gamma_{Landau}^{dy} . \tag{55}$$

From Table III one also sees that if one uses the paramagnon model and approximates the specific heat by its dynamical contribution, as was done in Ref. 6, one overestimates the coefficient of the $T^3\ln T$ term by a factor $\Gamma_{par}^{dy}/\Gamma_{Landau}$, which is 4.247 at zero pressure and 4.376 at 27 bars.

To investigate how well the specific heat is described by $T^3\ln T$ terms at 27 bars, we first describe calculations in which the long-wavelength form of the effective interaction is used. q_c is taken to be equal to $1.6p_F$, which is the value used by Brinkman and Engelsberg for the high pressure case. We find that the $T^3\ln T$ and T^3 contributions to the specific heat and the dynamical contribution to it calculated from the paramagnon model and Landau theory are

$$\begin{aligned}\Delta C_{Landau}&=\Gamma_{Landau}T^3\ln(T/186\text{ mK}) ,\\ \Delta C_{par}&=\Gamma_{par}T^3\ln(T/138\text{ mK}) ,\\ \Delta C_{Landau}^{dy}&=\Gamma_{Landau}^{dy}T^3\ln(T/138\text{ mK}) ,\end{aligned} \tag{56}$$

and

$$\Delta C_{par}^{dy}=\Gamma_{par}^{dy}T^3\ln(T/102\text{ mK}) .$$

The temperatures at which ΔC deviates by 10% from the logarithmic forms in Eq. (56) are 70 mK for ΔC_{Landau}, about 55 mK for both ΔC_{par} and ΔC_{Landau}^{dy}, and about 40 mK for ΔC_{par}^{dy}. The T_c's for the paramagnon theory calculations are lower than those calculated using the Landau theory parameters, and are lower for the dynamical contribution to the specific heat than for the specific heat itself. We have also carried out calculation where $\chi(q,\omega)$ is taken to be the full Lindhard function, and find

$$\begin{aligned}T_{Landau}^{st}&=81\text{ mK} ,\\ T_{Landau}^{dy}&=56\text{ mK} ,\\ T_{par}^{st}&=33\text{ mK} ,\end{aligned} \tag{57}$$

and

$$T_{par}^{dy}=24\text{ mK} .$$

The values of T_{Landau}^{st} and T_{Landau}^{dy} in Eq. (57) are about half their values in Eq. (56) and the values of T_{par}^{st} and T_{par}^{dy} in Eq. (57) are about a quarter of their values in Eq. (56). In the paramagnon model the ΔC are described by the logarithmic terms corresponding to those in Eq. (56) up to lower temperatures than in the analogous calculations using the Landau theory parameters. The deviation of ΔC from the logarithmic form is 10% at 25 mK for ΔC_{Landau}, at 14 mK from ΔC_{Landau}^{dy}, at 10 mK for ΔC_{par}, and at 7 mK for ΔC_{par}^{dy}. The reason for this large effect is that finite wave number effects become important for smaller q for larger values of the interaction parameter, as may be seen by considering the value of the momentum at which the leading momentum correction is equal to the zero-momentum term. The important momentum dependence in the effective interaction comes from the quantity $1-N(0)V_{q=0}\text{Re}\chi(q,0)$ which is given to second order in q by

$$\begin{aligned}1-N(0)V_{q=0}\text{Re}\chi(q,0)\\ =[1+N(0)V_{q=0}]-N(0)V_{q=0}q^2/12p_F^2 .\end{aligned} \tag{58}$$

The value of the momentum where the two terms are equal, q_{corr}, is given by

$$q_{corr}=\left[12\frac{1+N(0)V_{q=0}}{|N(0)V_{q=0}|}\right]^{1/2}p_F . \tag{59}$$

In the paramagnon case q_{corr} is approximately $0.8p_F$, while for the Landau case it is about $2p_F$. This crude estimate shows that because of the size of the paramagnon and Landau parameters the effect of momentum dependence in the Lindhard function becomes important at smaller momenta in the paramagnon case than in the Landau case. We therefore see that two reasons for the low values of T_c in the Brinkman and Engelsberg calculations are the use of the paramagnon model, and approximating the specific heat by its dynamical quasiparticle contribution.

Finally we compare the values of Γ obtained in the paramagnon model with experiment. From Table I we see that at low pressure Γ is 91.6 K^{-3}, and at 27 bars it

TABLE III. Comparison of the values of Γ and Γ^{dy} for Landau and paramagnon cases at two pressures. m_b is the effective mass of the quasiparticles making up the spin fluctuations. In Landau theory this is the full effective mass taken from experiment and in the paramagnon model it is the bare mass.

Pressure (bar)	m_b/m	ϵ_F^0 (K)	$F_0^a,\bar{I}$	Γ (K^{-3})	Γ^{dy} (K^{-3})
0	2.76	5.0	−0.700	58.59	135.62
0	1.0	5.0	0.891	91.6	250.33
27	5.17	6.216	−0.759	449.1	1097.24
27	1.0	6.216	−0.953	680.4	1964.74

is 680.4 K^{-3}, which are, respectively, 2.5 and 3.6 times the experimental values. As we discussed in Sec. III, the corresponding difference in the Landau theory calculations may be accounted for by introducing higher-order Landau parameters, which take into account effects neglected in the paramagnon model. If one were to extend the paramagnon model to accommodate $l=1$ contributions to the interaction, that channel would also have to be rather close to instability to account for the experimental value of Γ.

$T^3\ln T$ terms have also been identified in the experimental measurements of the specific heat of various metallic alloys, namely, UAl_2,[27] $TiBe_2$,[28] and UPt_3.[29,30,31] This $T^3\ln T$ dependence has been analyzed[32] using the expression given in Ref. 6 for the dynamical contribution to the specific heat which arises from the paramagnon model. As we have pointed out above, the $T^3\ln T$ terms should be analyzed using the expression for the $T^3\ln T$ term in the specific heat given by Landau theory. This has recently been done by Pethick, *et al.*,[33] and by Coffey and Pethick.[34]

V. CONCLUSION

In this paper we have investigated statistical and dynamical quasiparticle spectra in a model in which the interaction effects come from spin fluctuations. These two spectra are the same at zero temperature on the Fermi surface, but are different at finite temperatures or away from the Fermi surface, the dynamical quasiparticle spectrum being significantly more strongly temperature and momentum dependent than the statistical one. The temperature and energy dependence of the effective mass is quite strong and should be taken into account in calculations of transport and other properties at finite temperatures.

The calculations of the specific heat were carried out from two different starting points, the statistical quasiparticle spectrum and the thermodynamic potential. The corrections to the linear temperature dependence of the specific heat are well described by a $T^3\ln T$ behavior up to a temperature of about 100 mK. We find that this temperature is somewhat higher at low pressure than at high pressure. When A_1^a is included, the $T^3\ln T$ behavior survives to higher temperatures than when only one Landau parameter is considered and there is good agreement with experiment up to about 150 mK at both high and low pressure if q_c is chosen approximately. However, our calculations cannot account for a T_c at low pressure as high as that which Greywall finds experimentally, 450 mK. The highest value we find is 340 mK when T_c is calculated from the thermodynamic potential for $q_c=1.249p_F$, which is the largest value we investigated at $P=0$. The largest value of $T_c^{\rm st}$, calculated from the statistical quasiparticle spectrum with $q_c=1.249p_F$, is 380 mK. However, the difference between the specific heat calculated from the quasiparticle spectrum with $q_c=1.249p_F$ and the experimental specific heat is larger than the corresponding difference for the specific heat calculated from the thermodynamic potential for temperatures greater than 120 mK. A larger value of q_c would lead to a higher T_c, but, for such cutoffs, corrections to our results are likely to be important because of the approximations we have made. Another point that must be borne in mind is that in analyzing experimental data, the low-temperature values of $\gamma=C_v/T$ and Γ are treated as fitting parameters. This will tend to make a T plus $T^3\ln T$ fit to the data possible to higher temperatures than would be the case if γ and Γ were fixed. By contrast, in our analyses of the calculations, γ and Γ are determined analytically, and are not free parameters.

Our investigation revealed the reasons why the Brinkman and Engelsberg calculations suggested that the $T^3\ln T$ dependence in the specific heat would be seen only at low temperatures. The first is that they approximated the specific heat by its dynamical contribution, the second is that they used the paramagnon model rather than one which reduces to the Landau theory at long wavelengths and low frequencies, and third is that they considered parameters appropriate for high pressure. Their calculations were also shown to lead to a large overestimate of the coefficient of the $T^3\ln T$ term.

Since completing our calculations it has come to our notice that Larkin and Melnikov[35] have calculated the spin-fluctuation contributions to the heat capacity. They started from the thermodynamic potential, and took the effective mass of the quasiparticles making up the spin fluctuations to be the experimentally measured one. Thus, the starting point of their calculations is similar to ours, but their conclusions about the temperature up to which the heat capacity should exhibit $T^3\ln T$ behavior are more pessimistic than ours. We have no ready explanation for this, but it could be due in part to the fact that a number of approximations were made in Ref. 35 to enable the calculations to be carried out largely analytically.

Using our results we have investigated an assumption made by BPZ about the two quasiparticle spectra. In their calculations of the specific heat they assumed that the statistical quasiparticle spectrum could be obtained from the dynamical quasiparticle one by using the relationship derived by Pethick and Carneiro for values of ξ_p close to the Fermi surface at zero temperature, namely that the ratio of contributions to the spectra beyond the leading term, ξ_p, is given by $B^{\rm dy}/B^{\rm st}$. Our calculation of the spectra shows that the ratio approaches its value for $\xi_p=0$ only asymptotically because of different characteristic energies in the logarithmic term, but that for ξ_p and T less than 150 mK, which were the values considered by BPZ, the ratio never differs from the $T=0$, low ξ_p value by more than 20%.

We now make some general remarks regarding effects which limit the validity of our approximations. First of all, in our calculations we have included only the leading contribution from spin fluctuations, whose size increases with both temperature and q_c. For parameters appropriate to zero pressure the correction term is half the linear contribution to the specific heat at 300 mK for $q_c=1.249p_F$, which implies that there are almost certainly significant corrections to our results at such temperatures. At higher pressures these corrections are important at lower temperatures: for the 27-bars parameters and $q_c = 1.550 p_F$, the calculated spin-fluctuation corrections are almost half the linear term at 100 mK. Further calculations are needed to determine the importance of contributions beyond the leading ones. In order to improve upon the present calculations higher-order spin-fluctuation effects must be studied including renormalization of the vertices and propagators. A step in this direction has been made by Mishra and Ramakrishnan,[36] who used a model which reduces to the paramagnon model at low temperatures. They calculated the specific heat up to temperatures of about 600 mK, assuming $\bar{I}$, but not $\chi(q,\omega)$, to be temperature dependent, with a value determined from the experimental magnetic susceptibility at each temperature. However, in general, one must allow for renormalization of both the propagators as well as the effective interaction, and it is important to carry out these renormalizations consistently. Carneiro and Pethick have shown in their calculation of the magnetic susceptibility[23] that the leading finite temperature contributions due to deviations of the quasiparticle spectrum from the form $\xi_p = (p - p_F) v_F$ exactly cancel those due to vertex corrections, and similar cancellations probably occur here.

A second approximation we have made is to assume that the lifetimes of quasiparticles may be neglected. In order to determine the temperature at which lifetime effects lead to significant corrections to the quasiparticle picture, we now estimate the temperature at which the width of a typical quasiparticle state is equal to its average energy, of order $\langle \xi_p^2 \rangle^{1/2}$, where $\langle\ \rangle$ denotes a thermal average. The relaxation time of a normal-state quasiparticle is given by [see Eq. (4.62) of Ref. 22]

$$\left\langle \frac{1}{\tau_p} \right\rangle = \frac{\langle \xi^2/T^2 \rangle + \pi^2}{2} \frac{1}{\tau} \simeq \frac{4}{3} \frac{1}{\tau(0)} , \tag{60}$$

where $\tau(0)$ is the quasiparticle lifetime at the Fermi surface. The width is $(\Gamma_p = \hbar \langle 1/\tau_p \rangle /2)$ and this will be equal to a typical quasiparticle energy $\langle \xi_p^2 \rangle^{1/2} \simeq (\pi/\sqrt{3}) k_B T$ when

$$\frac{\hbar}{2} \left\langle \frac{1}{\tau_p} \right\rangle = \langle \xi_p^2 \rangle^{1/2} . \tag{61}$$

To estimate $\tau(0)$ we take the value of the thermal conduction time τ_K deduced by Greywall from his measurements of the thermal conductivity[37] and the specific heat. For temperatures of the order of 50 mK and above, $\tau_K T^2$ is about 0.5×10^{-12} sec K^2 at zero pressure and 0.3×10^{-12} sec K^2 at 27 bars. Thus the quasiparticle width is equal to the average quasiparticle energy at about 200 mK at zero pressure and about 100 mK at 27 bars, and the width of the quasiparticle states could have important effects on the thermodynamics at such temperatures. However, there is evidence to suggest that this simple estimate may be too pessimistic. Firstly, for liquid ^{4}He the thermodynamic functions calculated from the standard quasiparticle expressions, evaluated using as quasiparticle energies the (temperature-dependent) positions of peaks observed in inelastic neutron scattering experiments, agree remarkably well with experiment even when the peaks are broad. Secondly, calculations based on microscopic theory[12] suggests that the damping has less effect on the spectral density that enters expressions for thermodynamic properties than it does on the spectral density of the single-particle propagator.

We now comment on the connection between the enhancement of the effective mass due to spin fluctuations and the $T^3 \ln T$ terms in the specific heat. The arguments in Sec. II B showed that spin fluctuations can account for only part of the effective-mass enhancement for liquid ^{3}He, and consequently we considered a range of values of q_c. The effective-mass enhancement is a rapidly varying function of q_c, but the temperature below which the deviation from linear temperature dependence can be described by a $T^3 \ln T$ behavior has a much slower, approximately linear, dependence on q_c. We find that the spin-fluctuation model can account for the persistence of the $T^3 \ln T$ behavior to temperatures where it can be experimentally detected, and it does not depend on whether or not spin fluctuations can account for all the enhancement of the effective mass. This is encouraging, given the existence of theoretical evidence for contributions to the effective mass from other sources. For example, Pandharipande and Itoh[38] have shown that there is an enhancement of the effective mass of ^{3}He in ^{3}He-^{4}He mixtures due to backflow and one would expect similar effects for pure ^{3}He. Also, in the induced-interaction approach[24] it is found that both backflow and spin fluctuations each contribute about 20% and that the direct interaction contributes about 60% to the value of F_1^s.

Among other calculations of the properties of liquid ^{3}He are those of Fantoni *et al.*,[39] who used a correlated-basis function approach, supplemented by a phenomenological spin-fluctuation contribution to the quasiparticle energy, and those based on the idea that ^{3}He is a nearly localized Fermi liquid.[40] Using the latter approach, Vollhardt[41] has considered zero-temperature properties, and Seiler *et al.*,[42] have considered the transition from Fermi liquid behavior at low temperatures to classical behavior at high temperatures. The relationship between the various approaches to the properties of liquid ^{3}He is not understood in detail, but for our present purposes the important point is that they all provide evidence for significant contributions to the quasiparticle effective from sources other than spin fluctuations.

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