Distribution of fracture strengths in disordered continua

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We have studied the distribution of fracture strengths in disordered continuous systems. The model that has been used is a network of nonlinear resistors which burn out and change irreversibly into insulators if their dissipated power becomes very large. Fracture occurs when a sample-spanning path of insulators is formed. The conductance of the resistors is distributed according to a probability density function (PDF). We find that if the first inverse moment of the PDF is finite, the fracture distribution is in the form of an exponential of an exponential recently derived by others, even if the system does not have the topology of a percolation cluster for which the distribution was intended. However, if the first inverse moment is infinite, neither this distribution nor the classical Weibull distribution can describe the distribution of fracture strengths, even if the system has the topology of a percolation cluster.

Electrical and mechanical failure in disordered media is of immense technological and economic importance. Microscopic failure plays a fundamental role in many systems of industrial importance ranging from aircraft structures and pressurized nuclear reactors to the propagation of cracks in underground oil reservoirs and ceramics and fibrous composites. There exists an extensive literature on the general problem of electrical and mechanical failure of disordered solids.[1–4] However, most of the models discussed in the literature incorporate artificial features such as preassigned fracture loci and very complex microscopic laws of fracture. The contribution of such complexities to the phenomena of failure may not be essential and, therefore, their introduction into the model may only complicate the study of such phenomena.

More recently, a few simple models have been introduced for both electrical[5–8] and mechanical[9] failure of disordered solids. De Arcangelis, Redner, and Hermann[7] introduced a percolation model for the electrical breakdown of disordered media. In this model, each bond of a network is a fuse with the probability p and an insulator with the probability $1-p$. Each fuse has a unit conductance and it burns out and becomes an insulator if a voltage drop of more than unity is imposed on it. For $p>p_c$, (p_c is the bond percolation threshold), a sample-spanning cluster of fuses is formed and macroscopic conduction takes place. If a sufficiently large voltage drop is imposed on the network, some of the fuses will burn out, and if enough fuses burn out the entire network will break down. In this situation, we say that "fracture" has occurred, and the macroscopic conductivity of the system is zero. In order to find the breakdown voltages, i.e., the voltages at which the fuses burn out, one finds the distribution of voltages at the nodes of the network using the Kirchhoff's law. The fuse with the largest voltage drop (i.e., the hottest fuse) is converted to an insulator. The procedure is then repeated until the macroscopic fracture is formed. In the class of models that was introduced by Sahimi and Goddard,[9] each bond of a *fully connected* network is a spring that breaks irreversibly if stretched beyond a critical length u_c. At each step of simulation, *all* springs whose lengths have exceeded u_c are broken (not just the one that has suffered the largest stretching). This breaking of several springs at a time enhances the formation of the macroscopic fracture, and introduces a subtle difference between this class of models and that of de Arcangelis *et al.*[7] Moreover, it was shown[9] that depending on the statistical distributions of u_c and the effective spring constants, a variety of macroscopic fracture behaviors appear, which are also in agreement with recent continuum theories of mechanical failure.[10]

Of particular interest in the study of failure phenomena is the distribution of breakdown strengths of the disordered media. This is the probability F, that on application of an external voltage gradient of V/L (L is the length of the system), the first fuse in the network will burn out. By fitting a probability distribution to the strength data, Weibull,[1] in 1939, predicted the average fracture strength of brittle materials. Weibull's distribution is of the following form:

$$F=1-\exp[-c_1L^d(V/L)]^{m_1} , \quad (1)$$

where c_1 and m_1 are constant and d is the dimensionality of the system. Equation (1) has been widely used in fitting breakdown distributions in engineering applications of materials. On the other hand, Duxbury, Beale, and Leath,[8] who analyzed the model of de Arcangelis *et al.*[7] argued that

$$F=1-\exp[-c_2L^d\exp(-m_2L/V)] , \quad (2)$$

where c_2 and m_2 are also constant. The derivation of (2) is based on a Lifshitz-type argument and the fact that the distribution of defect clusters in percolation networks above p_c is exponential.[12]

Real disordered continuous media are usually characterized by statistical distributions for the conductances, or the effective elastic constants of the transport paths.

Such inhomogeneities can give rise to flaws of different shapes, sizes, and orientations, which in turn will give rise to a large scatter of fracture strengths in nominally identical small-scale specimens. Moreover, disordered continuous media do not usually have percolationlike characteristics and, therefore, may behave quite differently than those systems that were analyzed by Duxbury *et al.*[8] The purpose of this paper is to report the results of a study of the effect of such inhomogeneities on the failure and fracture behavior of disordered materials, and to test the validity of Eqs. (1) and (2) for such systems.

We have used the following model, which is a hybrid of the models of de Arcangelis *et al.*[7] and Sahimi and Goddard,[9] to study the fracture behavior of disordered media. To each fuse a conductance g is assigned which is selected at random from a probability density function (PDF) $f(g)$. At the beginning, the network is fully connected so that percolation effects are totally absent. Since the conductances of the bonds are distributed quantities, the proper correlating variable in Eqs. (1) or (2) is the voltage of the bond that dissipates the largest electric power (because this is the hottest bond), and not the maximum bond voltage in the network. The distribution of fracture strengths is determined by the method described above. We then try to find the best fit to the data using Eqs. (1) or (2). We have used square networks of lengths $L=50$, 75, and 100 to test the effect of network size. We have also used up to 1200 different realizations of each network and, therefore, we believe that our data are sufficiently accurate and representative of the system. Two different PDF's have been employed. The first one is a power-law PDF

$$f(g)=(1-\alpha)g^{-\alpha}, \quad 0<\alpha<1 \tag{3}$$

where α is a constant. Halperin, Feng, and Sen[13] have shown that Eq. (3) describes the distribution of the conductance of the channels in the random-void ("Swiss cheese") model of continuous media. In this model, spherical holes are randomly placed in a medium having otherwise uniform transport properties. It was proposed many years ago[14,15] that such distributions would give rise to nonuniversal behavior of the conductivity of percolating systems near p_c. We have shown that [16–18] such distributions can alter diffusion, reaction, and random-walk processes in disordered systems. The applicability of such distributions to continuous media is one of our prime motivations for using them here. As the second PDF, we have used

$$f(g)=(2\lambda)^{-1}, \tag{4}$$

with the conductance g being uniformly distributed in the interval $(1-\lambda, 1+\lambda)$. Changing the parameters α and λ allows us to vary the broadness of the distributions and study its effect on the quantities of interest. Note that the PDF's (3) and (4) have very different properties. In particular, distributions such as (3) have the property that

$$f_{-1}=\int_0^{\infty} f(g)/g\,dg \tag{5}$$

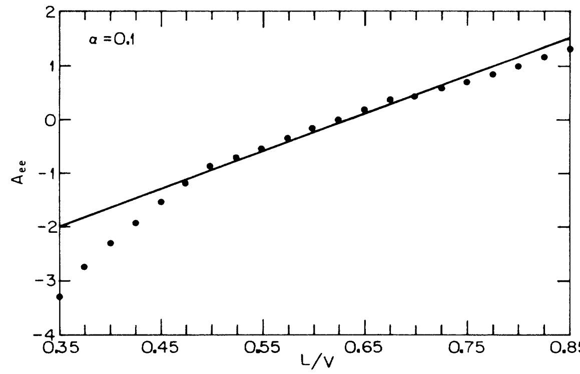

FIG. 1. Comparison of simulation data (solid circles) with the predictions of distribution function (2) (straight line) for the PDF (3).

is divergent, whereas f_{-1} is finite for distributions such as (4).

As it has already been pointed out,[8] the best way to distinguish between distributions (1) and (2) is to isolate V/L. From Eq. (1), we find that

$$A_w=\ln[-\ln(1-F/c_1L^d)]=-m_1L/V, \tag{6}$$

whereas the corresponding expression for Eq. (2) is given by

$$A_{ee}=\ln[-\ln(1-F/c_2L^d)]=-m_2\ln(V/L). \tag{7}$$

We now plot A_w and A_{ee} against L/V and $\ln(V/L)$ and see which plot provides a straight-line fit to the simulation data. In Figs. 1 and 2 we compare the data with the best fit which were obtained by using the PDF (3) using $\alpha=0.1$ and $L=75$. As can be seen, neither plot provides a good fit to the data, especially in the end tails of the curves, with the distribution (2) performing slightly better than the Weibull distribution. We carried out many statistical tests and various methods of fitting to see if we could find a better fit for the data using the distribution function (1) or (2). In no case could we find a satisfactory fit to the data. We also obtained similar results with $L=50$ and 100. Similar results were obtained

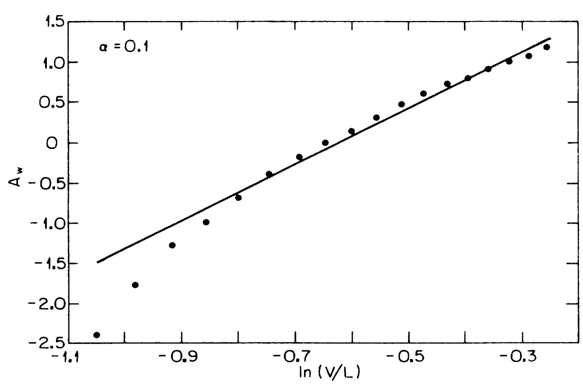

FIG. 2. Comparison of simulation data (solid curves) with the prediction of the Weibull distribution (straight line) for the PDF (3).

with other values of α (the details will be given elsewhere). In all cases that we studied the Weibull distribution provided a worse fit to the data than that of Duxbury *et al.*[8] The system that was studied by Duxbury *et al.*[8] was a percolation network with $p=0.9$, and all of the intact fuses had a unit conductance. When the distribution functions (1) and (2) were fitted to the data and were plotted against V/L, they provided virtually the same fit to the data, and they both appeared to be good fits. Using Eqs. (6) and (7) was the only way of distinguishing (1) from (2). For our system, even a plot of F versus V/L provided strong indications that neither distribution (1) nor (2) would provide a satisfactory fit to the data. Since for practical applications, the end tail of the distribution of fracture strengths is what is really needed, our results strongly indicate that microscopic disorder plays a fundamental role in the fracture behavior of disordered materials and one has to be cautious in using distribution function (1) or (2).

In order to compare with the results presented in Figs. 1 and 2, we present in Fig. 3 the best fit of our simulation data using the distribution (2) for the PDF (4) with $\lambda=\frac{1}{3}$. The agreement is good, and a good fit was also obtained with the Weibull distribution function, although Eq. (2) performs slightly better. These results indicate an important difference between the macroscopic fracture behavior of disordered continuous materials for which f_{-1} is divergent and those for which f_{-1} is finite. In the latter case, the distribution of fracture strengths derived by Duxbury *et al.*[8] (and, to a somewhat lesser degree of accuracy, the Weibull distribution) accurately predicts the onset of fracture, even if the system does not have the topology of a percolation cluster for which the distribution (2) was intended. However, in the former case for which f_{-1} is divergent, neither distribution performs well. We should point out that if we use the maximum bond voltages as the correlating variable, neither (1) nor (2) can provide a good fit to the data, regardless of the behavior of f_{-1}. Finally, in Figs. 4 and 5 we present the average breakdown voltages as a function of L for various values of α and λ. As can be seen, the effective slopes of the lines are sensitive to the values of α, but they show no apparent dependence on λ. This supports further the validity of our argument.

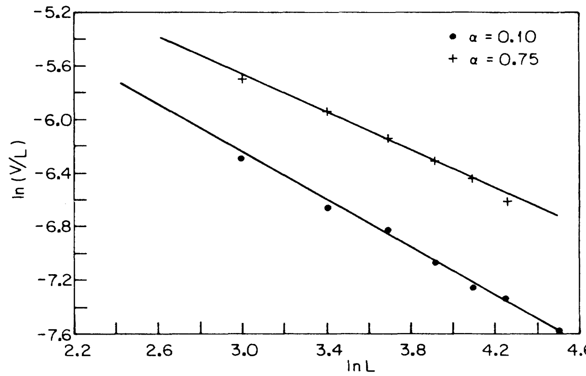

FIG. 4. The dependence of the fracture strength (i.e., the voltage of the first burnt-out resistor) on the length of the system for the PDF (3).

The reason that the distribution of Duxbury *et al.*[8] seems to provide an accurate prediction of the fracture strengths of the (nonpercolating) systems studied here may be the following. Although none of the resistors has a zero conductance, there are only a few paths of resistors that contribute significantly to the distribution of voltages and currents (or stresses in an elastic medium) in the system. The conductances of the rest of the resistors are small enough that such resistors cannot contribute significantly. Thus, the system may effectively have the topology of a percolating system and, thus, the distribution of Duxbury *et al.*[8] should be applicable. Similar ideas have been used to model successfully hopping transport in disordered solids[20] and single-phase flow in porous media,[21] where none of the systems seemed to have the topology of a percolation cluster.

One reason for the sensitivity of the fracture behavior of the system studied here to its microstructure is the fact that such systems are highly nonlinear and, therefore, are far more sensitive to the microstructure than the corresponding linear systems. Generally speaking, the nonlinear systems of interest here are in one of two groups. The first group is made of systems that contain

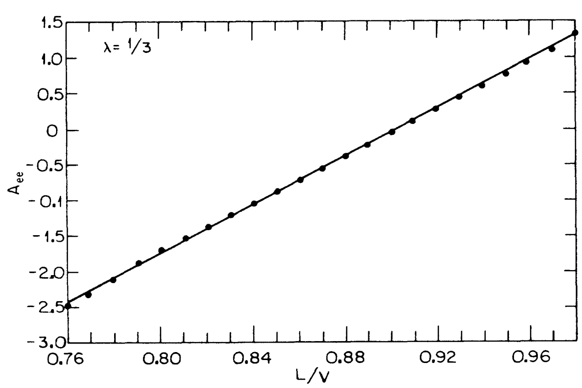

FIG. 3. Comparison of simulation data with the prediction of distribution function (2) for the PDF (4).

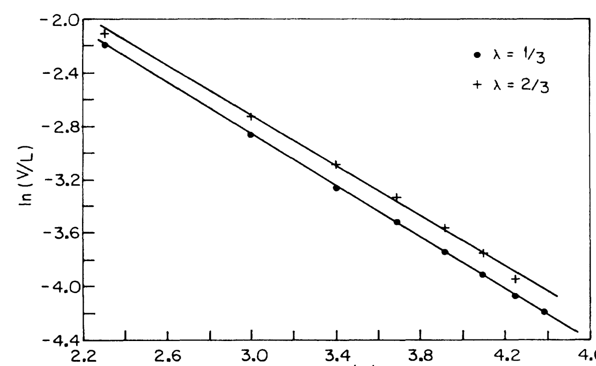

FIG. 5. The dependence of the fracture strength (i.e., the voltage of the first burnt-out resistor) on the length of the system for the PDF (4).

constitutive nonlinearity, i.e., the microscopic elements obey nonlinear laws. For example, the fuses of our network are in fact nonlinear resistors, because they break down if their dissipated power becomes too large. The second group consists of systems that are made of linear elements, but because a very large macroscopic driving force (e.g., a large external voltage) is imposed on them, their macroscopic response is also nonlinear. For example, the random resistor networks recently studied by Gefen *et al.*,[19] and the two-dimensional assemblage of nearly rigid disks that was recently used by Bashir, Sahimi, and Goddard[22] to study the shear flow of dense granular media, are in this group of nonlinear systems; the latter system can also contain constitutive nonlinearity. Thus, the two-dimensional system of disks with fluctuating radii that was recently studied by Herrmann, Stauffer, and Roux[23] falls in the first group of nonlinear systems, and it is not surprising that the macroscopic properties of the system do not follow linear elasticity (despite their claim that this is simply because of randomness in the radii of the disks).

In summary, we have found that the distribution of fracture strengths of continuous media depends on the behavior of the statistical distribution $f(g)$ of the conductance g (or the spring constant) of the elements of the system near $g=0$. If the first inverse moment f_{-1} of this distribution is finite, the distribution of fracture strengths of the system can be accurately described by the distribution function (2), derived by Duxbury *et al.*,[8] even if the system does not seem to have the topology of a percolation cluster. However, if f_{-1} is infinite, neither the distribution of Duxbury *et al.*,[8] nor the Weibull distribution, can accurately describe the fracture behavior of the system. Thus, to the extent that the present model represents the physics of failure phenomena in real materials, our results may have important implications for practical applications. In particular, they may provide a criterion based on which one may be able to predict *a priori* the fracture beahvior of such disordered continuous materials. On the other hand, it may be more reasonable to assume that fracture behavior in some real materials may be rate dependent,[24] in that there is a gradual "softening" of the microscopic elements of the system (i.e., the bond conductance or the effective spring constant decreases gradually), as opposed to the abrupt breakdown of the elements that was used here. It remains to be seen whether the fracture behavior of such materials is very different from what is found here. Work in this direction is in progress.

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