

## Multifractals and critical phenomena in percolating networks: Fixed point, gap scaling, and universality

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Analogies between critical phenomena and the continuous spectrum of scaling exponents associated with fractal measures are pointed out. The analogies are based first on the Hausdorff-Bernstein reconstruction theorem, which states that the *positive integer* moments suffice to characterize a probability distribution function with finite support, and second on the joint probability distribution for the positive integer moments. This joint probability distribution, which can be considered as a fixed point, is universal and exhibits both gap scaling and the infinite set of exponents. Monte Carlo simulations of the electrical properties of percolation clusters on the square and triangular lattices support this general result. Extensions to other fields where infinite sets of exponents have arisen, such as diffusion-limited aggregation and localization, should be straightforward.

The study of infinite sets of exponents, which originated in the field of turbulence,<sup>1</sup> has recently become the focus of attention in a number of fields involving fractal or scaling objects, ranging from random resistor networks,<sup>2-5</sup> turbulence,<sup>1</sup> dynamical systems,<sup>6,7</sup> diffusion limited aggregates (DLA),<sup>8</sup> to localization.<sup>9,10</sup> What is common to these different fields is that one wants to characterize the properties of a "weight" or "measure" associated to different parts of a fractal object.

In this paper, we wish to point out that the infinite set of exponents of fractal objects, or their continuous spectrum of scaling indices,<sup>1,6,7</sup> is analogous to a subset of the infinite set of irrelevant exponents associated with symmetry-breaking operators in critical phenomena. The infinite set of exponents for percolation networks has been justified by the possibility of their experimental observation,<sup>2,3</sup> which suggests that they should be considered instead as relevant exponents. But even in the field of critical phenomena, "irrelevant" exponents, in the renormalization-group sense, are also becoming experimentally accessible.<sup>12</sup>

To simplify the discussion, we restrict ourselves, from now on, to percolating random resistor networks and discuss the current distribution, but it should be clear that most of our results are applicable to DLA<sup>8,11</sup> and localization<sup>9,10</sup> as well. The analogy with critical phenomena proceeds through two steps. First, a reconstruction formula known from probability theory is used to argue that positive integer moments of the appropriate probability distribution<sup>13</sup>  $\mathfrak{P}$  suffice to recover all moments. Second, we propose a scaling form for the joint probability distribution which summarizes all known results and exhibits both the infinite set of exponents and the analog of "gap scaling" in critical phenomena (i.e., the sufficiency of a small number of exponents to characterize certain classes of observables). This scaling hypothesis and its consequences are supported by extensive Monte Carlo simulations.

Let us consider lattices whose bonds are occupied by

resistances  $r$  with probability  $p$  or by insulators with probability  $1-p$ . Recall that in these random resistor networks close to the percolation threshold, the *measurable* quantities of interest<sup>2,3</sup> are of the form  $\sum i_a^n$ , where  $n$  is an integer and  $ri_a^2$  ( $r=1$  in the following) is the power dissipated in the branch  $a$  of the network and the sum is over all branches  $a$  of the backbone. It is these (in principle) *experimentally* accessible moments<sup>2,3</sup> of the current distribution which have been shown<sup>2-5</sup> to each scale with different exponents  $-x_n$ :  $-x_0$  is the fractal dimension of the conducting backbone,  $-x_1$  is related to the conductivity exponent  $t$ , and the other  $x_n$  to correlation functions for the resistance fluctuations. From the probability distribution  $\mathfrak{P}$  for the current, one can obtain expectation values for all the various moments of interest:

$$\begin{aligned} \overline{i^{2n}} &\equiv \int_0^1 di^2 \mathfrak{P}(i^2, \xi, L) i^{2n} \\ &\equiv \left\langle \frac{\sum i_a^{2n}}{\sum i_a^0} \right\rangle \approx a_n L^{-x_n + x_0}, \end{aligned} \quad (1)$$

with  $\xi \approx (p-p_c)^{-\nu}$  the bulk correlation length,  $L$  the system size, and  $a_n$  an amplitude for the asymptotic scaling behavior. The angular brackets refer to averages over the sample realizations with  $L \ll \xi$ . As before, sums are over the backbone branches, while the overbar will from now on be a shorthand notation for moments of  $\mathfrak{P}$ .

We first discuss  $\mathfrak{P}(i^2, \xi = \infty, L)$ , which we denote as  $\mathfrak{P}(i^2, L)$  and interpret as the probability that a branch of a system of size  $L$  carries a current (squared)  $i^2$ . The finite correlation length case may be obtained from ordinary scaling arguments and is discussed in the last section.

While negative moments of the current distribution can be physically meaningful<sup>15</sup> and noninteger moments have been considered, we want to stress that all the information can be extracted from the positive integer moments.

This follows from a theorem due to Hausdorff which can be reformulated as follows: The positive integer moments of a probability distribution concentrated on a *finite interval* determine that distribution uniquely.<sup>14</sup> Since  $i^2$

is larger than zero and smaller than the square of the total input current,  $\mathfrak{P}$  satisfies the conditions of the theorem. For convenience, we can always choose the total input current to be unity, in which case  $i^2$  varies between 0 and 1. The reconstruction of the probability distribution from its positive integer moments  $i^{2n}$  proceeds in practice from the so-called Bernstein polynomials: Let  $k$  be a positive integer and  $\delta$  be Dirac's delta function; then the moments (positive or negative) of the series of distributions

$$\mathfrak{P}_N(i^2, L) = \sum_{k=0}^N \frac{N!}{k!(N-k)!} \times i^{2k} \overline{(1-i^2)^{N-k}} \delta(i^2 - (k/N)) \quad (2)$$

converge to those of the actual distribution  $\mathfrak{P}$  in the limit  $N \rightarrow \infty$ .

To be more specific,  $\mathfrak{P}_N$  converges weakly to  $\mathfrak{P}$  in the sense of  $\mathcal{Q}^2$ .<sup>15</sup> In other words, one can show that

$$\lim_{N \rightarrow \infty} \int_0^1 di^2 \mathfrak{P}_N(i^2) f(i^2) = \int_0^1 di^2 \mathfrak{P}(i^2) f(i^2) \quad (3)$$

where  $f(i^2)$  is a continuous function on the interval (0,1].

In practice, if negative moments are needed,<sup>16</sup> it may be better to compute or measure them directly instead of deducing them from the positive moments. Nevertheless, for the above-mentioned reasons, it is useful to know that this is in principle possible.

$$\mathcal{P}(a_0 M_0, a_1 M_1, a_2 M_2, \dots, p - p_c, \sigma_I / \sigma_m, L) = \lambda^{x_0} \lambda^{x_1} \lambda^{x_2} \dots \lambda^{x_{\infty}} \mathcal{P} \left( \frac{a_0 M_0}{\lambda^{-x_0}}, \frac{a_1 M_1}{\lambda^{-x_1}}, \frac{a_2 M_2}{\lambda^{-x_2}}, \dots, \frac{p - p_c}{\lambda^{-1/\nu}}, \frac{\sigma_I / \sigma_m}{\lambda^{-\phi/\nu}}, \frac{L}{\lambda} \right), \quad (4a)$$

where the  $a_n$  are nonuniversal numbers analogous to metric factors in critical phenomena,<sup>22</sup>  $\lambda$  is a scale factor, and

$$M_n \equiv \sum_a i_a^{2n} \quad (4b)$$

For generality, we have added the ratio of the conductivities  $\sigma_I / \sigma_m$  of the two components, which plays a role

$$\mathcal{P}_1 \left( \frac{a_1 M_1}{L^{-x_1}}, p (= p_c), \sigma_I / \sigma_m (= 0), 1 \right) \equiv \int \dots \int \prod_{k=2}^{\infty} d \left( \frac{a_k M_k}{L^{-x_k}} \right) \mathcal{P} \left( \frac{a_0 M_0}{L^{-x_0}}, \frac{a_1 M_1}{L^{-x_1}}, \frac{a_2 M_2}{L^{-x_2}}, \dots, p (= p_c), \sigma_I / \sigma_m (= 0), 1 \right), \quad (5)$$

do scale as predicted in Eq. (4). Equation (4) predicts that different  $M_n$  scale differently while at the same time, cumulants of a given  $M_n$  obey gap scaling, i.e.,  $\langle M_n^k \rangle_{\text{cum}} \approx L^{-kx_n}$ .

To check the general scaling form proposed in Eq. (4a), let us first consider the multivariant cumulant averages:

$$\langle M_m^k M_n^l \rangle_{\text{cum}} \approx L^{-x(m,n;k,l)} \quad (6)$$

We performed Monte Carlo calculations using the program described in Ref. 25. Our results on both square and triangular lattices show that for finite lattice sizes  $L$

A few analytical renormalization groups for multifractals have appeared.<sup>5,10,11,17,18,20</sup> In some cases, exponents for the positive integer moments are calculated<sup>10,11,17,18</sup> and the noninteger moments are obtained by analytic continuation, while in other cases, the scaling equations may be written directly for all the moments.<sup>5,19,20</sup> The previous section shows from very general principles that the positive integer moments of the current distribution suffice to characterize completely the problem. Since these moments scale near the percolation threshold, a renormalization group for the positive integer moments is a complete description of the current distribution. The countable set of  $-x_n + x_0$ , which characterizes the scaling properties of these moments,<sup>21</sup> measures how the probability distribution tends to its fixed point as the size  $L$  tends to infinity. Since all moments for  $n > 0$  vanish in this limit [see Eq. (5)], the fixed-point distribution is a  $\delta$  function of  $i^2$ .

To exhibit both the gap scaling familiar in critical phenomena, and the infinite set of exponents, we proceed as follows. Let us first introduce the quantity  $\mathcal{P}$  which, for system sizes less than, or of the order of, the correlation length  $\xi$ , may be interpreted as giving the joint probability distribution for the moments of a given realization of the random network. Note that the quantity  $\mathfrak{P}$  in Eq. (1) contains less information than  $\mathcal{P}$  in Eq. (8). In other words, knowing  $\mathcal{P}$  suffices to deduce the properties of  $\mathfrak{P}$ . We claim that the quantity  $\mathcal{P}$  is a universal function with the following general finite-size scaling behavior:

analogous to the magnetic field. The conductivity of the insulator  $\sigma_I$  is a relevant perturbation at the critical point. The scaling of the moments for  $p \neq p_c$  proposed in Ref. 5, is a special case of Eq. (4). One only needs to know that  $M_n$  scales with size  $L$  as  $L^{2n(1-d)+d}$  in the Euclidean limit to work out all the scaling laws. It has been shown<sup>23</sup> and verified<sup>4,24</sup> numerically that partial distribution functions such as that for the resistance, i.e.,

at  $p = p_c$  and  $\sigma_I / \sigma_m = 0$ , the following result holds:

$$x(m, n; k, l) = kx_m + lx_n, \quad (7)$$

for  $1 \leq k, l \leq 2$  and for  $0 \leq m, n \leq 3$ . A synopsis of the results is presented in Table I. Equations (6) and (7) demonstrate the analog of "gap scaling" in critical phenomena.

The scaling form for the joint probability distribution of the four variables  $M_0, M_1, M_2$ , and  $M_3$  was verified directly by applying the Kolmogorov-Smirnov test<sup>25</sup> for sizes  $L = 15$  and  $L = 31$  on a square lattice. In all but one of the 153 projections studied, we found  $Q \geq 0.01$ , which

TABLE I. Exponents and amplitudes for bond percolation at  $p = p_c$  on the square and triangular lattices. Rough estimates of the numerical uncertainties are in parentheses throughout. The first four lines display the exponents and amplitudes defined in Eq. (1). The columns  $-x(m,n;k,l)$  and  $-kx_m + lx_n$  show that gap scaling, as defined by Eqs. (6) and (7), is obeyed. The universal amplitude ratios  $A(m,n;k,l)$  defined by Eq. (8) were averaged over system sizes for each lattice type to reduce statistical fluctuations. For the square lattice, we used the Fourier accelerated conjugate-gradient technique of Ref. 25 with top and bottom bus bars to study sizes  $L = 7, 15, 31, 63$  with the corresponding number of samples  $N = 6000, 2500, 2500, 500$ . For the triangular lattice, we used the same boundary conditions for  $L = 6, 8, 10, 12, 14$  with the corresponding sample numbers,  $N = 10000, 5000, 5000, 4000, 4000$ .

| $n$       | Square lattice |                  |              | Triangular lattice |                  |              |
|-----------|----------------|------------------|--------------|--------------------|------------------|--------------|
|           | $-x_n$         | $a_n$            | $-x_n$       | $a_n$              | $a_n$            |              |
| 0         | 1.63 (0.01)    | 1.06(0.03)       | 1.67(0.11)   | 1.0 (0.40)         |                  |              |
| 1         | 0.982(0.004)   | 1.01(0.02)       | 0.96(0.01)   | 0.88(0.02)         |                  |              |
| 2         | 0.818(0.009)   | 1.00(0.03)       | 0.78(0.02)   | 0.83(0.03)         |                  |              |
| 3         | 0.773(0.01)    | 1.01(0.03)       | 0.73(0.02)   | 0.81(0.03)         |                  |              |
| $m,n;k,l$ | $-x(m,n;k,l)$  | $-(kx_m + lx_n)$ | $A(m,n;k,l)$ | $-x(m,n;k,l)$      | $-(kx_m + lx_n)$ | $A(m,n;k,l)$ |
| 1,1;1,1   | 1.97 (0.03)    | 1.96             | 0.13(0.005)  | 1.96(0.05)         | 1.92             | 0.15(0.01)   |
| 2,2;1,1   | 1.72 (0.03)    | 1.64             | 0.43(0.05)   | 1.70(0.05)         | 1.56             | 0.51(0.03)   |
| 3,3;1,1   | 1.64 (0.03)    | 1.55             | 0.57(0.09)   | 1.63(0.05)         | 1.46             | 0.67(0.05)   |
| 1,2;1,1   | 1.85 (0.03)    | 1.80             | 0.23(0.01)   | 1.83(0.05)         | 1.74             | 0.26(0.01)   |
| 1,3;1,1   | 1.80 (0.03)    | 1.76             | 0.26(0.02)   | 1.79(0.05)         | 1.69             | 0.30(0.02)   |
| 2,3;1,1   | 1.68 (0.03)    | 1.59             | 0.50(0.05)   | 1.66(0.05)         | 1.51             | 0.58(0.04)   |
| 0,0;1,2   | 4.69 (0.04)    | 4.89             | 0.01(0.003)  | 3.9 (0.40)         | 5.01             | 0.18(0.01)   |
| 1,1;1,2   | 2.9 (0.10)     | 2.95             | 0.03(0.01)   | 3.0 (0.25)         | 2.88             | 0.04(0.01)   |
| 2,2;1,2   | 2.7 (0.15)     | 2.45             | 0.18(0.05)   | 2.8 (0.20)         | 2.34             | 0.23(0.05)   |
| 3,3;1,2   | 2.6 (0.15)     | 2.32             | 0.28(0.09)   | 2.7 (0.20)         | 2.19             | 0.37(0.10)   |
| 1,2;2,1   | 2.8 (0.10)     | 2.78             | 0.05(0.01)   | 2.9 (0.25)         | 2.70             | 0.07(0.01)   |
| 1,2;1,2   | 2.8 (0.15)     | 2.36             | 0.10(0.02)   | 2.8 (0.25)         | 2.52             | 0.13(0.03)   |
| 1,3;2,1   | 2.76 (0.06)    | 2.74             | 0.20(0.01)   | 2.7 (0.10)         | 2.65             | 0.18(0.02)   |
| 1,3;1,2   | 2.70 (0.10)    | 2.53             | 0.13(0.03)   | 2.8 (0.20)         | 2.42             | 0.17(0.04)   |
| 2,3;2,1   | 2.70 (0.15)    | 2.41             | 0.20(0.06)   | 2.8 (0.20)         | 2.29             | 0.27(0.06)   |
| 2,3;1,2   | 2.60 (0.15)    | 2.36             | 0.24(0.08)   | 2.7 (0.20)         | 2.24             | 0.31(0.08)   |

means that the hypothesis that the two distributions are identical cannot be rejected with high probability.

The joint probability distribution in Eq. (4) may also be characterized by the ratios

$$\frac{\langle M_m^k M_n^l \rangle_{cum}}{\langle M_m \rangle^k \langle M_n \rangle^l} \equiv A(m,n;k,l) . \tag{8}$$

Table I shows that these amplitude ratios are universal numbers, independent of both system size and lattice type. Table I also contains the amplitudes  $a_n$  defined in Eq. (1).

In the work of Park, Harris, and Lubensky,<sup>17</sup> the scaling properties are obtained by means of a field theory for an effective Hamiltonian. The fields conjugate to the  $M_n$  break the replica space symmetry. Increasing values of  $n$  correspond to lower symmetries.<sup>27</sup> The  $M_n$  are not eigenoperators,<sup>17,18</sup> but their scaling is given by increasingly smaller exponents and not by a few relevant exponents. They are nevertheless experimentally accessible through resistance and resistance noise measurements.<sup>2,3</sup> The infinite set of irrelevant exponents recently studied by Aharony *et al.*<sup>13</sup> behaves similarly: They are associated with symmetry-breaking fields in hexatic liquid crystals and are experimentally accessible through the fluctuations of the order parameter. We should emphasize then that the sign of the exponents ( $-x_n \geq 0$ ) associated with the moments  $M_n$  ( $n \geq 2$ ) of the noisy resistor network is not

the appropriate criterion for relevance or irrelevance: While the renormalization-group flows associated with the  $M_n$  are influenced by the geometry of percolation, the reverse is not true; all the cluster statistics and the usual percolation exponents are completely independent of the parameter space describing  $M_n$  ( $n \geq 2$ ) which then appear as akin to irrelevant operators. This ambiguity in the significance of the sign of the exponents is a direct manifestation of the ambiguity in the choice of the electrical boundary conditions on the network. One can rescale while keeping the current equal to unity, as in the calculation of the  $-x_n$ , or one can keep the voltage or the power constant, etc. The field theoretical approach in its present form<sup>17</sup> corresponds to the unit current normalization but it can also be formulated with the constant power normalization. In the latter case, the current moments would be moments for the fraction of dissipated power, and the associated exponents  $-x_n + nx_1$  would be negative for  $n \geq 2$ .

In conclusion, we have first pointed out that the positive integer moments of the current distribution in a random resistor network suffice to compute all other moments. The significance of this result is two-fold. First it is noteworthy since these are the moments which are (in principle) experimentally accessible through noise measurements;<sup>2,3</sup> second it shows that even though only positive integer moments are computed in the work of Parks,

Harris, and Lubensky,<sup>17</sup> as is usually done in critical phenomena, their approach also contains the information about all the other moments. In the present work, we have considered directly the current distribution function. This distribution exhibits critical behavior at the percolation threshold. The fixed point is a  $\delta$  function at zero current and the infinite set of exponents describes how one tends to this fixed point. We have also introduced the joint probability distribution for the sample to sample fluctuations of the current distribution. This joint probability distribution can also be considered as a fixed point distribution because of its scale invariance properties. Through simulations on both the square and triangular lattice, we have given strong evidence that apart from nonuniversal "metric" factors, this joint probability distribution is a universal function of the moments  $M_n$ . The corresponding generating function is then the analog of the free energy in critical phenomena, exhibiting both gap scaling and an infinite set of exponents<sup>28</sup> akin to the

infinite set of symmetry related exponents in critical phenomena.

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<sup>15</sup> $\Omega^2$  is the space of square-integrable functions on the interval  $[0, 1]$ . Equation (3) is Eq. (3.6) of Ref. 12, p. 225.

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