Probability distributions in a two-parameter scaling theory of localization

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Probability distributions for the resistance of two- and three-dimensional disordered conductors are studied using a Migdal-Kadanoff–type scaling transformation together with the author's previously derived distributions in one dimension. The present treatment differs from earlier work in two respects: On one hand, it includes the effect of an average potential barrier V experienced by an electron originating from the perfect leads which connect the conductor to a constant-voltage source; on the other hand, the input distribution for one-dimensional systems is based on an exact solution for the effect of the random potential on the complex reflection amplitude of an electron at a certain energy. The scaling equation for probability distributions and for their successive moments are parametrized in terms of the mean resistance, $\bar{\rho}$, and of a fixed parameter γ related to V. Hence they correspond to a special form of two-parameter scaling. A mobility edge, $\bar{\rho} \equiv \rho_c$, exists only for $d > 2$ and, for $d = 3$, detailed results for ρ_c, for the conductivity exponent ν, and for the fixed resistance distribution at ρ_c as a function of γ are presented. The asymptotic distribution of resistance away from the mobility edge for $d = 3$, and in both small- and large-resistance regimes for $d = 2$ are also studied. In the metallic regime for $d > 2$ our treatment yields two distinct distributions, one of which is characterized by Ohm's law for the mean resistance and the other one by Ohm's law for the mean conductance. In the latter case the fluctuations of conductivity are independent of sample size for large samples. The calculated distributions are generally broad and in the localized regime, for $d = 3$ and $d = 2$, the rms values of resistance dominate the mean values in the infinite-sample limit.

I. INTRODUCTION

The scaling theory of Abrahams *et al.*[1] has considerably changed our understanding of electronic localization and of metal-insulator transitions induced by disorder. It suffices to mention two of its fundamental predictions, namely the absence of a metallic transition in one and two dimensions and the absence of a minimum metallic conductivity in both two- and three-dimensional systems (for recent reviews see Refs. 2–4 and the references quoted therein). The theory of Abrahams *et al.* is based on the study of the transformation of the dimensionless conductance, $g(L)$, under increase of the edge size L of a d-dimensional hypercubic sample. However, this theory does not make direct reference in its formulation to the actual randomness of the potential but regards $g(L)$ as a suitable typical conductance instead.

This has recently led Shapiro[5] to develop a scaling theory at a more basic level where one studies the probability distribution of conductance (or resistance) in the actual random conductor. Shapiro's treatment parallels an earlier analysis by Kumar and Jayannavar[6] who studied the scaling behavior of the first two resistance moments rather than that of the full distribution. The authors of Refs. 5 and 6 imagine the hypercubic sample to be divided up into independent linear chains of identical cubes of smaller size in the direction of current flow. They argue that quantum interference effects are essentially one-dimensional, determining only the law of series combination of conductances (resistances) within a chain. If this is the case, the conductance of the hypercubic sample is given by the parallel combination of the above individual chain conductances, which is governed by the classical Ohm's law. This procedure for obtaining the conductance (resistance) leads to a Migdal-Kadanoff-type scaling transformation for the probability distribution under an infinitesimal increase of linear dimensions. Clearly, the success of this approach depends on the availability of a reliable description of the quantum resistance (conductance) of a disordered one-dimensional conductor.

The dimensionless resistance of a one-dimensional conductor is defined by the Landauer formula,

$$\rho = \frac{r}{1-r}, \quad r = |R(L)|^2 , \tag{1}$$

for any realization of the random potential. Here $R(L) = \sqrt{r}\, e^{i\theta(L)}$ is the complex amplitude of reflection of an electron of energy E by the conductor of length L. We recall that in Eq. (1) the disordered conductor is regarded as a macroscopic scatterer described by a tunneling barrier, which is connected to a constant voltage source via perfect conductors at both ends.[7,8] Now, in actual treatments of resistance (conductance) based on Eq. (1), one usually assumes the phase angle $\theta(L)$ to be randomized and uniformly distributed at sufficiently large length scales[5,9,10] (random phase model) as a result of the disorder. Furthermore these treatments do not include the effect of an average tunneling barrier, thus reducing to an ideal conductor when the disorder is "switched off." We recall that phase randomization due to the random potential has not been generally demonstrated and is not supported by numerical work on one-dimensional systems.[11] This somewhat unsatisfactory situation has prompted us recently to present a detailed study of the Landauer resistance for an alternative model which treats directly the effect of the random potential and includes the effect of an average barrier V (see Ref. 12, hereafter referred to as I). Such an average barrier is commonly introduced in related contexts[7,8] and is, of course, present in various physical systems such as, e.g., a disordered metallic alloy conductor connected to (perfect) leads made of one of the alloy constituents. In addition, the barrier V plays a crucial role in our model since it ensures the property of "additive mean" of $\ln\rho$,[9,12] whose importance has been stressed by Anderson *et al.*[9] While the results agree qualitatively with earlier work[10,13,14] in revealing strong fluctuations of the quantum resistance (leading to non-self-averaging behavior at large length scales) they differ significantly in detail, in particular because of the role of V.

The purpose of the present paper is to use the results of I to study resistance fluctuations and the metal-insulator transition for two- and three-dimensional systems using a Migdal-Kadanoff scaling transformation, as discussed above. The potential in the differential equation for $R(L)$ obtained from an "invariant imbedding" procedure in the one-dimensional case has the form[12]

$$V(L)=V+v(L), \quad \langle v(L)\rangle=0 \ , \tag{2}$$

where the conducting chain extends from $x=0$ to $x=L$ and $v(L)$ denotes the random potential at the edge $x=L$ with a Gaussian white-noise correlation:

$$\langle v(L)v(L')\rangle=\frac{1}{\xi}\delta(L-L') \ . \tag{3}$$

For such a symmetrically distributed potential the treatment of I is valid at the energy $E=V/2$. This is because a term of the form $[2E-V(L)]R(L)$ has been ignored in the equation for $R(L)$. We note that the residual value of this term $[-v(L)R(L)]$ is expected to have relatively minor effects only.[15] The scaling theory discussed below departs in important respects from previous scaling theories.[1,5,6] The average tunneling barrier, in particular, introduces an additional parameter in the scaling equation, which leads to nonuniversal values for, e.g., the mobility edge and the critical exponent for the conductivity.

A point of considerable interest in this study is to find out whether the strong fluctuations of the quantum resistance in one dimension get appreciably reduced by the multiple connectivity of higher space dimensions. In this respect our work corroborates the earlier conclusion based on analytical[5,6] as well as numerical[16] work that the distribution of resistance at the mobility edge in three-dimensional systems is rather broad. Moreover, we find that the variance dominates the mean resistance squared on the insulating side of the mobility edge in three dimensions, as well as in both the low- and large-resistance regimes in two dimensions. In particular, we obtain for the first time the explicit form of the leading L-dependent dimensionality effect in the asymptotic expression of the distribution in the insulating regime.

In Sec. II we use the results of I for the distribution $P_\rho^{(1)}(\rho,L)$ of the quantum resistance of a one-dimensional chain to set up the scaling equation for the distribution $P_\rho^{(d)}(\rho,L)\equiv P_\rho(\rho,L)$ (and of its lowest-order moments) in a d-dimensional specimen, using a Migdal-Kadanoff transformation. The detailed analysis of these equations is presented in Secs. III and IV. In Sec. III we study successively the mobility edge, the conductivity exponent ν and the fixed distribution $P_\rho^c(\rho)$ at the mobility edge for $d>2$. In Sec. IV we then discuss the asymptotic behavior of $P_\rho(\rho,L)$ in the insulating regime for arbitrary dimensionality and in the (quasi-) metallic regime for $d>2$, respectively. We also discuss the asymptotic distribution of resistance in the low-resistance, quasimetallic regime in two dimensions. Finally, some concluding remarks are presented in Sec. V.

II. SCALING EQUATION FOR THE PROBABILITY DISTRIBUTION OF RESISTANCE

Following Shapiro[5] we concentrate on the scaling properties of the probability distribution of the resistance ρ of a disordered d-dimensional cubic sample. We shall find the change in the probability distribution of resistance when the side of the hypercubic block is increased by an infinitesimal amount, $\Delta L\to 0^+$, from L to $L+\Delta L$ (the corresponding scaling factor is $b=1+\Delta L/L$), using a Migdal-Kadanoff-type scaling transformation.[17] This transformation involves the following two steps. We start from a d-dimensional cube of size L and first combine b such cubes (cells) in series to form a linear chain of length $bL=L+\Delta L$ whose resistance $\rho^{(1)}$ is described by the quantum probability distribution $P_\rho^{(1)}(\rho^{(1)},L+\Delta L)$ for a finite one-dimensional conductor, derived in I.[12] Thus by expanding in powers of ΔL we obtain from the first of Eqs. (34) of I

$$\frac{\partial P_\rho^{(1)}(\rho^{(1)},L)}{\partial l}=e^{\gamma l}\int_0^\infty dz\left[\frac{\partial P_y}{\partial l}+\gamma\frac{\partial}{\partial y}(yP_y)\right]_{y=ze^{\gamma l}} \times\delta\left[\rho^{(1)}-\frac{1}{4z}-\frac{z}{4}+\frac{1}{2}\right], \tag{4}$$

where $P_y(y,l)$ is the distribution of the variable y,

$$P_y(y,l)=(2\sqrt{2\pi l}\,y)^{-1}\exp\left[-\frac{(\ln y)^2}{8l}\right], \tag{5}$$

$$y=\exp\left[-\frac{\kappa}{V}\int_0^L v(L')dL'\right], \tag{6}$$

and

$$l=L/L_c, \quad \gamma=\kappa L_c \ , \tag{7}$$

are dimensionless quantities with

$$L_c=k_0^2\xi, \quad \kappa=2V/k_0, \quad k_0=\sqrt{2E} \ . \tag{8}$$

Thus l is a reduced length defining L in units of the localization length L_c for the one-dimensional conductor. γ is the ratio between L_c and the decay length (or penetration depth), $k_0/2V$, of eigenstates inside the barrier V, and we have $E=V/2$, as discussed above.

In the second step we then combine b^{d-1} independent chains (chosen to lie in the direction of current flow) with distributions $P_\rho^{(1)}(\rho,L+\Delta L)$ into a d-dimensional cube of edge $L+\Delta L$. In other words, the hypercubic block of side $L+\Delta L$ is assumed to be decomposed into b^d elementary cells (of side L), and "bonds" (current leads) between cells in the $d-1$ directions perpendicular to the current flow direction are assumed to be cut, leaving a collection of b^{d-1} independent chain resistances as the constituents of the block. The resistance of the block, $\rho^{(d)}(L+\Delta L)$, is thus given by the Ohm's-law combination of chain resistances in parallel. For simplicity we assume furthermore[5,6] that the b^{d-1} chains have the same resistance, $\rho^{(1)}(L+\Delta L)$, chosen from the distribution $P_\rho^{(1)}(\rho,L+\Delta L)$, thus obtaining

$$\rho^{(d)}(L+\Delta L)=\rho^{(1)}(L+\Delta L)b^{1-d} . \tag{9}$$

Our neglect of statistical fluctuations of resistance between chains (limit of extreme anisotropy[6]) has been justified by Shapiro[5] for dimensionalities close to 2. On the other hand, the neglect of quantum interference in directions perpendicular to the current flow (via the bond cutting procedure) has been discussed in Sec. I. Shapiro[5] has pointed out furthermore that the classical parallel combination of chain resistances yields the correct asymptotic behavior of a typical resistance in the metallic (weak scattering) regime. He has also noted the close similarity between the transformation described above and Anderson's so-called "fan" transformation.[18]

From Eq. (9) it follows now that the probability distribution of the resistance $\rho^{(d)}(L+\Delta L)\equiv\rho(L+\Delta L)$ of a d-dimensional block, $P_\rho^{(d)}(\rho,L+\Delta L)\equiv P_\rho(\rho,L+\Delta L)$, is given by

$$P_\rho(\rho,L+\Delta L)=b^{d-1}P_\rho^{(1)}(b^{d-1}\rho^{(1)}(L+\Delta L),L+\Delta L) . \tag{10}$$

By expanding Eq. (10) to linear order for $\Delta L\to 0^+$, using Eq. (4), one readily finds

$$\frac{\partial P_\rho(\rho,L)}{\partial \ln L}=(d-1)\frac{\partial}{\partial\rho}[\rho P_\rho(\rho,L)] +le^{\gamma l}\int_0^\infty dz\left[\frac{\partial P_y}{\partial l}+\gamma\frac{\partial}{\partial y}(yP_y)\right]_{y=ze^{\gamma l}} \times\delta\left[\rho-\frac{1}{4z}-\frac{z}{4}+\frac{1}{2}\right] , \tag{11}$$

where we have ignored, as usual, the difference between chain and block resistances on the right-hand side (rhs) in the sense of iteration.[5,6] As in Refs. 5 and 6 we choose the mean resistance $\bar\rho_L$ of the d-dimensional block as our scaling variable to parametrize the distribution. We note, however, that the latter will depend, in addition, on the fixed parameter γ. In order to parametrize the inhomogeneous term of (11) in terms of $\bar\rho_L$ (Ref. 5) we first express the reduced length l on the rhs as a function of the mean resistance $\bar\rho_L^{(1)}$ of a segment of length L given by[12]

$$\bar\rho_L^{(1)}=\tfrac{1}{2}[e^{2l}\cosh(\gamma l)-1] , \tag{12}$$

i.e., $l\equiv l(\bar\rho_L^{(1)})$ is the inverse function of $\bar\rho_L^{(1)}$. Next, when substituting this expression in Eq. (11), we replace $\bar\rho_L^{(1)}$ by $\bar\rho_L$ [i.e., we put $l\equiv l(\bar\rho_L)$] in the spirit of the renormalizations following a small change of length scale, as described by the finite difference forms of Eqs. (4) and (11). A closed equation for $\bar\rho_L$ is obtained by multiplying Eq. (11) by ρ and integrating over ρ, which yields

$$\frac{\partial\bar\rho_L}{\partial \ln L}=-(d-1)\bar\rho_L +l(\bar\rho_L)e^{\gamma l(\bar\rho_L)} \times\int_0^\infty dz\left[\frac{1}{4z}+\frac{z}{4}-\frac{1}{2}\right] \times\left[\frac{\partial P_y}{\partial l}+\gamma\frac{\partial}{\partial y}(yP_y)\right]_{y=ze^{\gamma l(\bar\rho_L)}} \tag{13}$$

provided $P_\rho(\rho,L)$ is sufficiently well-behaved that the boundary terms appearing on integration by parts vanish. An important example of a distribution which does not satisfy this condition will be encountered in Sec. IV. By performing the integrals over z, using Eqs. (5) and (12), we obtain the exact equation

$$\frac{\partial\bar\rho_L}{\partial \ln L}=-(d-1)\bar\rho_L+l(\bar\rho_L)(1+2\bar\rho_L) \times\left[1+\frac{\gamma}{2}\tanh[\gamma l(\bar\rho_L)]\right] , \tag{14}$$

where the scaling function on the rhs also depends on γ, unlike in Refs. 5 and 6 where it is a universal function of $\bar\rho_L$ alone. Finally, a similar exact calculation for the second moment, $\overline{\rho_L^2}$, of $P_\rho(\rho,L)$ leads to the equation

$$\frac{\partial\overline{\rho_L^2}}{\partial \ln L}=-2(d-1)\overline{\rho_L^2}-l(\bar\rho_L)e^{2l(\bar\rho_L)}\cosh[\gamma l(\bar\rho_L)]\left[1+\frac{\gamma}{2}\tanh[\gamma l(\bar\rho_L)]\right] +l(\bar\rho_L)e^{8l(\bar\rho_L)}\cosh[2\gamma l(\bar\rho_L)]\left[1+\frac{\gamma}{4}\tanh[2\gamma l(\bar\rho_L)]\right] . \tag{15}$$

Equations (11), (14) and (15) describe the evolution of the distribution and of its two lowest moments under a change of length scale. Note that Eqs. (11) and (15) involve $\bar{\rho}_L$ as a parameter [via the inverse function $l(\bar{\rho}_L)\equiv l(\bar{\rho}_L^{(1)})$ of $\bar{\rho}_L^{(1)}$ in (12)] which must be obtained explicitly from (14) before their solution can be studied. Thus once $P_\rho(\rho,L)$ has been found in explicit form it may be parametrized in terms of $\bar{\rho}_L$ and γ using the inverse function of the solution of (14). This property of the distribution, as well as Eq. (14), show that we are dealing with a special type of two-parameter scaling where one of the parameters (γ) is scale-invariant, being fixed by the model. Finally, we note that for length scales where the values of $\bar{\rho}_L$ are such that $\gamma l(\bar{\rho}_L)\ll 1$, Eq. (14) together with (12) reduces to the scaling equation for the mean resistance of Refs. 5 and 6. However, the distribution $P_\rho(\rho,L)$ and its higher moments remain different from those of Ref. 5 due to the difference between the models used, as discussed in Sec. I.

III. MOBILITY EDGE AND PROPERTIES NEAR THE MOBILITY EDGE

The spatial localization of electronic states in a disordered system is described by the localization length which diverges at the mobility edge. Thus in a scaling theory the mobility edge may be defined as a critical point where physical quantities remain fixed with respect to changes of length scale. Since our scaling parameter is the mean resistance, the mobility edge corresponds to the fixed-point resistance ρ_c given by

$$\frac{\partial \bar{\rho}_L}{\partial \ln L}=0 . \tag{16}$$

The fixed values P_ρ^c and ρ_2^c of the distribution and of its second moment are then determined in terms of ρ_c by

$$\frac{\partial P_\rho}{\partial \ln L}=\frac{\partial \overline{\rho_L^2}}{\partial \ln L}=0 . \tag{17}$$

The original scaling theory of Abrahams *et al.*[1,2] was formulated in terms of the conductance and so it is useful to define the conductance $g=\bar{\rho}_L^{-1}$ corresponding to $\bar{\rho}_L$. Equation (14) may then be rewritten in the form

$$\frac{d\ln g}{d\ln L}=\beta(g) , \tag{18a}$$

$$\beta(g)=d-1-(2+g)l(g^{-1})\left[1+\frac{\gamma}{2}\tanh[\gamma l(g^{-1})]\right] , \tag{18b}$$

where $l(x)$ denotes the inverse function of $x=\bar{\rho}_L^{(1)}$ given by Eq. (12). For $\gamma=0$ this β function reduces to the corresponding function for the variable $g=\bar{\rho}_L^{-1}$ obtained from Eq. (5) of Shapiro.[5] The limiting forms of $\beta(g)$ for small and large g obtained by inverting Eq. (12) are

$$\beta(g)=d-2-(1+\gamma^2/4)g^{-1}+\cdots,\quad g\to\infty , \tag{19a}$$

and

$$\beta(g)=(1+g/2)\ln g+(d-1-\ln 4)+\cdots,\quad g\to 0 . \tag{19b}$$

Equation (16) defining the mobility edge $\rho_c=g_c^{-1}$ now takes the form

$$\beta(g_c)=0 , \tag{20}$$

and one easily verifies that real positive solutions exist only for $d>2$, which shows that all states are localized in one- and two-dimensional systems for any disorder. The above results show that the behavior of the function (18b) is qualitatively similar to that of the β function discussed by Abrahams *et al.*[1] and displayed in their Fig. 1. Finally we recall that[1,2] the mobility edge ρ_c (for $d>2$) separates the insulating regime where the resistance scales from an initial value ρ_c towards exponentially large values, from the metallic regime where the scaling is, towards arbitrarily small values when L is increased indefinitely. This is because $\partial\bar{\rho}_L/\partial\ln L$ $(\partial\ln g/\partial\ln L)$ changes sign at the mobility edge, being positive for $\bar{\rho}_L>\rho_c$ and negative for $\bar{\rho}_L<\rho_c$.

Near the mobility edge g_c one may write[1,2]

$$\frac{d\ln g}{d\ln L}=s\ln\left[\frac{g}{g_c}\right]\simeq s\left[\frac{g-g_c}{g_c}\right] , \tag{21}$$

where

$$s=g_c\left[\frac{\partial\beta}{\partial g}\right]_{g=g_c} . \tag{22}$$

The parameter s characterizes the critical behavior of the conductivity and of the localization length as a function of $g-g_c$,[1]

$$\sigma=\frac{g_c}{\xi_{\rm loc}^{d-2}}\propto(g-g_c)^{(d-2)\nu},\quad \nu=1/s . \tag{23}$$

From Eqs. (18b) and (22) it follows that in our two-parameter scaling theory the critical conductance g_c as well as the critical exponent ν depend on the fixed parameter γ, while having universal values, $g_c\simeq(1.96)^{-1}$ and $\nu\simeq 1.68$ (Ref. 17) in the earlier one-parameter theories.[5,6]

The scaling theory of localization is valid for length scales larger than the mean free path, L_0, for scattering of an electron of energy E_F by the random potential.[2] Above this lower length cutoff the motion is diffusive[19] and the Thouless scaling argument[1] applies. The conductance $g_0(L_0,E_F)$ at the scale L_0 is a microscopic measure of disorder and provides the boundary condition for the first-order equation (18a). In particular, it fixes the value of the arbitrary constant in the solution, $\ln(g/g_c)=CL^s$ of Eq. (21) close to g_c, namely $C=L_0^{-s}\ln g_0/g_c$, which must change sign as g changes from values $g<g_c$ to values $g>g_c$, while being zero for $g=g_c$ (since L cannot become smaller than L_0). The condition $C=0$ for $g=g_c$ is satisfied as a result of the connection between g_c and the energy of the mobility edge[2] E_c: E_c corresponds to the special value of the Fermi energy (where conductances are measured) for which g_0 takes the value g_c for $d=3$, i.e.,

$$g_0(L_0,E_F=E_c)=g_c . \tag{24}$$

On the other hand, for small deviations of E_F from E_c one has[2]

$$\ln\left[\frac{g}{g_c}\right] \simeq \frac{g_0-g_c}{g_c}\left[\frac{L}{L_0}\right]^s \simeq (E_F-E_c)\frac{g_0'}{g_c}\left[\frac{L}{L_0}\right]^s , \quad (25)$$

which displays the required sign change as E_F moves across the mobility edge. Finally, we note that the solution of (21) written in the form

$$g = g_c(1+CL^s), \quad g \to g_c \quad (26)$$

implies that there exists a small region on the insulating side of the mobility edge where the eigenstates are no longer exponentially localized.

As in Sec. II and in earlier work[5,6] we now proceed with our discussion using again the mean resistance as the basic scaling variable. While we could have formulated the treatment of Sec. II directly in terms of the (random) conductance, $g=\rho^{-1}$, the Landauer formula leads to a singular distribution for g with infinite moments[12] at any scale for a linear chain. For this reason the mean conductance $\bar{g}_L = \overline{\rho^{-1}}$, in particular, does not seem to be a meaningful scaling variable, at least in one dimension. As observed by Shapiro,[5] the singular form of the distribution of g might be the origin of difficulties encountered in scaling theories based on the calculation of conductance moments.[20]

In Fig. 1 we have plotted the mobility edge $\bar{\rho}_L \equiv \rho_c$ as given by Eqs. (14) and (16), as a function of γ. Our results show that ρ_c decreases monotonically from the value $\rho_c \simeq 1.96$ to the value $\rho_c \simeq 0.602$ as γ increases from $\gamma=0$ to a limiting value $\gamma\equiv\gamma_1=4.678$. No mobility edge transition is found when γ exceeds γ_1. The absence of a transition for $\gamma>\gamma_1$ is a consequence of the fact that when L_c is sufficiently large (weak disorder) compared to the barrier penetration distance, the eigenstates remain exponentially localized near the edge of the barrier, at energies $E\simeq V/2$. Also, the fact that the critical resistance decreases with increasing γ corresponds to the fact that, at fixed penetration length, the conductance is expected to be larger, the larger the localization length. Further information about the mobility edge may be obtained, in principle, from Eq. (24) with $g_c=\rho_c^{-1}$. Since in the present case $E_F=E_c\simeq V/2$, and g_c depends on the disorder through $\gamma=2L_c\sqrt{V}$, Eq. (24) determines the value of the disorder parameter L_c for which the mobility edge lies at $E_c=V/2$. Note, incidentally, that, unlike in one-parameter scaling theories, the disorder no longer enters exclusively (and indirectly) through the parameter g_0 in the properties near the mobility edge. Finally, we note that Eqs. (14) and (16) also have a trivial fixed point, $\bar{\rho}_L=0$, for any γ. This fixed-point characterizes the perfect metal phase towards which the system scales on the metallic side of the mobility edge.

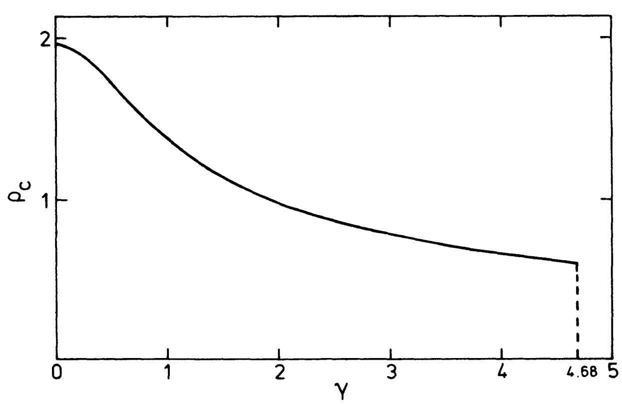

FIG. 1. Resistance at the mobility edge versus parameter γ defined in the text, for $d=3$.

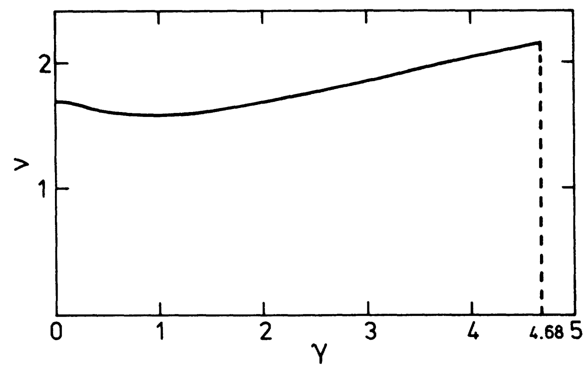

FIG. 2. Conductivity exponent near the mobility edge versus parameter γ defined in the text, for $d=3$.

A parameter of much experimental interest is the critical conductivity exponent ν given by Eqs. (18b), (22) and (23). The latter is plotted numerically in Fig. 2. The values of ν decrease from $\nu=1.68$ for $\gamma=0$ to a minimum $\nu\simeq1.58$ for $\gamma\simeq1$ while for larger values of γ they increase monotonically to a limiting value $\nu=2.15$ for $\gamma=\gamma_1$. These values are generally larger than previous estimates based on scaling theories: $1.25<\nu<1.75$,[21] $\nu=1.5$,[22] and $\nu=1$.[23] Note, however, that the value $\nu=1$ is usually obtained by extrapolating results for $d=2+\epsilon$ dimensions, $\epsilon\ll1$, to $d=3$.[23] As shown below, this procedure also yields $\nu=1$ when used in connection with the present treatment and with Shapiro's. Various experimental results reviewed by Thomas[23] are consistent with either $\nu=0.5$ or $\nu=1$. As will be shown in Sec. IV the fluctuations of resistance are large on both sides of the metal-insulator transition for $d=3$, making the resistance a non-self-averaging quantity. This may affect the accuracy of the analysis of experimental results for the resistance and, in particular, the determination of the exponent ν.

Finally, we discuss the form of the invariant probability distribution of resistance, $P_\rho^c(\rho)$, at the mobility edge $\bar{\rho}\equiv\rho_c$. This fixed distribution may be obtained in closed form from Eq. (17), using the Laplace transform of the rhs of Eq. (11) and inserting Eq. (5). After performing the necessary integrals we obtain the final expression

$$P_\rho^c(\rho)=\frac{1}{4(d-1)[2\pi l(\rho_c)]^{1/2}}\frac{1}{\rho}\left\{[\ln(e^{-\gamma l(\rho_c)}z_+)]\exp\left[-\frac{[\ln(e^{\gamma l(\rho_c)}z_+)]^2}{8l(\rho_c)}\right]-[\ln(e^{-\gamma l(\rho_c)}z_-)]\exp\left[-\frac{[\ln(e^{\gamma l(\rho_c)}z_-)]^2}{8l(\rho_c)}\right]\right\}, \tag{27}$$

where we have defined

$$z_\pm=1+2\rho\pm2\sqrt{\rho(\rho+1)}\ . \tag{28}$$

The distribution (27) depends parametrically on γ and on ρ_c via the inverse function $l(\rho_c)$ of $\bar{\rho}\equiv\rho_c$ given by (12). It is instructive to analyze the expression of $P_\rho^c(\rho)$ in the asymptotic limits $\rho\gg1$ and $\rho\ll1$. Noting that in the range of interest ($\gamma<\gamma_1$) the quantities ρ_c and $l(\rho_c)$ are of order unity and $\gamma l\lesssim1$, we obtain successively

$$P_\rho^c(\rho)\propto1/\rho,\quad \rho\ll1 \tag{29}$$

and

$$P_\rho^c(\rho)\simeq\{2(d-1)[2\pi l(\rho_c)^{1/2}\}^{-1}\rho^{-1}\ln(4\rho)\times\exp\left[-\frac{[\ln(4\rho)]^2}{8l(\rho_c)}\right],\quad \rho\gg1 \tag{30}$$

which may be rewritten in the form

$$P_\rho^c(\rho)=\frac{8}{d-1}\left[\frac{2l(\rho_c)}{\pi}\right]^{1/2}\frac{[\alpha(\rho)-1]}{(4\rho)^{\alpha(\rho)}},\quad \alpha(\rho)=1+\frac{\ln(4\rho)}{8l(\rho_c)}\ . \tag{31}$$

This expression is formally similar to the limit $\rho\gg1$ of the distribution at the critical point obtained by Shapiro[5] in terms of an exponent α [analogous to our $\alpha(\rho)$] which is, however, constant and of order 2.5 for $d=3$.

An important consequence of the limiting form (30) is that the finite-order moments of the distribution $P_\rho^c(\rho)$, $\rho_n^c=\int_0^\infty d\rho\,\rho^nP_\rho^c(\rho)$, are finite and, for example, the first- and second-order moments given by Eqs. (14)–(17) are

$$\rho_1^c\equiv\rho^c=[2(d-1)]^{-1}le^{2l}[\gamma\sinh(\gamma l)+2\cosh(\gamma l)]\ , \tag{32}$$

and

$$\rho_2^c=[2(d-1)]^{-1}le^{2l}\times\left[\frac{\gamma}{4}[e^{6l}\sinh(2\gamma l)-2\sinh(\gamma l)]+e^{6l}\cosh(2\gamma l)-\cosh(\gamma l)\right], \tag{33}$$

where $l\equiv l(\rho_c)$ is the function defined by Eq. (12). Since ρ_c (Fig. 1) and $l(\rho_c)$ are of order unity and $\gamma l(\rho_c)\lesssim1$ these equations yield $\rho_2^c/\rho_c^2\sim1$, which shows that $P_\rho^c(\rho)$ is quite broad. We note that Shapiro's distribution at the mobility edge differs qualitatively from Eqs. (27) and (31) in that all its moments beyond ρ_c are infinite. Our conclusion concerning the width of $P_\rho^c(\rho)$ may be checked explicitly for dimensionalities $d=2+\epsilon$, $\epsilon\ll1$, where solutions in the form of expansions in powers of ϵ exist. By expanding the rhs of Eqs. (14) and (15) we obtain the finite values $\rho_c\simeq\epsilon(1+\gamma^2/4)^{-1}$, $\rho_2^c\simeq3\epsilon^2(1+\gamma^2/4)^{-2}$, which lead to $\rho_2^c/\rho_c^2\simeq3$. Similarly, from Eqs. (22), (18b) ($g_c=\rho_c^{-1}$) and the above value of ρ_c we get $\nu=\epsilon^{-1}+O(1)$. This shows that to leading order the conductivity exponent is independent of γ and coincides with the universal value of Shapiro's one-parameter scaling theory.

IV. ASYMPTOTIC DISTRIBUTIONS OF RESISTANCE FOR DIMENSIONALITIES $d>1$

If the initial value of $\bar{\rho}_L$ at some arbitrary scale L_0 is less than ρ_c (for $d>2$), the system scales towards metallic behavior with $\bar{\rho}_L$ varying as $\bar{\rho}_L=AL^{2-d}$ for $L\to\infty$. If, on the other hand, $\bar{\rho}_{L_0}>\rho_c$, then the system scales towards insulating behavior with an exponentially increasing $\bar{\rho}_L$ reflecting localization of eigenstates on scales less than L. These results based on Eq. (14) assume that $P_\rho(\rho,L)$ is sufficiently well-behaved, as discussed in Sec. II. An important singular distribution for the metallic regime will, however, be obtained below by using the mean conductance as scaling variable instead of $\bar{\rho}_L$. In this section we study in detail the distribution of resistance on both the metallic and the insulating side of the mobility edge, in the infinite sample limit. Since the one-dimensional case has been discussed elsewhere,[12] we confine ourselves to dimensionalities $d>2$ and to the study of the insulating state for $d=2$.

A. Metallic regime, $\rho<\rho_c$

In this case Eq. (14) shows that $\bar{\rho}_L$ decreases with increasing L. More precisely, for $L\to\infty$ we have $\bar{\rho}_L\ll1$ and Eqs. (12) and (14) successively reduce to

$$l(\bar{\rho}_L)\simeq\bar{\rho}_L\ , \tag{34}$$

and

$$\frac{\partial\bar{\rho}_L}{\partial\ln l}=-\epsilon\bar{\rho}_L+O(\bar{\rho}_L^2),\quad \epsilon=d-2\ , \tag{35}$$

whose solution,

$$\bar{\rho}_L=AL^{2-d},\quad L\to\infty\ , \tag{36}$$

displays typical metallic behavior for $d>2$. On the other hand, the scaling transformation leading from a linear chain distribution $P_\rho^{(1)}(\rho,L)$ to the distribution $P_\rho(\rho,L)$ for a d-dimensional block is expressed by the equation

$$\frac{\partial P_\rho}{\partial \ln L}=(d-1)\frac{\partial}{\partial\rho}(\rho P_\rho)+l(\bar\rho_L)\frac{\partial P_\rho^{(1)}}{\partial l(\bar\rho_L)}\ , \quad (37)$$

which follows from Eq. (10) for $\Delta L\to 0$, with the usual parametrization of the rhs in terms of the mean resistance[5] via Eq. (12). In the present case $l(\bar\rho_L)$ is given by (34) and we require the linear chain distribution for $\rho\ll 1$ [where both $\bar\rho_L\ll 1$ and $l(\bar\rho_L)\ll 1$] given by the expansion of Eq. (34) of I for $\rho\to 0$, $l\to 0$:

$$P_\rho^{(1)}(\rho,l)=(2\pi l\rho)^{-1/2}e^{-\rho/2l}\times\left[1-\frac{\rho}{2}+\frac{\gamma^2}{8}(\rho-l)+\frac{1}{6}\frac{\rho^2}{l}+\cdots\right],\quad \rho\ll 1,\ \ l\ll 1\ , \quad (38)$$

with $l\equiv l(\bar\rho_L)$. The leading term of this expression yields

$$\frac{\partial P_\rho^{(1)}}{\partial \ln l}=-\frac{\partial}{\partial\rho}(\rho P_\rho^{(1)})\ ,$$

whose substitution into Eq. (37), with the replacement of $P_\rho^{(1)}$ by P_ρ in the sense of iteration, gives

$$\frac{\partial P_\rho}{\partial \ln L}=\epsilon\frac{\partial}{\partial\rho}(\rho P_\rho)\ , \quad (39)$$

which is independent of the disorder and of the average barrier. The important point to observe is that this linear equation admits *two* distinct solutions of interest for large L, namely a regular one,

$$P_\rho(\rho,L)\equiv P_\rho^{(1)}(\rho,L)=(2\pi AL^{-\epsilon}\rho)^{-1/2}e^{-\rho/2AL^{-\epsilon}}\ , \quad (40)$$

which leads to Eq. (36), and a singular one given by

$$P_\rho(\rho,L)\equiv P_\rho^{(2)}(\rho,L)=(2\pi BL^{\epsilon}\rho^3)^{-1/2}e^{-1/2BL^{\epsilon}\rho}\ . \quad (41)$$

Here A and B are arbitrary constants. We note that Shapiro's distribution for the metallic case differs from (40) by the form of the preexponential factor. We also note that while Eq. (40) reduces to the leading term of Eq. (38) for $d=1$, this is not a necessary requirement for the solution of (39) since in one dimension the general equation (37) is just an identity. While Eq. (40) describes a system where the mean resistance obeys Ohm's law, we show below that (41) implies Ohm's law for the mean conductance, which thus becomes the natural scaling variable, instead of $\bar\rho_L$, in this case.

The moments of Eq. (40) in the infinite sample limit are finite and are given by

$$\rho_n=\int_0^\infty d\rho\,\rho^n P_\rho(\rho,L)=(2AL^{-\epsilon})^n\pi^{-1/2}\Gamma(n+\tfrac{1}{2})\ ,\quad n=1,2,\ldots \quad (42)$$

where ρ_1, reduces to Ohm's law (36). We note that the distribution (40) is relatively broad with an rms to mean-value ratio of $\sqrt{2}$, which shows that ρ is non-self-averaging. On the other hand, all moments of Eq. (41) are unbounded. In order to discuss this distribution further we consider the properties of the random conductance $g=\rho^{-1}$. The distribution of g is defined in terms of $P_\rho(\rho,L)$ by

$$P_g(g,L)=\int_0^\infty d\rho\,P_\rho(\rho,L)\delta(g-\rho^{-1})\ , \quad (43a)$$

$$=g^{-2}P_\rho(g^{-1},L)\ . \quad (43b)$$

By performing the change of variable $g=\rho^{-1}$ in Eq. (39), using (43b), we obtain

$$\frac{\partial P_g}{\partial \ln L}=-\epsilon\frac{\partial}{\partial g}(gP_g)\ . \quad (44)$$

The formal similarity between Eqs. (44) and (39) then leads to solutions analogous to Eqs. (40) and (41):

$$P_g(g,L)\equiv P_g^{(1)}(g,L)=(2\pi AL^{-\epsilon}g^3)^{-1/2}e^{-1/2AL^{-\epsilon}g}\ , \quad (45)$$

$$P_g(g,L)\equiv P_g^{(2)}(g,L)=(2\pi BL^{\epsilon}g)^{-1/2}e^{-g/2BL^{\epsilon}}\ . \quad (46)$$

One readily verifies that $P_g^{(1)}(g,L)$ and $P_g^{(2)}(g,L)$ are related to Eqs. (40) and (41), respectively, by the transformation (43a), as required. While the moments of $P_g^{(1)}(g,L)$ are unbounded, those of $P_g^{(2)}(g,L)$ are given by

$$g_n=\langle g^n\rangle=(2BL^{\epsilon})^n\pi^{-1/2}\Gamma(n+\tfrac{1}{2}),\quad n=1,2,\ldots \quad (47)$$

where $\bar g_L\equiv g_1$ is seen to obey Ohm's law for conductance:

$$\bar g_L=BL^{d-2}\ . \quad (48)$$

Of course, due to statistical fluctuations we have $\langle g\rangle=\langle\rho^{-1}\rangle\neq\langle\rho\rangle^{-1}$ so that for a given system Ohm's law cannot be obeyed simultaneously by $\langle\rho\rangle$ and by $\langle g\rangle$. Our treatment, based on the Migdal-Kadanoff transformation, gives rise to two solutions (for $d>2$), Eqs. (40) and (46), which distinguish between physical situations where $\bar\rho_L$ and $\bar g_L$, respectively, are given by Ohm's law. Equations (40) and (46) thus correspond to distinct boundary conditions for the *general* solution of the first-order partial differential equation (44) (which is expressed in terms of an arbitrary function): finite resistance moments in the case of (40) and finite conductance moments in the case of (46). We note that recent first-principles perturbation theory analyses[24] have led to Ohm's law for the mean conductance. The latter is also a basic ingredient of the scaling theory of Abrahams *et al.*[1] These results would suggest that the metallic conductance is described by the regular distribution (46), rather than by the distribution (45) corresponding to the regular resistance distribution (40). Finally, it is of interest to examine the distributions of the resistivity $\lambda=\rho L^{\epsilon}$ and of the conductivity $\sigma=gL^{-\epsilon}$ obtained from Eqs. (40) and (46), respectively. These are

$$P_\lambda(\lambda,L)=(2\pi\bar\lambda\lambda)^{-1/2}e^{-\lambda/2\bar\lambda}\ , \quad (49)$$

and

$$P_\sigma(\sigma,L)=(2\pi\bar\sigma\sigma)^{-1/2}e^{-\sigma/2\bar\sigma}\ , \quad (50)$$

where $\bar\lambda\equiv\langle\lambda\rangle=A$ and $\bar\sigma\equiv\langle\sigma\rangle=B$ are the mean resistivity and the mean conductivity. Thus we find that the variance of σ is constant,

$$\langle\sigma^2\rangle-\langle\sigma\rangle^2=2B^2\ , \quad (51)$$

instead of going to zero for $L \to \infty$ (for $d=3$),[24] as required if metallic behavior is to be characterized by a well-defined conductivity. The fact that conductance (conductivity) fluctuations are not properly described within our improved treatment of the metallic domain may be due to the neglect of statistical fluctuations between the independent Migdal-Kadanoff chains, as discussed above.

B. Insulating regime, $\rho > \rho_c$

In this case, the rhs of Eq. (14) is positive to that $\bar{\rho}_L$ increases with L and becomes exponentially large, thus reflecting exponential localization of eigenstates. This limit corresponds to realizations of the random potential with $\rho \gg 1$. In this case the distribution of resistance obeys Eq. (37) with $\partial P_\rho^{(1)}/\partial l$ given by the Fokker-Planck equation (60) of Ref. 12, as corrected in the Erratum. In the asymptotic range of interest, $\bar{\rho} \gg 1$,

$$l(\bar{\rho}_L) \simeq \frac{1}{2+\gamma} \ln(4\bar{\rho}_L) \ . \tag{52}$$

After replacing, as usual, $P_\rho^{(1)}$ by P_ρ in the expression for $\partial P_\rho^{(1)}/\partial l$ the equation for $P_\rho(\rho,l)$ takes the final form

$$\frac{\partial P_\rho}{\partial \ln l} = (d-1)\frac{\partial}{\partial \rho}(\rho P_\rho) + \frac{\ln(4\bar{\rho}_L)}{2+\gamma} \times \left[(2-\gamma)P_\rho + (6-\gamma)\rho\frac{\partial P_\rho}{\partial \rho} + 2\rho^2 \frac{\partial^2 P_\rho}{\partial \rho^2} \right] , \quad \rho \gg 1 , \tag{53}$$

where for $\bar{\rho}_L$ we substitute the explicit solution of Eq. (14). For $\bar{\rho}_L \gg 1$ the latter equation reduces to

$$\frac{\partial \bar{\rho}_L}{\partial \ln l} = -(d-1)\bar{\rho}_L + \bar{\rho}_L \ln(4\bar{\rho}_L) + O(\bar{\rho}_L^{-1} \ln \bar{\rho}_L) , \tag{54}$$

which is solved by

$$\bar{\rho}_L = \tfrac{1}{4} e^{Al+d-1} , \tag{55}$$

where A is an arbitrary constant. The coefficient of the term in large parentheses in (53) is therefore

$$\frac{1}{2+\gamma}\ln(4\bar{\rho}_L) = \frac{1}{2+\gamma}(Al+d-1) \equiv A_1 l + B \ . \tag{56}$$

Equation (53) with the definition (56) is conveniently studied by converting it into an equation for the moments ρ_n:

$$\frac{\partial \rho_n}{\partial l} = -\frac{d-1}{l} n\rho_n + (A_1 + B/l) n(2n+\gamma)\rho_n , \quad n = 1,2,\ldots \tag{57}$$

whose solution reads

$$\rho_n \sim l^{-n(d-1)+Bn(2n+\gamma)} e^{A_1 n(2n+\gamma)l} , \tag{58}$$

where, e.g., ρ_1 displays the expected exponential growth of the mean resistance. Equations (58) show, in particular, that the exponential form of the relative rms deviation obtained earlier in the one-dimensional case[10,12–14] is retained to leading order at higher dimensionalities,

$$\frac{(\rho_2 - \rho_1^2)^{1/2}}{\rho_1} \simeq l^{2(d-1)/(2+\gamma)} e^{2A_1 l} , \tag{59}$$

although the rate of exponential growth determined by $(2A_1)^{-1}$ may depend on d. The power-law prefactor alone even implies an enhancement of the dispersion at higher dimensionalities. The probability distribution of resistance,

$$P_\rho(\rho,L) = \frac{1}{2\pi}\int_{-\infty}^{\infty} dk\, e^{-ik\rho}\phi(k) , \tag{60}$$

is now obtained by applying an earlier procedure[15] of exact linearization of the exponential exponent in (58) with respect to n. This enables us to express the characteristic function in the form

$$\phi(k) = \sum_{n=0}^{\infty} \frac{(ik)^n}{n!}\rho_n = (8\pi\tau)^{-1/2} e^{-\tau/2} \times \int_{-\infty}^{\infty} dx\, e^{x/2 - x^2/8\tau} \times e^{(ike^{-2\tau+x+A_1\gamma l} l^{-(d-1-B\gamma)})} , \tag{61}$$

$$\tau = A_1 l + B \ln l , \tag{62}$$

and by substituting this expression into (60) and performing the integrals, we finally obtain

$$P_\rho(\rho,l) = \frac{1}{\sqrt{8\pi\tau}\rho} \times \exp\left[-\frac{1}{8\tau}\{\ln\rho - [\gamma\tau - (d-1)\ln l]\}^2 \right] . \tag{63}$$

Like in the one-dimensional case,[12] the resistance thus has a log-normal distribution where, however, the variance and the mean value include logarithmic corrections to the characteristic linear variations with length scale in one dimension.[12] This shows that at higher dimensionalities the property of "additive mean" of $\ln\rho$ (Ref. 9) is valid only to leading order,

$$\overline{\ln\rho} = \gamma A_1 l - 2\frac{d-1}{2+\gamma}\ln l \ . \tag{64}$$

In particular, this equation defines a typical (most probable) resistance

$$\rho_s = l^{-2(d-1)/(2+\gamma)} e^{\gamma A_1 l} , \tag{65}$$

which is expected to be observed, at least in sufficiently long chains.[9] Finally, Eq. (63) with l related to $\bar{\rho}_L$ via (52) displays explicitly the parametrization of the distribution in terms of the two parameters $\bar{\rho}_L$ and γ. In spite of this difference with respect to the one-parameter scaling theory,[5] we confirm that $\ln\rho$, unlike ρ itself, is a proper self-averaging quantity since its relative rms deviation

$$\frac{1}{\overline{\ln\rho}}[\overline{\ln^2\rho}-(\overline{\ln\rho})^2]^{1/2}=\frac{2\sqrt{\tau}}{\gamma\tau-(d-1)\ln l}\ , \tag{66}$$

decreases with increasing length scale at any dimensionality for $L \to \infty$.

We now turn to the special case $d=2$ where there is no mobility edge and all eigenstates are localized. As noted by Abrahams *et al.*,[1] there is, however, a smooth crossover from logarithmic to exponential behavior of the mean conductance as the length scale is increased from a sufficiently small initial value to values $L \to \infty$. Of course, the above discussion for the range $L \to \infty$ ($\bar{\rho}_L$, $\rho \gg 1$) remains valid for $d=2$ indicating, in particular, an exponential growth of $\rho_1 \equiv \bar{\rho}_L$ for $L \to \infty$. On the other hand, in the low-resistance, quasimetallic regime, $\bar{\rho}_L \ll 1$, the expansion of the right-hand sides of Eqs. (12) and (14) to second order gives for $d=2$

$$\frac{\partial \bar{\rho}_L}{\partial \ln L}=(1+\gamma^2/4)\bar{\rho}_L^2\ , \tag{67}$$

which yields a conductance

$$\frac{1}{\bar{\rho}_L}=C-(1+\gamma^2/4)\ln l\ , \tag{68}$$

where C is a positive constant corresponding to the resistance of Eq. (36). This expression is of the form predicted by Abrahams *et al.*[1] at short scales. Here we are interested, more generally, in the explicit form of the distribution $P_\rho(\rho,L)$ at these short scales for $d=2$. By following the same steps as in Sec. IV A but including the corrections linear in ρ and/or l of Eq. (38) we obtain from Eq. (37)

$$\frac{\partial P_\rho}{\partial \ln l}=\left[\frac{\gamma^2}{8}(\rho-l)-\frac{\rho}{2}+\frac{1}{6}\frac{\rho^2}{l}\right]\frac{e^{-\rho/2l}}{\sqrt{2\pi l\rho}}\ ,\qquad \rho \ll 1,\ \ l \ll 1\ , \tag{69}$$

where we again express l on the rhs in terms of $\bar{\rho}_L^{(1)}$ using Eq. (12) and replace $\bar{\rho}_L^{(1)}$ by $\bar{\rho}_L$. Thus for $\bar{\rho}_L \ll 1$ we have $l(\bar{\rho}_L) \simeq \bar{\rho}_L$ with $\bar{\rho}_L$ defined by Eq. (68). By performing these substitutions in Eq. (69) we get

$$\frac{\partial P_\rho}{\partial y}=-\frac{\sqrt{y}}{1+\gamma^2/4}\left[\frac{\gamma^2}{8}\left[\rho-\frac{1}{y}\right]-\frac{\rho}{2}+\frac{1}{6}\rho^2 y\right]\frac{e^{-\rho y/2}}{\sqrt{2\pi\rho}}\ ,\qquad y=\bar{\rho}_L^{-1} \tag{70}$$

and finally

$$P_\rho(\rho,l)=\frac{2}{4+\gamma^2}\frac{e^{-\rho/2\bar{\rho}_L}}{(2\pi\rho\bar{\rho}_L)^{1/2}}\times\left\{1+\frac{\gamma^2}{2}+\frac{\rho}{3\bar{\rho}_L}-\left[\frac{\pi\bar{\rho}_L}{2\rho}\right]^{1/2}e^{\rho/2\bar{\rho}_L}\times\operatorname{erf}\left[\left[\frac{\rho}{2\bar{\rho}_L}\right]^{1/2}\right]\right\},\quad \rho \ll 1,\ \ \bar{\rho}_L \ll 1\ , \tag{71}$$

where

$$\operatorname{erf} z=\frac{2}{\sqrt{\pi}}\int_0^z e^{-t^2}dt\ .$$

This shows that, like in the one-dimensional case,[12] the distribution of resistance crosses over from a modified exponential form for $L \ll L_c$ to the log-normal form (63) for $L \gg L_c$. The length-scale dependence of the distribution in the range $L \ll L_c$ is, however, quite different in the cases $d=1$ and $d=2$. Finally, it is of interest to study the relative rms deviation in the present quasimetallic regime for $d=2$.[20] This is done most accurately by expanding Eq. (15) for ρ_2 in the limit $\bar{\rho}_L \to 0$, using Eq. (12). This yields

$$\frac{\partial \rho_2}{\partial \ln l}=-2\rho_2+6\bar{\rho}_L^2+O(\bar{\rho}_L^3)\ , \tag{72}$$

where $\bar{\rho}_L$ is given by Eq. (68). From the solution of (72) we obtain, to leading order,

$$\left[\frac{\rho_2}{\bar{\rho}_L^2}-1\right]^{1/2}\sim\frac{-\ln l}{l},\quad l \to 0\ , \tag{73}$$

which diverges for $l \to 0$. This divergency is to be compared with the stronger exponential divergency (59) obtained in the asymptotic large resistance regime.

V. CONCLUDING REMARKS

In this paper we have presented a first-principles analysis of the scaling properties of probability distributions of resistance, $P_\rho(\rho,L)$, in terms of two parameters: the mean resistance $\bar{\rho}_L$ [whose scaling behavior enters as an input for the study of $P_\rho(\rho,L)$] and a fixed parameter γ giving the ratio of the localization length L_c and of the penetration depth inside the average tunneling barrier V. In this connection we note that some evidence has been gathered recently against the validity of the simple one-parameter scaling ansatz of Abrahams *et al.*[1] On the one hand, Kumar[25] has questioned the internal consistency of the one-parameter scaling ansatz on general theoretical grounds. Kumar's work[25] has subsequently developed into a controversy which has been recently reviewed.[26] On the other hand, Kaveh and Mott[27,3] have calculated the critical conductance and found it to be significantly nonuniversal as a result of various effects. These authors have also summarized experimental results supporting this nonuniversality. Furthermore, from their numerical results on the conductivity and the participation ratio in three-dimensional systems, Ioffe *et al.*[16] have concluded that it is unlikely that scaling is governed by a single parameter.

In the two-parameter scaling theory discussed above both the critical resistance and the conductivity exponent are nonuniversal in that they depend on γ. This theory predicts, however, a universal value, $\gamma=\gamma_1$, beyond which a mobility-edge transition at an energy $E_c=V/2$ ceases to exist. We note that the parameter γ may also have important effects away from the critical point: for example, it controls the leading linear variation of $\overline{\ln\rho}$ with length scale in the asymptotic infinite sample limit in the insulating regime. In fact, the relation $\gamma l \sim \overline{\ln\rho}$ provides a useful interpretation of the parameter γ describing the average barrier V. As mentioned in Sec. III a basic feature of the scaling theory of Abrahams *et al.*[1] is the description of the microscopic disorder by the conductance g_0 (or resistance g_0^{-1}) at the cutoff length scale L_0.[2] Likewise we may identify γ with $1/l$ times the average of the logarithm of the resistance measured at a sufficiently long scale, $l \gg 1$. This shows that now the microscopic aspects of the system are measured by the two resistance parameters g_0^{-1} and $\overline{\ln\rho}$ at scales L_0 and $L \gg L_c$, respectively.

For completeness we mention that a simple two-parameter scaling theory has been proposed previously by McMillan[28] for the case where electron-electron interactions are included in the description of a disordered system. In the McMillan theory the new scaling parameter in addition to the conductance is the dimensionless interaction constant which, however, depends on length scale, unlike our parameter γ. Another example of a scaling theory with two scale-dependent parameters is the localization model with added percolation disorder discussed by Shapiro.[17] In fact, an alternative way of viewing the scaling equation (14) of Sec. II is to regard the rhs as a one-parameter scaling function depending on an additional constant parameter γ. From this point of view our treatment might be referred to as a "nonuniversal one-parameter scaling theory" to distinguish it from more general two-parameter scaling theories such as those of Refs. 27 and 17. Another reason for distinguishing the two types of two-parameter models would be the fact that in a scaling theory with two scale-dependent parameters one expects the exponents of the Anderson transition to be universal,[17] in contrast to the results of Sec. II. However, the use of the term "two-parameter scaling theory" in Sec. II has the advantage of not anticipating on the detailed form of the results concerning the mobility edge. On the other hand, we believe that one should be cautious in using the term universality (nonuniversality) borrowed from phase-transition theory, in the context of the mobility-edge transition. While universality in phase transitions refers indeed to the universality of critical exponents, this concept draws part of its importance, of course, from the concomitant intrinsic nonuniversality of the critical temperature. In contrast to this, *both* the critical exponent ν and the critical resistance are found to be universal in the one-parameter scaling theory of localization[5], while being nonuniversal in the two-parameter theory of Sec. II.

Note added. Our treatment of the metallic regime in Sec. IV A may be rationalized in terms of a modified description of the Migdal-Kadanoff chains (Sec. II), taking into account the fact that they are made up of d-dimensional blocks rather than of true one-dimensional units. The d-dimensional ($d>1$) aspect of the chains is included by assuming them to be described by a distribution of conductance with finite moments. On the other hand, their one-dimensional character is described by assuming their distribution of resistance to obey the differential scaling relation for a linear chain [Eq. (39) for $d=1$]. The internal consistency of the above picture follows from the existence of a solution of the scaling equation with finite conductance moments, as shown in Sec. IV A [Eq. (46) for $d=1$]. Our treatment of the metallic regime based on the Migdal-Kadanoff approach has been recently generalized to include the transverse fluctuations between independent chains (J. Heinrichs, A. M. Jayannavar, and N. Kumar, unpublished). In contrast to the analysis of Sec. IV A, this leads to a constant variance for the metallic conductance in 3D, as expected on the basis of first-principles studies.[24]

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