Gap solitons in nonlinear periodic structures

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In a medium with a dielectric constant periodic in one spatial direction, there are gaps in the dispersion relation of electromagnetic waves which propagate in the structure. If the index of refraction also contains a term proportional to the local-field intensity, we show that for frequencies within the gap, there exist soliton solutions to the nonlinear wave equation of the medium. This is demonstrated by analytic methods; the results agree with conclusions reached earlier, in numerical studies of power-dependent transmission through a finite superlattice, with a nonlinear element in each unit cell.

I. INTRODUCTION

A superlattice may be viewed, from the perspective of an electromagnetic wave which propagates within it, as a structure in which the dielectric constant exhibits a periodic variation with the coordinate in a direction normal to the interfaces. When an electromagnetic wave propagates in this direction, its dispersion relation is modified by the periodicity. Most particularly, gaps in the dispersion relation open at the various special points in the Brillouin zone, in a manner well known from the theory of wave propagation in periodic structures.

Recently Chen and one of us calculated the transmissivity of a finite superlattice, illuminated by radiation which propagates normal to the interfaces, and with a frequency which lies within a forbidden gap.[1] Interest was focused on the nonlinear optical response of the structure, when one film in each unit cell has a dielectric constant whose value depends on the local intensity of the optical wave. Dramatic transmission resonances were found at rather modest input power, with the transmissivity driven to a value of unity at the peak. The resonant peak had a skewed shape familiar from the theory of optical bistability, so such samples are predicted to exhibit hysteretic behavior, with switching from a state of low to high transmissivity as one sweeps through resonance by increasing the incident power. It was argued that for frequencies in the gap, soliton solutions to the nonlinear wave equation of the structure exist, and the resonance has its origin in soliton-mediated self-induced transparency.

These conclusions were based on numerical calculations carried out for a model superlattice. The purpose of the present paper is to explore the nonlinear wave equations for such structures by analytic methods, and we demonstrate that solitons with properties identical to those in Ref. 1 emerge from the analysis, under the conditions described below.

The present analysis is based on a model introduced in the work of Winful, Marburger, and Garmire,[2] who were interested in bistability in distributed feedback structures. This model is somewhat simpler than that used in Ref. 1, if we have application to superlattices in mind, but it clearly contains all the essential physical effects and can be applied directly to superlattices if the parameters are interpreted appropriately. Winful *et al.* based their analysis on a slowly varying envelope approximation, and we do so also. However, the set of equations that we obtain differs from that presented in Ref. 2. In our view, an essential feature that should be incorporated into the discussion is a proper description of the band gap and wave mixing produced by the periodic modulation of the dielectric constant of the structure, in the absence of nonlinearities in the response. This is absent from the discussion in Ref. 2, as discussed below.

In Sec. II we derive the equations central to our analysis, and Sec. III examines the analytic description of the solitons. Section IV concludes with brief comments.

II. THE BASIC EQUATIONS

We consider propagation of plane electromagnetic waves with frequency ω which propagate down a structure with a dielectric constant that varies periodically with path length z. As elsewhere,[2,3] we ignore harmonic generation on the assumption that the interactions which lead to its presence are not phase matched, and these beams are therefore weak.

The dielectric constant $\epsilon(z)$ is taken to have the sinusoidal spatial variation

$$\epsilon(z)=\epsilon+\Delta\epsilon\cos(Gz)\ . \tag{2.1}$$

If we wish to consider more complex spatial variations of $\epsilon(z)$, then we are to replace $\Delta\epsilon$ by the appropriate spatial Fourier coefficient of the profile. Only the particular spatial Fourier coefficient displayed in Eq. (2.1) enters our discussion importantly. A similar statement applies to the nonlinear coefficient λ defined below. If λ varies with position, it is its average value (the $G=0$ spatial Fourier component) that is to be identified with the parameter λ of the present paper.

Then the wave equation we wish to study is identical to that used earlier:[1,2]

$$\frac{d^2E}{dz^2}+\frac{\omega^2}{c^2}\epsilon(z)E+\frac{\omega^2}{c^2}\epsilon\lambda\,|E|^2E=0\;, \tag{2.2}$$

where $E(z)$ is the electric field in the plane-polarized wave.

We seek solutions of Eq. (2.2) of the form

$$E(z)=E_+(z)e^{ikz}+E_-(z)e^{i(k-G)z}\;, \tag{2.3}$$

where $E_\pm(z)$ are envelope functions assumed to vary slowly in space, on the length scale that is set by the periodic variation of $\epsilon(z)$. We shall be interested in wave vectors k that lie very close to the Brillouin zone boundary at $k=G/2$, so we write

$$k=\tfrac{1}{2}G-\Delta k\;, \tag{2.4}$$

where $\Delta k \ll G/2$. Then Eq. (2.3) becomes

$$E(z)=e^{-i\Delta kz}[E_+(z)e^{i(G/2)z}+E_-(z)e^{-i(G/2)z}]\;. \tag{2.5}$$

It is at this early point that our discussion departs from that presented by Winful *et al.*[2] These authors also considered a solution in the form of right and left running waves, each modulated by an envelope function that varies slowly in space. However, the running waves were assumed to have the form $\exp(\pm ik_0z)$, where $k_0=\omega\sqrt{\epsilon}/c$ is the wave vector of a wave propagating in a *uniform* dielectric, with $\Delta\epsilon=0$. In any periodic medium, in fact, the frequency and wave vector of the basic waves are coupled through a dispersion relation which must emerge from the analysis of the underlying wave equation. This is true also for waves in a nonlinear medium as well; the dispersion relation then depends on field intensity.[4] As a consequence of the procedure just described, a dispersion relation does not emerge from the analysis in Ref. 2, but one appropriate to the perfectly uniform medium is imposed at the outset. We shall see that in the present treatment, the proper dispersion relation emerges when the strength λ of the nonlinearity is set to zero. Without this feature, the soliton solution discussed in Sec. III would not be contained in the theory.

The discussion here, and that in Ref. 2, assumes implicitly that both $\Delta\epsilon$ and λ are small; if $\Delta\epsilon$ were not small, for example, for $\lambda=0$ and for wave vectors near the zone boundary, the eigensolution could not be represented as a linear combination of only two counterpropagating waves, as in Eq. (2.5). It is our view that the treatments here and in Ref. 2 contain different assumptions about the relative magnitude of the two small parameters $\Delta\epsilon$ and λ. Our theory is built around a solution that is exact when $\lambda=0$, and is appropriate when the nonlinearity is regarded as a weak perturbation on the mixing provided by the corrugation in the dielectric constant. On the other hand, when $\Delta\epsilon=0$, then the approach in Ref. 2 provides a complete description (within the slowly varying envelope approximation) of the strong backscattering present by virtue of the nonlinear term controlled by λ. The equations describe a situation where the amplitude of the backscattered wave generated by the nonlinearity is so strong that depletion of the pump wave is large, and feedback between the two counterpropagating waves is important. The approach is appropriate for examining the influence of small-amplitude corrugations in the dielectric constant on this picture. Thus, in essence, Ref. 2 assumes implicitly that $\Delta\epsilon \ll \lambda$, and results appropriate to the limit $\lambda\to 0$ are not contained in the scheme. We believe the present approach is the one suitable for discussing distributed feedback devices, for frequencies near a gap, where a proper description of the strong Bragg reflections provided by the corrugation of the dielectric constant must be incorporated into the theory.

We proceed by inserting the form in Eq. (2.5) into Eq. (2.1), ignoring second derivatives of E_+ and E_-, and retaining only terms proportional to $\exp(\pm i\tfrac{1}{2}Gz-i\Delta kz)$. Equating the coefficients of each term to zero leads to two differential equations for the pair of functions E_+ and E_-. These have the form, for $\Delta k \ll G/2$,

$$\left[\frac{\omega^2}{c^2}\epsilon-\left[\frac{G}{2}\right]^2+G\Delta k\right]E_+ +\frac{\omega^2}{2c^2}\Delta\epsilon E_- +iG\frac{dE_+}{dz}+\frac{\omega^2}{c^2}\epsilon\lambda(|E_+|^2+2|E_-|^2)E_+=0\;, \tag{2.6a}$$

and

$$\left[\frac{\omega^2}{c^2}\epsilon-\left[\frac{G}{2}\right]^2-G\Delta k\right]E_- +\frac{\omega^2}{2c^2}\Delta\epsilon E_+ -iG\frac{dE_-}{dz}+\frac{\omega^2}{c^2}\epsilon\lambda(|E_-|^2+2|E_+|^2)E_-=0\;. \tag{2.6b}$$

From Eqs. (2.6) it is straightforward to establish that the quantity

$$W=|E_+|^2-|E_-|^2 \tag{2.7}$$

is independent of position, a property shared also by the envelope functions studied in Ref. 2.

If the parameter λ is set to zero, then $E_+(z)$ and $E_-(z)$ are independent of z. The dispersion relation of waves in the structure is obtained by setting the appropriate 2×2 determinant equal to zero. One finds the two-branch dispersion relation, with $\omega_G=cG/2\sqrt{\epsilon}$,

$$\omega_\pm^2(k)=\omega_G^2\pm\tfrac{1}{2}\omega_G^2\left[\left[\frac{\Delta\epsilon}{\epsilon}\right]^2+\left[\frac{4\Delta k}{G}\right]^2\right]^{1/2}. \tag{2.8}$$

The gap at the zone boundary $\Delta k=0$ thus is bounded from below by

$$\omega_-=\omega_G\left[1-\frac{1}{2}\frac{\Delta\epsilon}{\epsilon}\right]^{1/2} \tag{2.9a}$$

and above by

$$\omega_+ = \omega_G \left[1 + \frac{1}{2}\frac{\Delta\epsilon}{\epsilon} \right]^{1/2} . \tag{2.9b}$$

Suppose we consider solutions of Eqs. (2.6), again linearized with $\lambda = 0$, for frequencies ω within the forbidden gap, $\omega_- < \omega < \omega_+$. With $\Delta k = 0$, these are exponentially damped, with the spatial variation $\exp(\pm\alpha z)$, where α is given by

$$\alpha = G \frac{(\omega^2 - \omega_-^2)^{1/2}(\omega_+^2 - \omega^2)^{1/2}}{\omega_G^2} . \tag{2.10}$$

This completes our discussion of the general structure of the equations that form the basis of the analysis in Sec. III.

III. GAP SOLITONS

We explore solutions of the full nonlinear equations, Eqs. (2.6), for frequencies ω which lie in the forbidden gap, $\omega_- \le \omega \le \omega_+$. The parameter $W = |E_+|^2 - |E_-|^2$ is related to the rate of energy transport in the wave, and the solitons of interest are stationary and transport no energy. Thus for these solutions $W=0$, and $|E_+| = |E_-| = \mathcal{E}(z)$. We may rewrite Eqs. (2.6) in the form (we set $\Delta k = 0$ for the stationary solution)

$$(\omega^2 - \omega_G^2)E_+ + \frac{\omega_G^2}{2}\frac{\Delta\epsilon}{\epsilon}E_- + i4\frac{\omega_G^2}{G}\frac{dE_+}{dz} + 3\omega_G^2\lambda\mathcal{E}^2 E_+ = 0 , \tag{3.1a}$$

$$(\omega^2 - \omega_G^2)E_- + \frac{\omega_G^2}{2}\frac{\Delta\epsilon}{\epsilon}E_+ - i4\frac{\omega_G^2}{G}\frac{dE_-}{dz} + 3\omega_G^2\lambda\mathcal{E}^2 E_- = 0 . \tag{3.1b}$$

Upon taking the complex conjugate of Eq. (3.1b), then comparing with Eq. (3.1a), one may establish that $E_- = (E_+)^*$. Hence, we seek solutions of the form $E_+(z) = \mathcal{E}(z)\exp[i\phi(z)]$, $E_-(z) = \mathcal{E}(z)\exp[-i\phi(z)]$. Upon inserting these forms into Eq. (3.1a), and separating real and imaginary parts, we have a pair of differential equations for $\mathcal{E}(z)$ and $\phi(z)$:

$$\frac{4}{G}\frac{d\phi}{dz} = \left[\frac{\omega^2 - \omega_G^2}{\omega_G^2} \right] + \frac{\Delta\epsilon}{2\epsilon}\cos(2\phi) + 3\lambda\mathcal{E}^2 \tag{3.2a}$$

and

$$\frac{4}{G}\frac{d\mathcal{E}}{dz} = \frac{\Delta\epsilon}{2\epsilon}\mathcal{E}\sin(2\phi) . \tag{3.2b}$$

One may now differentiate Eq. (3.2a) with respect to z, then use Eq. (3.2b) to eliminate $d\mathcal{E}/dz$. This leads to an equation for ϕ alone:

$$\left[\frac{4}{G}\right]^2 \frac{d^2\phi}{dz^2} + \frac{\Delta\epsilon}{\epsilon}\left[\frac{\omega^2 - \omega_G^2}{\omega_G^2}\right]\sin(2\phi) + \left[\frac{\Delta\epsilon}{2\epsilon}\right]^2 \sin(4\phi) = 0 . \tag{3.3}$$

This possesses a first integral. We write this

$$\frac{1}{2}\left[\frac{4}{G}\right]^2 \left[\frac{d\phi}{dz}\right]^2 + \frac{1}{2}\frac{\Delta\epsilon}{\epsilon}\left[\frac{\omega^2 - \omega_G^2}{\omega_G^2}\right][1 - \cos(2\phi)] + \frac{1}{4}\left[\frac{\Delta\epsilon}{2\epsilon}\right]^2 [1 - \cos(4\phi)] = C , \tag{3.4}$$

where C is a constant of integration.

The differential equation displayed in Eq. (3.3) is a well-known form in the study of nonlinear systems. It is known as the double sine-Gordon equation; a particularly clear tabulation of its soliton solutions has been given recently by Campbell and collaborators.[5] In what follows, we cast the equation in a form that places it in contact with their results. It is convenient to consider first frequencies in the upper half gap, $\omega_G \le \omega < \omega_+$, then frequencies in the lower half gap, $\omega_- < \omega \le \omega_G$.

(i) The case $\omega_G \le \omega < \omega_+$: We let $z = \alpha_+ x$, where

$$\alpha_+^2 = \frac{8}{G^2}\frac{\epsilon}{\Delta\epsilon}\frac{\omega_G^2}{\omega^2 - \omega_-^2} , \tag{3.5a}$$

and define

$$\eta_+ = -\frac{\Delta\epsilon}{8\epsilon}\frac{\omega_G^2}{\omega^2 - \omega_G^2} , \tag{3.5b}$$

and we set $\phi = \psi/4$. The Eq. (3.3) assumes the form of Eq. (1.1) of Ref. 5:

$$\frac{d^2\psi}{dx^2}\frac{4}{1+4|\eta_+|}[\tfrac{1}{2}\sin(\tfrac{1}{2}\psi) - \eta_+ \sin\psi] = 0 . \tag{3.6}$$

If we regard ψ as the displacement of a particle as a function of a fictitious time x, then Eq. (3.6) describes the motion of the particle in the effective potential

$$V(\psi) = 4[\cos(\psi/2) - \eta_+ \cos\psi]/(1 + 4|\eta_+|) .$$

One sees easily that as ω is swept through the frequency range $\omega_G \le \omega < \omega_+$, we have $-\infty \le \eta_+ \le -\frac{1}{4}$ which corresponds to region I in the classification scheme employed in Ref. 5. In this parameter regime, one encounters two distinctly different types of kink solution. These are written, for the kink (K) solution or the antikink ($\bar{K}$),

$$\psi^{\rm I}_{K(\bar{K})}(x) = (2n+1)2\pi \pm 4\tan^{-1}\left\{ \left[\frac{4|\eta_+| - 1}{4|\eta_+| + 1}\right]^{1/2} \tanh\left[\left[\frac{4|\eta_+| - 1}{16|\eta_+|}\right]^{1/2} x \right] \right\} , \tag{3.7a}$$

and for that of type II,

$$\psi^{\rm II}_{K(\bar K)}(x)=(2n)2\pi\pm4\tan^{-1}\left\{\left[\frac{4|\eta_+|+1}{4|\eta_+|-1}\right]^{1/2}\tanh\left[\left[\frac{4|\eta_+|-1}{16|\eta_+|}\right]^{1/2}x\right]\right\}. \tag{3.7b}$$

In the above expressions, the plus sign describes the kink, and the minus sign the antikink.

We shall examine the properties of these solutions in more detail below, but first we turn to the solutions in the lower half gap, $\omega_- < \omega \le \omega_G$.

(ii) The case $\omega_- < \omega \le \omega_G$: We let $\psi = \Theta + 2\pi$, so Eq. (3.3) becomes, after introducing

$$\alpha_-^2 = \frac{8}{G^2}\frac{\epsilon}{\Delta\epsilon}\frac{\omega_G^2}{\omega_+^2-\omega^2} \tag{3.8a}$$

and

$$\eta_- = -\frac{\Delta\epsilon}{8\epsilon}\frac{\omega_G^2}{\omega_G^2-\omega^2}, \tag{3.8b}$$

identical to Eq. (3.8), but with η_+ simply replaced by η_-, and ψ by Θ. One sees also that $-\infty \le \eta_- \le -\frac{1}{4}$, as ω is swept from ω_G down to ω_-. Hence we again have two distinct classes of kinks and their associated antikinks:

$$\Theta^{\rm I}_{K(\bar K)}(x)=(2n+1)2\pi\pm4\tan^{-1}\left\{\left[\frac{4|\eta_-|-1}{4|\eta_-|+1}\right]^{1/2}\tanh\left[\left[\frac{4|\eta_-|-1}{16|\eta_-|}\right]^{1/2}x\right]\right\} \tag{3.9a}$$

and

$$\Theta^{\rm II}_{K(\bar K)}(x)=(2n)2\pi\pm4\tan^{-1}\left\{\left[\frac{4|\eta_-|+1}{4|\eta_-|-1}\right]^{1/2}\tanh\left[\left[\frac{4|\eta_-|-1}{16|\eta_-|}\right]^{1/2}x\right]\right\}. \tag{3.9b}$$

Now notice the following identities:

$$\left[\frac{4|\eta_-|-1}{4|\eta_-|+1}\right]^{1/2}=\left[\frac{4|\eta_+|+1}{4|\eta_+|-1}\right]^{1/2}=\frac{(\omega^2-\omega_-^2)^{1/2}}{(\omega_+^2-\omega^2)^{1/2}}, \tag{3.10a}$$

and

$$\left[\frac{4|\eta_-|-1}{16|\eta_-|}\right]^{1/2}\frac{1}{\alpha_-}=\left[\frac{4|\eta_+|-1}{16|\eta_+|}\right]^{1/2}\frac{1}{\alpha_+}\equiv\frac{1}{d}, \tag{3.10b}$$

where an explicit expression for the dimensionless length d is

$$d=\frac{4\omega_G^2}{G}\frac{1}{(\omega^2-\omega_-^2)^{1/2}(\omega_+^2-\omega^2)^{1/2}}. \tag{3.11}$$

If the expressions given above are compared, and rewritten in terms of the original angle ϕ, as ω is varied through the entire gap from ω_- to ω_+, we have only two distinct solutions. The pair $\Theta^{\rm I}_{K(\bar K)}$ evolve into $\psi^{\rm II}_{K(\bar K)}$ as ω is swept through the midgap frequency ω_G, while $\Theta^{\rm II}_{K(\bar K)}$ evolve into $\psi^{\rm I}_{K(\bar K)}$. We are left with the following two kink-antikink pairs, throughout the gap ω_G:

$$\phi^{\rm I}_{K(\bar K)}(z)=n\pi\pm\tan^{-1}\left[\frac{(\omega^2-\omega_-^2)^{1/2}}{(\omega_+^2-\omega^2)^{1/2}}\tanh\frac{z}{d}\right] \tag{3.12a}$$

and

$$\phi^{\rm II}_{K(\bar K)}(z)=(2n+1)\frac{\pi}{2}\pm\tan^{-1}\left[\frac{(\omega_+^2-\omega^2)^{1/2}}{(\omega^2-\omega_-^2)^{1/2}}\tanh\frac{z}{d}\right]. \tag{3.12b}$$

We next turn our attention to the envelope function $\mathcal{E}(z)$, which may be calculated most easily by rearranging Eq. (3.2a):

$$\mathcal{E}^2(z)=\frac{1}{3\lambda}\left[\frac{4}{G}\frac{d\phi}{dz}+\left[\frac{\omega_G^2-\omega^2}{\omega_G^2}\right]-\frac{\Delta\epsilon}{2\epsilon}\cos(2\phi)\right]. \tag{3.13}$$

The study of $\mathcal{E}^2(z)$ will lead us to new constraints on the allowed solutions.

For instance, for the type-I soliton, with $\phi(z)$ given by Eq. (3.12a), one finds

$$\mathcal{E}_{\rm I}^2(z)=-\frac{\omega^2-\omega_-^2}{3\lambda\omega_G^2}\frac{(\mp1+1)\,{\rm sech}^2(z/d)}{1+\left[\dfrac{\omega^2-\omega_-^2}{\omega_+^2-\omega^2}\right]\tanh^2(z/d)}, \tag{3.14a}$$

where the upper choice of sign is appropriate for the kink solution, and the lower choice for the antikink. Clearly, $\mathcal{E}_{\rm I}^2(z)\neq0$ only for the antikink solution. Furthermore, we find this solution only if the nonlinear coupling constant $\lambda<0$. Hence, with this choice we have

$$\mathcal{E}_{\rm I}^2(z)=\frac{2(\omega^2-\omega_-^2)}{3\,|\,\lambda\,|\,\omega_G^2}\,\frac{\operatorname{sech}^2(z/d)}{1+\left[\dfrac{\omega^2-\omega_-^2}{\omega_+^2-\omega^2}\right]\tanh^2(z/d)}$$

$$(\lambda<0;\ \text{antikink})\ . \qquad (3.14b)$$

Similarly, if we consider the type-II kink and antikink pair, Eq. (3.13) gives

$$\mathcal{E}_{\rm II}^2(z)=\frac{\omega_+^2-\omega^2}{3\lambda\omega_G^2}\,\frac{(\pm 1+1)\operatorname{sech}^2(z/d)}{1+\left[\dfrac{\omega_+^2-\omega^2}{\omega^2-\omega_-^2}\right]\tanh^2(z/d)}\ . \qquad (3.15a)$$

Now we achieve a nontrivial solution if we choose the kink solution, and also $\lambda>0$. Thus, we have

$$\mathcal{E}_{\rm II}^2(z)=\frac{2(\omega_+^2-\omega^2)}{3\lambda\omega_G^2}\,\frac{\operatorname{sech}^2(z/d)}{1+\left[\dfrac{\omega_+^2-\omega^2}{\omega^2-\omega_-^2}\right]\tanh^2(z/d)}$$

$$(\lambda>0;\ \text{kink})\ . \qquad (3.15b)$$

This completes our discussion. For each choice of λ, we have one solitary wave solution of the nonlinear wave equation, for all frequencies within the gap. For the antikink solution given in Eq. (3.14b), the spatial variation of the phase is given in Eq. (3.12a), with the lower choice of sign, while for the kink solution in Eq. (3.15a), the spatial variation of the phase is given by Eq. (3.12b), with the upper sign chosen.

IV. DISCUSSION

In this section, we make contact with the solitons studied in Ref. 1.

First note that the spatial scale of the soliton is controlled by the length d, defined in Eq. (3.10). Near either the lower gap edge, $\omega\cong\omega_-$, or the upper gap edge, $\omega\cong\omega_+$, d becomes very large compared to the lattice constant a_0 of the periodic structure. Recall that G is related to the lattice constant by the relation $G=2\pi/a_0$, so we may replace Eq. (3.10) by

$$d=\frac{2a_0}{\pi}\,\frac{\omega_G^2}{(\omega^2-\omega_-^2)^{1/2}(\omega_+^2-\omega^2)^{1/2}}\ . \qquad (4.1)$$

For frequencies near midgap, $\omega\cong\omega_G$, we have

$$d\cong\frac{8a_0}{\pi}\,\frac{\epsilon}{\Delta\epsilon}\ . \qquad (4.2)$$

The numerical calculations in Ref. 1 examined a parameter set where $\Delta\epsilon\cong\epsilon$, in the present language. Then Eq. (4.2) shows that in midgap, there are no spatially extended solitary waves with envelope-function many-lattice constants in extent. As one moves away from either gap edge toward midgap, the envelope function shrinks, to become the size of the lattice constant a_0 itself in the midgap region, when $\Delta\epsilon\cong\epsilon$. (Note, incidentally, that the slowly varying envelope approximation used in the present paper is valid only when $d\gg a_0$, so we can only regard the predictions found here as approximate, when $\Delta\epsilon\sim\epsilon$. The present treatment is valid for *all* frequencies in the gap for a weakly modulated structure, for which $\Delta\epsilon\ll\epsilon$. Such weakly modulated structures were of primary interest to Winful, Marburger, and Garmire.[2]) In Ref. 1, the numerical studies found spatially extended solitons (i.e., solitons with $d\gg a_0$) only near either gap edge, and it was noted that as one moved away from the edge into the gap, the envelope indeed shrank in size. No solitons were found near midgap, where numerical stability problems were encountered in the solution. The method of starting the procedure of integrating the equations is readily seen to be inappropriate to the study of highly localized solitons, so this is not surprising.

Consider the case $\lambda<0$, and the antikink solution described Eqs. (3.12a) and (3.14b). Near the lower band edge $\omega\cong\omega_-$, one may overlook the $\tanh^2(z/d)$ in the denominator, and replace Eq. (3.14b) by

$$\mathcal{E}_{\rm I}^2(z)\cong\frac{4(\omega-\omega_-)}{3\,|\,\lambda\,|\,\omega_G}\operatorname{sech}^2(z/d)\ , \qquad (4.3a)$$

while the phase angle is given by

$$\phi_K^{\rm I}(z)=n\pi-2\left[\frac{\epsilon}{\Delta\epsilon}\right]^{1/2}\frac{(\omega-\omega_-)^{1/2}}{\omega_G^{1/2}}\tanh\left[\frac{z}{d}\right]\ . \qquad (4.3b)$$

In Ref. 1, for $\lambda<0$, spatially extended solitons were found only near the lower band edge, $\omega\cong\omega_-$; it was noted that here $\mathcal{E}(z)$ was accurately fitted by the functional form $\operatorname{sech}(z/d)$, as displayed in Eq. (4.3a). Also, examination of the numerical results (see, for example, Fig. 2 of Ref. 1) shows that $\phi_K^{\rm I}(z)$ is very small for all z. The results are consistent with $\phi_K^{\rm I}(z)\cong n\pi$, as given by Eq. (4.3b) for $\omega\cong\omega_-$.

Thus the above limiting forms nicely describe the soliton states explored near the lower gap edge in Ref. 1 for the case $\lambda<0$. When $\lambda>0$, spatially extended solitons were found only near the upper gap edge, $\omega\cong\omega_+$. One sees that properties of the type-II kink soliton [Eqs. (3.15b) and (3.12b)] reproduce the results obtained in these studies, near the upper gap edge.

For $\lambda<0$, as we have seen, the type-I soliton also exists near the upper gap edge, $\omega\cong\omega_+$, though the numerical work in Ref. 1 yielded solitons here only in the opposite case $\lambda>0$. For $\lambda<0$ and for $\omega\cong\omega_+$, the spatial profile of the type-I soliton is very different from the limiting forms that apply near the bottom of the gap [Eqs. (4.3a) and (4.3b)] despite the fact that scale length d becomes very long near ω_+, as it does near ω_-. The point is that now the denominator in Eq. (3.14b) plays a crucial role, while it may be set to unity to good approximation for $\omega\cong\omega_-$, as we have seen. For ω close to ω_+, and values of z comparable to the scale length d, the denominator in Eq. (3.14b) becomes very large and drives $\mathcal{E}_{\rm I}(z)$ to zero everywhere except in the region $z\ll d$. There the soliton does have a "core," in the region $z\sim a_0\ll d$, within which $\mathcal{E}_{\rm I}(z)$ is large. A description of this core is found by replacing $\tanh^2(z/d)$ by $(z/d)^2$ in Eq. (3.14b), when $z\ll d$. Note that for any fixed z, as ω approaches ω_+, this approximation will always become valid since then d approaches infinity. One finds

$$\mathcal{E}_{\mathrm{I}}^2(z)=\frac{8a_0^2}{3\pi^2|\lambda|}\frac{\epsilon}{\Delta\epsilon}\frac{1}{z^2+\frac{4a_0^2}{\pi^2}\left[\frac{\epsilon}{\Delta\epsilon}\right]^2} \quad (z \ll d,\ \omega\cong\omega_+) \ . \tag{4.4}$$

The "core" of the type-I soliton is thus localized to a spatial region on the order of a_0, for $\Delta\epsilon \sim \epsilon$, and once again the methods of Ref. 1 prove inadequate for the study of such states.

The discussion presented here shows how the solitons found in Ref. 1 emerge from an analytic discussion when the frequency is near either edge of the forbidden gap. If we have a structure within which $\Delta\epsilon \ll \epsilon$, then the description obtained here applies for all frequencies within the gap, and we see that either choice of the nonlinear coupling constant λ allows a solitary wave soliton with an envelope function that has a large spatial extent compared to the lattice constant a_0, for all frequencies within the gap. If $\Delta\epsilon \sim \epsilon$, both the discussion presented here and the method used in Ref. 1 prove inadequate to study the type-I solution away from the lower gap, or the type-II solution away from the upper gap edge.

The earlier work[1] elucidated the relationship between the spatially extended gap solitons and the nonlinear response of superlattice structures, as remarked in Sec. I.

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[4]The nonlinear dispersion relation of surface polaritons, in the presence of a nonlinear medium, has been discussed by A. A. Maradudin, in *Proceedings of the Second International School on Condensed Matter Physics, Varna, Bulgaria,* edited by M. Borissov (World Scientific, New York, 1982). A rather general treatment of this problem has been given by K. M. Leung, Phys. Rev. B **32**, 5093 (1985). While these references both discuss surface polaritons, similar considerations apply to electromagnetic waves which propagate in the bulk of nonlinear media.

[5]David K. Campbell, Michel Peyrard, and Pasquale Sodano, Physica **19D**, 165 (1986).