Growth and equilibration in the two-dimensional random-field Ising model

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The nonequilibrium and equilibrium behavior of the two-dimensional Ising model are studied after rapid cooling in a random field. Extensive Monte Carlo simulations are presented, covering a wide range of temperature and random-field strength. Quantitative comparison is made with several recent theories of domain growth and equilibration. In particular, strong support is given to the Villain-Grinstein-Fernandez theory of logarithmic growth.

I. INTRODUCTION

The addition of local, static, random fields to the ordinary Ising model results in the drastic modification of its behavior in both equilibrium and nonequilibrium situations. The random fields give rise to many local minima of the free energy, obscuring the true equilibrium state of the system and making analysis of its ordering characteristics extremely difficult.[1,2] Whereas the zero-field Ising model is well known to exhibit long-range order at finite temperature for all dimensionalities $d>1$, it has only recently been demonstrated that the ground state of the three-dimensional random-field Ising model (RFIM) is ordered.[3] In one dimension, the RFIM does not order at all,[4] and for $d=2$, the lower critical dimension d_c,[2,5-9] there is strong evidence that it is disordered even at zero temperature.

The complexity of the random-field free energy also gives rise to long relaxation times as the system lingers in a succession of local minima on its way to the lowest energy state. The physical manifestation of the free-energy barriers are local clusters of random fields which influence the formation of domains and then hinder their subsequent evolution. This is clearly seen in Fig. 1, which shows the time development of a spin system in the presence of random fields, after a quench to a low temperature. The initial state is uncorrelated, but the spins rapidly coalesce to form domains in the early stages of growth [Figs. 1(a) and 1(b)]. The domains then expand and compact at the expense of their smaller neighbors [Figs. 1(b) and 1(c)], but their continued growth is greatly diminished [Figs. 1(c)–1(e)]. The remaining small domains and rough interfaces, stable due to local concentrations of the random fields, disappear only after extremely long times [Figs. 1(e) and 1(f)]. This slow relaxation can be contrasted with the zero-field Ising model: While the initial growth is much the same as in Figs. 1(a)–1(c), the subsequent development is much faster, with the domains rapidly approaching the size of the system. For a small lattice such as in Fig. 1, one domain will dominate and long-range order will be established for $t\approx 400$.

Nonequilibrium theories of the RFIM (Refs. 10–15) have concentrated their attention on the kinetics of the interfaces between domains, in the spirit of the work of Lifshitz and of Allen and Cahn[16] (LAC) on the zero-field Ising model. In the pure system, the driving force behind domain growth is the reduction in surface tension which results from the flattening of domain walls; the linear domain size L is found to obey the well-known growth law $L\sim t^{1/2}$. The added ingredient in the RFIM is the roughening of domain walls as they wander to gain energy from local concentrations of random fields, as seen in Fig. 1. The walls may then be temporarily pinned, until thermal fluctuations drive them to new positions with lower energy. The subsequent domain growth will be reduced from the LAC result, and if the fields are strong enough, it will be halted altogether. If the domains continue to gradually increase in size, the system may be considered metastable, and, after very long times, it will attain long-range order. This is applicable to the three-dimensional RFIM, for small fields. But if the domains eventually stop growing, the system will be in a disordered phase (although the domain size may be very large). This describes the two-dimensional RFIM for any field strength. In the intermediate-time regime, however, metastability and disorder can be difficult to distinguish.

This similarity has been responsible for a large part of the controversy over the RFIM, with field-cooled diluted antiferromagnet systems [experimental realizations of the RFIM (Ref. 17)] in both two and three dimensions appearing to be stabilized in a glassy, domain state,[18–20] or exhibiting a slow, logarithmic increase in the ordering.[21] Given that the domains are approximately static, their size is found to depend algebraically on the field strength h,

$$L\sim h^{-\nu_h}, \tag{1.1}$$

where ν_h is a decreasing function of temperature. In both the two-dimensional system[18] $Rb_2Co_xMg_{1-x}F_4$ and the three-dimensional system[19] $Fe_xZn_{1-x}F_2$, ν_h has a value $\gtrsim 2$ at low temperatures and $\lesssim 1$ near the critical temperature (although larger exponents have been observed in other materials).[20]

Because equilibrium theories of the RFIM (Refs. 4, 6, and 7) predict such a power-law behavior for $d<d_c$, these experiments were initially taken to be evidence that the lower critical dimensionality was at least three.

However, if the system is not in equilibrium, but rather is very slowly ordering, (1.1) may instead describe the size of metastable domains, in both two and three dimensions. This was demonstrated by Bruinsma and Aeppli,[10] who generalized the LAC theory of interfaces to include the effects of random fields, and calculated the maximum size that a domain may reach before becoming pinned. Villain[11] and Grinstein and Fernandez[12] arrived at a similar result by estimating the free-energy barriers which must be overcome for a domain wall to change its position. Their results also included an explicit logarithmic time dependence,

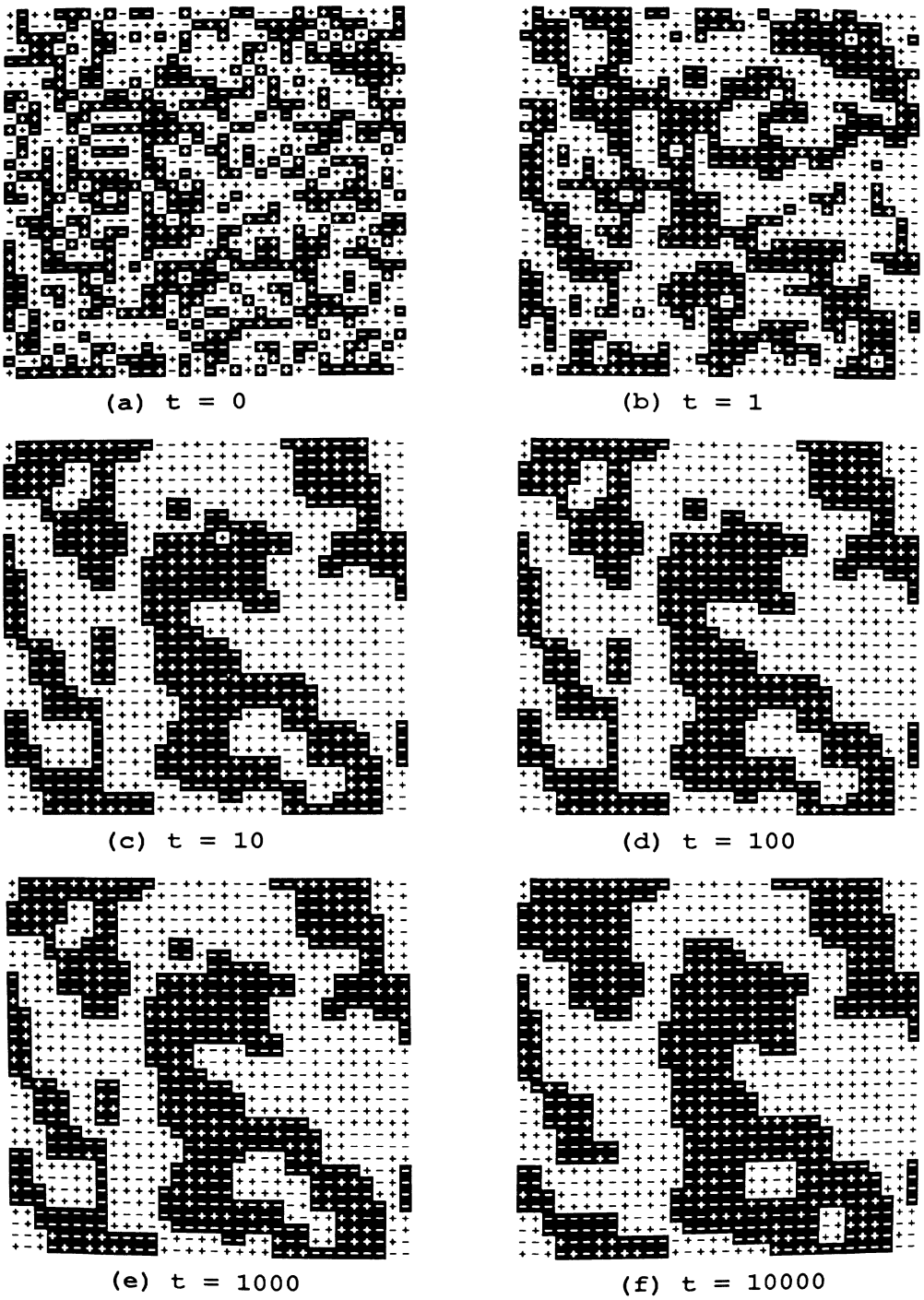

FIG. 1. The time development of a random-field Ising spin system after a quench to a temperature $T=0.10$. The static random fields of strength $h=0.5$ are represented by $+$ and $-$; white and black areas are "up" and "down" domains, respectively. Indicated times are in Monte Carlo steps per spin.

$$L \sim h^{-\nu_h} \ln t \ . \tag{1.2}$$

For each of these theories the calculated exponent, $\nu_h \lesssim 2$, was consistent with the experimental measurements. Subsequently, Nattermann[13] and Andelman and Joanny[14] were able to unify the work of Villain and of Grinstein and Fernandez with that of Bruinsma and Aeppli, finding the latter to be a low-temperature, early-time limit. These theories were therefore very successful in explaining both the algebraic field dependence (1.1) and the logarithmic growth of the ordering observed in some experiments. Also, because the results derived for $d=2$ and $d=3$ are not qualitatively different, the close resemblance of two- and three-dimensional experimental systems is accounted for.

A somewhat different analysis of the nonequilibrium RFIM was performed by Grant and Gunton,[15] who, like Bruinsma and Aeppli, considered a generalization of the LAC theory of curvature-driven growth, but instead derived a time-dependent description. In three dimensions, they found that the $t^{1/2}$ LAC result was modified only by a field-dependent reduction in amplitude, while in two dimensions the growth law contained a logarithmic correction term,

$$L \sim t^{1/2}[A - Bh^2 \ln(t/\tau)]^{1/2} \ , \tag{1.3}$$

where A, B, and τ are field-independent constants. This derivation therefore predicts a reduced growth in the presence of random fields and a maximum domain size in two dimensions, which is in qualitative agreement with the scenario outlined above. It also naturally reduces to the LAC $t^{1/2}$ growth law in the limit of zero random field. However, it differs in detail from the other theories, and implies a more rapid equilibration; it has therefore been suggested that it may only be applicable in the earlier stages of growth.[12,15] Grant and Gunton[22] have also recently examined the random-field problem from the point of view of classical nucleation theory, reproducing the metastable growth law (1.2) in two dimensions, but again finding $L \sim t^{1/2}$ for $d > 2$.

It is clear, then, that each of these theories explains some aspects of the experimental situation, while complementing each other to varying degrees. There is still some question about their applicability to diluted antiferromagnets, however, especially given the continued claims that there is no observable time dependence in the domain state of these materials.[18–20] There is also recent numerical work questioning a fundamental assumption of these theories, viz., the root-mean-square fluctuation of random fields within a domain.[23] In what follows, Monte Carlo simulations are used to examine the growth kinetics of the RFIM and provide numerical support for these theories. Several other simulations have already been performed to test various aspects of these theories: Pytte and Fernandez[24] studied a small-domain version of the theory of Grinstein and Fernandez, while Gawlinski *et al.*[25] attempted to verify the theory of Grant and Gunton. Chowdhury and Stauffer[26] also examined these two theories. However, the extent of these simulations was limited, typically to one temperature or one field strength, or both. This makes it difficult or impossible to get a complete understanding of the complex interaction between these parameters and the predicted time dependence. Their results, therefore, have generally been consistent with the theories, but not strongly supportive.

The simulation here differs from these others by studying a wide range of temperature and random-field strength, to determine at what times and in which part of the phase diagram each of these theories may apply. In particular, it is found that several of these theories can be verified using larger values of temperature and field strength than might otherwise be expected. This makes them accessible to finite-time simulations. The work here has also been restricted to two dimensions, since more of the phase diagram can be studied for longer times, while still providing useful verification of the underlying principles of these theories.

The most important result of this study is the close correspondence found to the theories of Villain and of Grinstein and Fernandez, in every dependence on time, temperature, and random-field strength. Qualitative support is also given to the low-temperature results of Bruinsma and Aeppli, of Nattermann, and of Andelman and Joanny. In addition, some evidence is found for the theory of Grant and Gunton, but with important differences from (1.3). Supplementary to these nonequilibrium results, the equilibrium state has been studied for larger field strengths, and the domain size is found to be consistent with an exponential dependence on random-field strength, as predicted by Binder[7] and as would be expected at the lower critical dimension of the RFIM.

In Sec. II, the model and Monte Carlo methods used are described, with some discussion of previous simulations of the RFIM. In Sec. III, the results of the simulation are presented and directly compared with the applicable theories in the early-, intermediate-, and late-time regimes, and in the low-temperature region of the phase diagram. Section IV contains a final discussion.

II. MONTE CARLO SIMULATIONS

In principle, because Monte Carlo simulations share the underlying model of the theories, they should provide a more direct verification than experimental systems, which have "nonideal" effects such as slow cooling and random exchange, in addition to random fields.[1] Their parameters are also more easily adjustable, and can be varied over a wider range. In practice, however, simulations also share the long relaxation times of experimental random-field systems, and the subsequent difficulties in determining static and dynamic properties. These difficulties are exacerbated by finite lattice sizes, which limit the time over which a simulation may be run before the growing domains saturate the lattice.

In simulating the growth kinetics of the RFIM, the standard procedure is to use a set of spins σ_i and static fields h_i on a lattice of size N, with the initial spin configuration characteristic of the high-temperature phase. A random configuration, corresponding to $T_{\text{initial}} = \infty$, is typical. The fields are randomly distributed from site to site, so that $\langle h_i \rangle = 0$ and $\langle h_i h_j \rangle = h^2 \delta_{ij}$.

A bimodal distribution, $h_i = \pm h$, was used here, and by Pytte and Fernandez and Chowdhury and Stauffer, but a Gaussian distribution of fields was used by Gawlinski *et al.* The spins are randomly chosen and subjected to a flipping algorithm based on the RFIM Hamiltonian

$$\mathcal{H}/J = -\sum_{\langle i,j \rangle} \sigma_i \sigma_j - \sum_i h_i \sigma_i \ , \tag{2.1}$$

and representing a temperature T below the zero-field critical temperature T_c^0. The Glauber flipping probability is used here,

$$W_i = \frac{1}{2}\left[1 - \tanh\frac{\delta\mathcal{H}_i/J}{2T}\right] , \tag{2.2}$$

where $\delta\mathcal{H}_i$ is the change in energy if σ_i is flipped. This process can be described as an infinitely fast "field-cooling" process. Note that the temperature T and random-field strength h are expressed in dimensionless units of J/(Boltzmann constant) and J/(magnetic moment), respectively, so that $T_c^0 = 2.27$ in two dimensions and $h < 1$ is a small field strength. After the quench, the growth of order is observed as a function of time, which is measured in Monte Carlo steps per spin (MCS). A large number R of these "runs" is made, each with a different, noncorrelated set of initial spins and fields. The quantities of interest may then be calculated with a time-dependent ensemble average.

The domain size is determined by the relation[25,27]

$$L(t) = \left\langle \frac{1}{N}\left[\sum_i \sigma_i\right]^2 \right\rangle^{1/d} , \tag{2.3}$$

which measures the fluctuation of the magnetization $\langle \sum_i \sigma_i \rangle \equiv 0$, and corresponds to the structure factor at zero wave vector. This formula was also used by Gawlinski *et al.*, along with several other measures of the domain size. Chowdhury and Stauffer used a slightly different version of this formula, $L = l\langle\,|\sum_i \sigma_i|\,\rangle^{2/d}$, where $l = N^{1/d}$ is the lattice size. The disadvantage of this expression is that l becomes yet another parameter in the simulation. Pytte and Fernandez took the entire lattice to be the "domain" and compared its size with the time for the growth to saturate the lattice; they also studied the decay of single, predefined domains. When using (2.3), L must be, in some sense, much smaller than l to avoid finite-size effects. A useful "rule-of-thumb" in this regard has been established[27,28] which states that such effects can be avoided if the growth is only allowed to continue up to a point where

$$L \lesssim 0.4l \ . \tag{2.4}$$

When (2.4) is satisfied, any differences due to lattice size are subordinate to the statistical error.

Using these methods, much of the RFIM's low-temperature phase diagram has been studied, from fields as small as $h = 0.1$ to as large as $h = 4$, and for temperatures $0 \le T \le 1.27 \approx 0.56 T_c^0$. The domain size has been simulated for times of up to 10 000 MCS, using a range of lattice sizes $16 < l < 128$; the number of runs typically varied from 200 for the larger lattices to 800 for the smaller.

Most of this work was performed as low-priority background jobs on a small network of Sun Workstations over the course of many months. On a Sun-3 computer, which is approximately twice as fast as a VAX 780, 800 runs to 10 000 MCS on a 16^2 lattice could be performed in about half a day, whereas 200 runs on a 128^2 lattice would take close to a week. This requires some patience, but it puts the excess CPU cycles to good use, while providing a local, flexible, and inexpensive alternative to supercomputers or special purpose processors.[29] This will be increasingly true as minicomputers such as the Sun become more powerful and more widely available.

III. RESULTS

The basic characteristics of the RFIM can be seen in Fig. 2, which shows the domain size as a function of time for several field strengths at a temperature $T = 0.63$. It is immediately apparent that there is a wide range of behavior, from the fast, $t^{1/2}$ growth of zero field to almost immediate equilibration in a disordered state for the larger fields. The initial time period displays very rapid growth as the domains coalesce, as was also seen in Fig. 1. In the early-time regime, the fully formed domains continue growing with a slower, random-field-impeded expansion. This is followed by the intermediate-time regime, with a possibly logarithmic increase in domain size. For field strengths $h \gtrsim 1.7$, the onset of the late-time regime is also observable, when the domains cease their growth and the final equilibrium state is obtained. The latter becomes more easily visible as the field strength increases, while the extent of the early- and intermediate-time regimes is correspondingly reduced. The growth thus breaks very naturally into several parts; each of them shall be considered in turn.

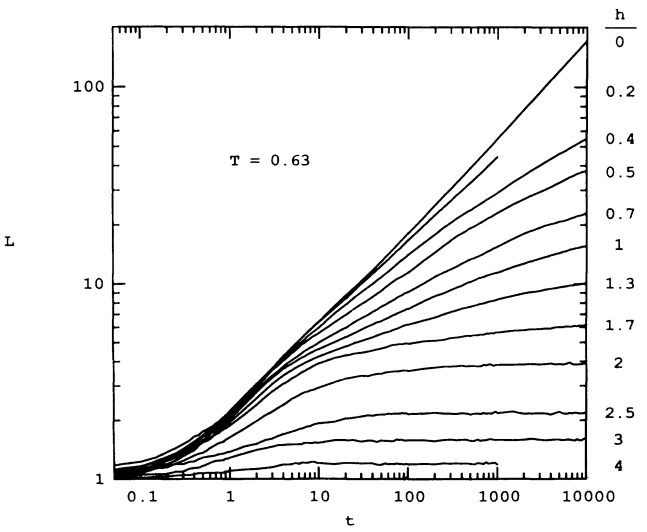

FIG. 2. The domain size L as a function of time at $T = 0.63$, labeled by the field strength h. There are 20 data points per decade, providing essentially continuous curves.

A. Initial growth and early times

For times $t \lesssim 4$ MCS the domains are only beginning to form and expand, and for $h \lesssim 1$, the growth is approximately independent of field strength. Under these conditions, a small random field h_i will simply appear as another "spin" in the environment of σ_i, rather than a pinning force [cf. Eq. (2.1)]. This can be easily seen in Fig. 3, where the spin–random-field correlation

$$\eta = \langle \sigma_i h_i \rangle / h \tag{3.1}$$

increases over this time range as the spins partially align with the local field. For larger field strengths, the growth is noticeably reduced, as the spins are more likely to follow their local random field than their neighboring spins. The growth is essentially nonexistent for $h > 4$, when the spins merely switch from their initial random configuration to that of the random fields.

After this initial time period, the curves in Fig. 2 separate and become manifestly field dependent, as curvature-driven growth commences, retarded by the random fields. In Fig. 3, the spin-field correlation peaks and then decreases somewhat as the spins begin to act collectively as domains. The position of this maximum is approximately independent of field strength, $t_{\max} \approx 4$ MCS. Throughout this time regime the growth is relatively fast, especially for the smaller field strengths, which continue to follow the $t^{1/2}$ law for some time. In the limit of zero random field, then, it is the early-time regime which reduces to the LAC growth law; the onset of the intermediate-time regime is delayed to later and later times.

It is here that the theory of Grant and Gunton[15] should be an appropriate description, since it is a generalization of the LAC theory of curvature-driven growth.[16,30] Their equation for the growth law in two dimensions can be written in the form

$$[L(h,t)/L(0,t)]^2 = A - Bh^{\mu} \ln t \ , \tag{3.2}$$

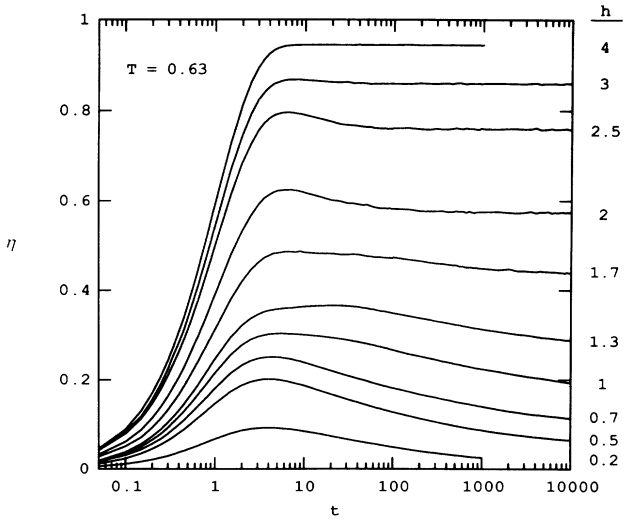

FIG. 3. The spin–random-field correlation η as a function of time at $T = 0.63$, labeled by the field strength h.

where $A = 1 - aT + bh^{\mu}$, a, b, and B are constants, and $\mu = 2$. If the data are expressed in this format, they should then appear as a straight line when plotted using a logarithmic time scale, as in Fig. 4. The data are relatively noisy, because it is the quotient of two statistically independent quantities. However, for each field strength it can be seen that there is indeed a time range which is approximately logarithmic, as indicated by the solid lines. For $h \lesssim 1$, the beginning of this time range is approximately the same as the peak in the spin-field correlation η. Its extent decreases with increasing field strength, covering a very narrow range for $h \gtrsim 1$. If the Grant-Gunton theory is to hold, then, it must be in this time range.

To make a quantitative comparison with (3.2), the beginning and end of the apparent logarithmic interval can be estimated, as in Fig. 4. This is obviously somewhat arbitrary, since it is not clear exactly where the "real" interval lies. The method used here is to maximize the interval while maintaining a good visual fit; the resulting statistical fit is then perfect. Because of finite resolution, there will always be a small amount of curvature at the ends. This is much less than would be present, however, if a statistical fit alone were used, since the errors allow "good" fits to a much larger interval.

The amplitude of the logarithm term is found to be an increasing function of field strength for $0 \leq h \lesssim 0.7$. For larger values of h, however, it is approximately constant. This can be seen from the slopes of the lines in Fig. 4. Over its increasing range, which was measured in intervals of $h = 0.1$, the amplitude can be fit by an algebraic form as in (3.2), albeit somewhat poorly because of the noise. The resulting values of B and μ are shown in Fig. 5 for several different temperatures. The exponent is consistent with a constant value $\mu = 0.8 \pm 0.1$, but not with the predicted value of 2. Grant and Gunton did suggest the possibility, however, that μ is not a universal quantity, so that it might, in fact, have a value in the

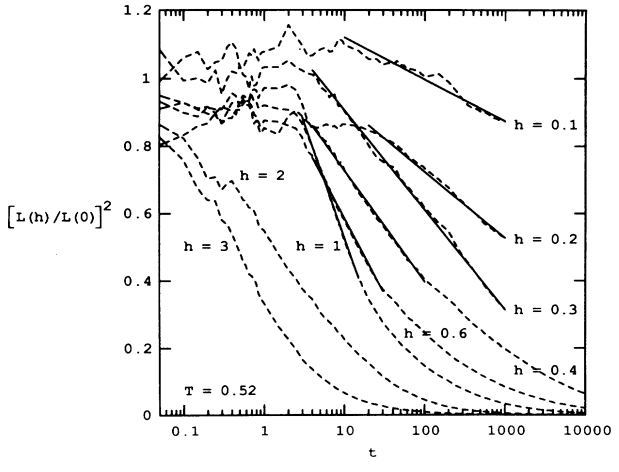

FIG. 4. The domain size L for nonzero random field relative to the zero-field growth, $[L(h)/L(0)]^2$, as a function of time. Several different random-field strengths h are shown, at a temperature $T = 0.52$. The data are plotted on a semilog scale to show the intervals of approximately logarithmic behavior, indicated by the solid lines.

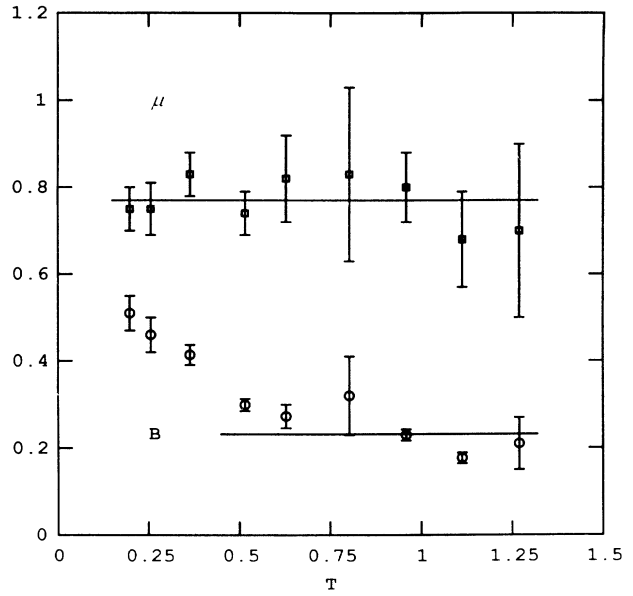

FIG. 5. The coefficient B and exponent μ of the curvature-driven growth law $L(h)=L(0)(A-Bh^{\mu}\ln t)^{1/2}$, for several temperatures T. The coefficient B is approximately constant for $T\gtrsim 0.5$, with a value 0.23 ± 0.03 (indicated by a horizontal line). The exponent μ is constant for all temperatures shown, with a value 0.77 ± 0.05 (again indicated by a horizontal line).

lattice-based RFIM which is reduced from its value in the continuum (where their model was derived). This is in analogy to the lattice effects found in equilibrium by Binder[7] and described in the late-time section below. Another apparent discrepancy is the coefficient B, which is seen to be a decreasing function of temperature. Its increase at lower temperatures may be due to the "field discretization" that occurs when using a bimodal distribution of random fields, as discussed in the low-temperature section below. A fit to (3.2) also yields the quantity $A=1-aT+bh^{\mu}$. Again, it is fairly noisy, and over the range $h\lesssim 0.7$ it does not have any noticeable temperature or field dependence. However, it is approximately constant, with a value $A=1.1\pm 0.1$; this is consistent with its predicted value if a and b are small. For larger field strengths, though, it is a decreasing function of field.

Previous simulations of the RFIM (Refs. 25 and 26) have been able to fit their data in some time intervals to a power law $L(t)\sim t^{\alpha}$, with a field-dependent exponent $\alpha(h)\lesssim 0.5$. It has been pointed out[15,25] that for small fields, (3.2) will also be similar to a power law, which may explain these observations. It is evident from Fig. 2, however, that any algebraic growth is necessarily limited in extent, since it otherwise would be a straight line such as the zero-field curve.

In their simulation, Gawlinski *et al.*[25] also attempted to fit their data to (3.2) with $\mu=2$. They considered one temperature $T=1$, four field strengths in the range $0.3\lesssim h\lesssim 0.8$, and times of 500 MCS (5000 MCS for the largest field strength). Although they were able to get good individual fits within the stated errors, the resulting values of the coefficient B were found to depend on the field strength, indicating, as here, that $\mu=2$ is not consistent. An analysis of their published values of B, however, actually reveals an inverse dependence on h^2, i.e., the logarithmic amplitude Bh^{μ} was found to be approximately constant with a value 0.12. As mentioned previously, a constant amplitude is also seen here, but only for larger field strengths. However, it does have a similar value at this temperature. The smaller value of the "crossover" field strength may be due to the use of different random-field distributions, which would suggest that the Gaussian distribution used by Gawlinski *et al.* is somewhat better at reducing the growth than the bimodal distribution used here. A more mundane possibility is that their data were fit over too wide a time range, which would have the effect of reducing the amplitude measured at the larger field strengths (cf. Fig. 4). A third possibility is that statistical fluctuations occurred in their data; this would prevent meaningful conclusions with so few data points.

Even with the larger number of data points used here, the noisiness apparent in Fig. 4 makes the analysis difficult. While the time dependence is appropriate over a certain range, it is clear that the exponent μ is not consistent with the predicted value of 2. The data therefore provide only highly qualified support for the Grant-Gunton theory.

B. Intermediate times

As the domains expand and their interfaces flatten, the driving force per unit area will decrease.[16] At the same time, the interfaces are roughened (over a smaller scale) to take advantage of the local fluctuations in the random fields; this results in a constant "pinning" force per unit area.[10] Eventually a crossover occurs, and continued growth becomes dependent on thermal fluctuations to overcome the random-field-induced barriers.[11,12] The rapid domain expansion of the early-time regime

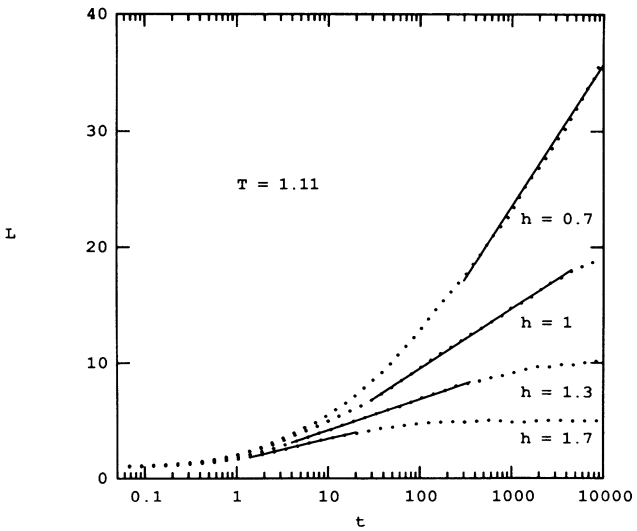

FIG. 6. The domain size L as a function of time at $T=1.11$, for several field strengths h. As in Fig. 4, the intervals of approximately logarithmic growth are indicated by solid lines.

thus gives way to a much slower growth. This can be more easily seen by plotting the data using a logarithmic time scale, as in Fig. 6. It is then apparent that, for each field strength, there is a time range where the growth is approximately logarithmic, as indicated by the solid lines. This region is sandwiched between the fast, early-time growth and the final approach to equilibrium, and it becomes larger and more visible with decreasing field strength.

The theories of Villain[11] and Grinstein and Fernandez[12] were independently developed to describe this intermediate-growth regime, by estimating the energy barriers which pin an interface in place. It is then the time necessary to cross the largest barrier, $t \sim e^{E_h/T}$, that limits the overall growth. To compare with the results reported here, the analysis of Grinstein and Fernandez is most appropriate, as they used a discrete-lattice model and explicitly considered the two-dimensional case. They predicted that for low temperatures the domain size L would increase as

$$L \sim 2Th^{-\nu_h} \ln(t/\tau) , \tag{3.3}$$

with $\nu_h = 2$. Villain's result, derived in the continuum, differed only in the numerical factor of 2, suggesting that it is a nonuniversal feature of (3.3). He also predicted that, in the critical region, the exponent would be smaller, $\nu_h \gtrsim 1$.[31] The direct temperature dependence of (3.3) indicates the important role of thermal fluctuations in producing the growth, in contrast to the equilibrium state, where increasing temperature will reduce the ordering.

As in the preceding section, a quantitative comparison can be made with (3.3) by maximizing the logarithmic interval, while maintaining a good fit to $L(t) = A \ln(t/\tau)$. The amplitude A can then be extracted,

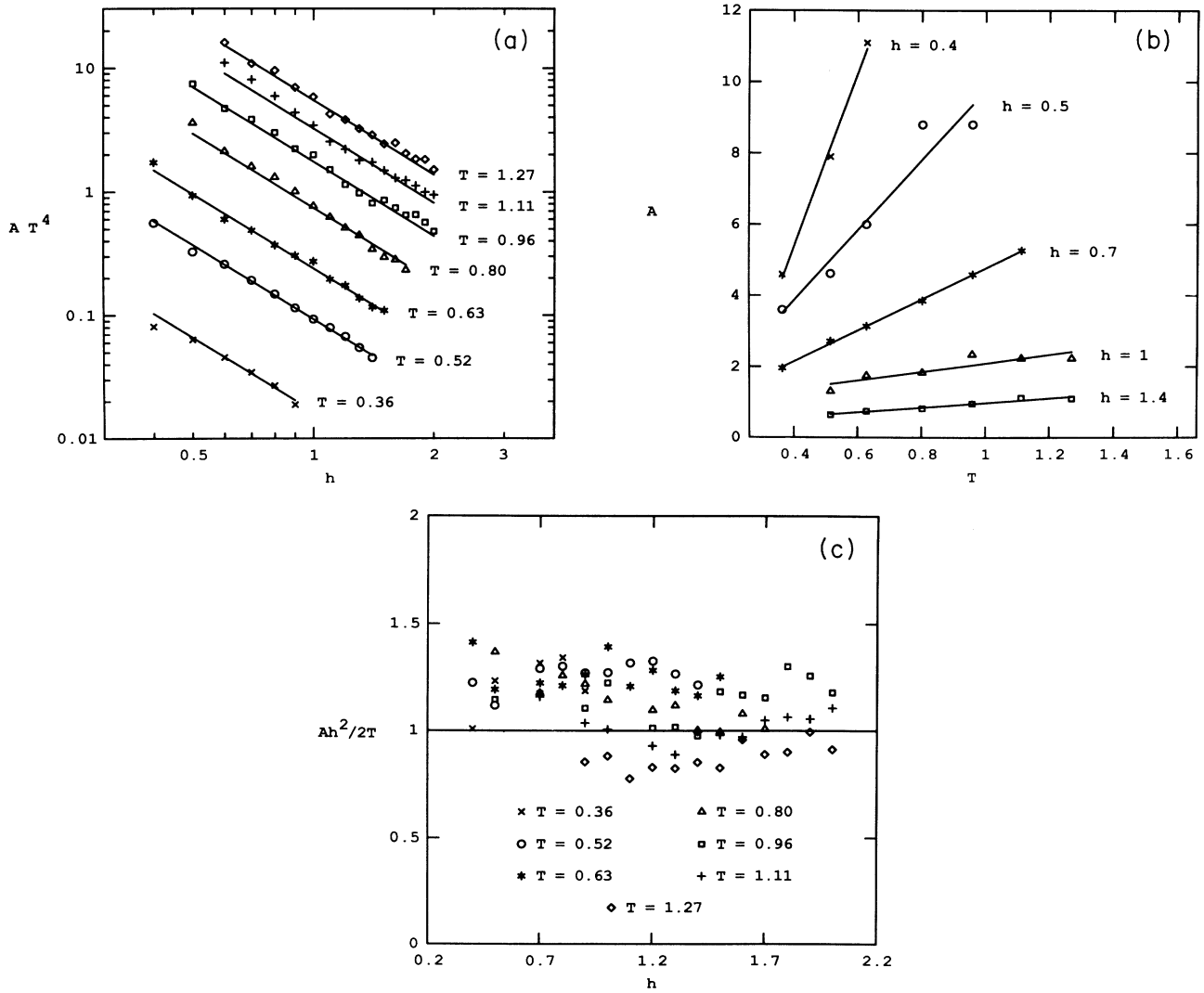

FIG. 7. The amplitude A of the logarithmic growth $L(t) = A \ln(t/\tau)$: (a) *As a function of field strength h.* The data are scaled by temperature to make them more easily visible. For each of the temperatures indicated, the solid line is a best fit assuming a slope of -2. (b) *As a function of temperature T.* Best-fit straight lines are shown for several different field strengths. (c) *Multiplied by $h^2/2T$, for several temperatures and field strengths.* A constant value of 1 is expected.

and compared with the predicted value $2T/h^2$. The dependence on h is shown in a log-log plot in Fig. 7(a), for several different temperatures; in each case, the displayed lines have slope -2. Figure 7(b) shows the dependence on T, for several different field strengths; as predicted, good linear fits can be made. Finally, while one would not be surprised if the constant factor 2 was incorrect, it is interesting to note that the product $Ah^2/2T$ is, indeed, approximately equal to one for a wide range of temperature and field strength, as seen in Fig. 7(c). In total, therefore, the results shown in Fig. 7 are strong evidence for (3.3).

Grinstein and Fernandez also obtained an expression somewhat different from (3.3) to describe small domains, $L \lesssim (4/h)^2$, which can decay more rapidly,

$$L \sim \left[\frac{T}{2h} \ln(t/\tau) \right]^2 . \tag{3.4}$$

This should describe an earlier-time regime than (3.3), when the ordering is dominated by such small domains. To compare with (3.4), the analysis technique used previously can be applied to $L^{1/2}$. Again, intervals can be identified that exhibit a $\ln^2 t$ dependence. To a certain extent, these intervals overlap both the earlier-time regime described by the Grant-Gunton theory and the later-time regime of (3.3), indicating that the intervals used are somewhat too large. It also shows how subtle the crossovers are between the different regimes. The measured amplitude is indeed found to have an h dependence consistent with (3.4), for field strengths in the range $0.2 \lesssim h \lesssim 1.5$. However, its temperature dependence is not consistent, but is better fit by $L \propto T$, as in (3.3). Also, if this temperature dependence is used, the numeric prefactor is found to have a value 0.34 ± 0.02 rather than $\frac{1}{2}$. A better description of the data in this regime is therefore given by $L \approx [(T^{1/2}/3h) \ln t]^2$, in partial disagreement with (3.4). (This expression will also be considered in a somewhat different context in the low-temperature section below.)

The simulation of Pytte and Fernandez[24] was devoted to the study of (3.4), using a single temperature $T = 0.1$ and several field strengths in the range $0.01 \leq h \leq 0.4$. The measured equilibration times were as large as $\sim 10\,000$ MCS. They found their results to be consistent with both the time and random-field dependencies, as here. Because they only used a single temperature, though, they could not examine the T dependence of (3.4). Chowdhury and Stauffer[26] also compared their Monte Carlo results to (3.3) and (3.4), but examined only the time dependence, up to $t \sim 1000$, at $T = 1.5$ and $h \sim 1$. For later times they found the growth to be consistent with (3.4), but apparently did not consider long enough times to be able to compare with (3.3). Both of these simulations included three-dimensional studies as well, which were found to be very similar to the two-dimensional results, as predicted by Grinstein and Fernandez.

Along with these two simulations, the Monte Carlo data here are very supportive of the time- and random-field dependence of the small-domain relation (3.4). The data are not consistent, however, with the temperature dependence of this relation. On the other hand, the simple logarithmic growth law (3.3) is found to describe very well the growth of the ordering, in the appropriate time range. The results here are the first numerical support it has received.

C. Low temperatures

The results presented so far have avoided low temperatures, $T \lesssim 0.35$. This is because of a freezing effect which delays the onset of the intermediate-growth regime, thereby making it difficult to observe in the finite amount of time available to Monte Carlo simulations. This is clearly represented in Fig. 8, which shows the domain growth for several different temperatures at a single field strength $h = 0.5$. All of these curves follow the same path for times $t \lesssim 4$ MCS, which indicates that the early-time growth, when the domains are forming and then roughening, is not only field independent, as seen in Fig. 2, but temperature independent as well. After this time, however, the domain walls have maximally roughened, and thermal fluctuations are required to overcome the random-field-induced pinning centers, as described in the preceding section. The zero-temperature domains are therefore completely frozen in a metastable state. At small but finite temperatures, the domains are also frozen at the same size, until some later time when the thermally induced growth sets in and the domains resume their expansion.[13] The subsequent growth appears much the same as at higher temperatures, although it is difficult to make this quantitative because of the large time scales involved. The transition to this intermediate-growth regime is relatively rapid, once large enough thermal fluctuations arrive to push the system out of the frozen state. This entire process can also be seen in Fig. 1, which shows an example of the domain growth at $T = 0.10$; as in Fig. 8, there is little activity for $10 \lesssim t \lesssim 1000$.

The maximum size L_0 of the frozen domains was calculated by Bruinsma and Aeppli[10] by comparing the

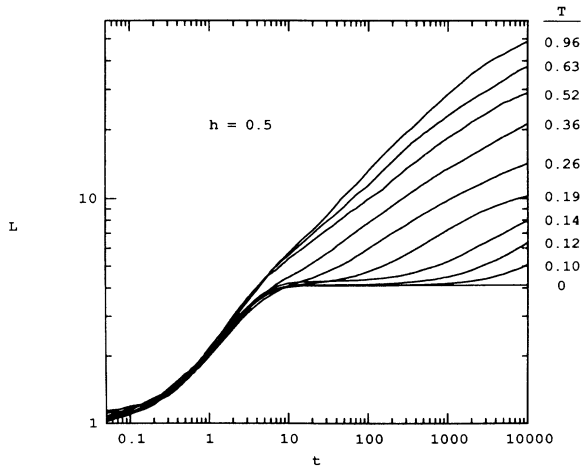

FIG. 8. The low-temperature domain growth as a function of time at $h = 0.5$, labeled by the temperature T.

curvature-induced driving force[16] proportional to L^{-1} with the random-field pinning force. The result is

$$L_0 \sim h^{-4/(5-d)} , \tag{3.5}$$

which is independent of temperature and has the same form as (1.1). When $d=3$, the exponent is 2, consistent with experiment,[19] but when $d=2$, (3.5) predicts a smaller value, $\frac{4}{3}$, which does not adequately describe the experimental data.[18] By generalizing the analysis of Villain[11] to include the effects of curvature, Nattermann,[13] and Andelman and Joanny[14] were also able to derive (3.5) for low temperatures, where the effects of thermal fluctuations could be ignored. As pointed out by Nattermann, this indicates that after a quench to low temperature, the domain size will initially be the same as at $T=0$, and then begin to increase after a time $t \sim \exp(\text{const}/T)$. This is exactly the effect seen in Fig. 8. Nattermann also considered the effects of a discrete lattice, and found a low-temperature result corresponding to the derivation of Grinstein and Fernandez, viz.,

$$L_0 \sim h^{-2/(3-d)} . \tag{3.6}$$

In contrast to (3.5), this result is consistent with experiment in two dimensions, but not in three.

The zero-temperature state seen in Fig. 8 was studied in a previous simulation[32] in both two and three dimensions, and it was demonstrated there that, for $h \lesssim 2$, a completely ordered state has a lower energy. Hence, this state is indeed metastable and does not represent the true ground state of the system. It was also pointed out that a bimodal distribution of random fields will give rise to a discretization effect at low temperatures, such that the growth kinetics is identical within certain ranges of field strength, as seen for L_0 in Fig. 9. While (3.6) would be the appropriate theory in this regime, this effect makes direct comparison impossible. However, a curve following (3.6) is shown in Fig. 9 to indicate the gross

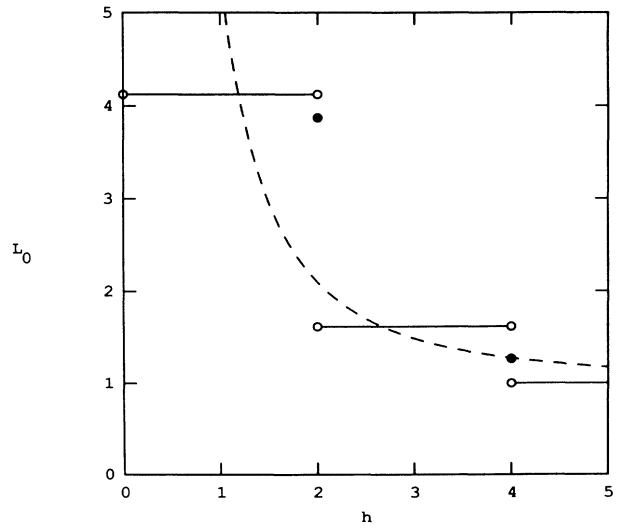

FIG. 9. The metastable domain size L_0 at zero temperature, as a function of field strength h. The bands of constant size do not include their endpoints. The dashed line is the curve $1+4.3h^{-2}$.

similarity. This discretization effect has not been considered by any theories, as they usually assume a Gaussian distribution of fields. However, together with the temperature independence at early times, it may be related to the freezing behavior observed in experimental systems at very low temperature, where the scattering intensity is found to be insensitive to changes in random-field strength or temperature.[18–20,32]

When the frozen state begins to "melt," it is the remaining small antiphase droplets and barriers between larger domains which will decay away first, as seen in Fig. 1. These will be smaller than the typical size $L_0 \lesssim 4$. As discussed by Grinstein and Fernandez,[12] a small domain L will disappear in a relatively "monotonic" fashion, since the time scale for overcoming field barriers,

$$t_0 = \tau_0 \exp(2L^{1/2}h/T) , \tag{3.7}$$

is shorter than the time scale for an unfavorable increase in surface area $\sim \exp(2/T)$. Equation (3.7) should then describe the time at which the frozen state destabilizes. Estimating t_0 from the data is somewhat difficult, since a small fluctuation can change its apparent value by a significant amount. A reasonable fit can still be found over the range $h \leq 1$, however, given by

$$t_0 = (0.06 \pm 0.02) \exp[(1.5 \pm 0.1)h/T] .$$

Unlike the preceding section, where (3.7) was used to describe the overall growth of domains as the smaller domains decayed away, the temperature dependence is found to be appropriate here. The numeric factor 1.5 ± 0.1 does not compare particularly well with $2L^{1/2}$ (since $L \geq 1$), but this may very well be due to a systematic bias in the location of t_0. The derivation of (3.7) also does not include detailed analysis of the geometry of the decaying droplets, which might result in a difference in numeric factors.

Although the rest of the description of the low-temperature regime is qualitative, it is supportive of the physical principles underlying the Bruinsma-Aeppli theory of frozen domains. The reasonable fits to (3.7) give further evidence for the importance of thermal fluctuations in producing the subsequent growth, in agreement with Nattermann.

D. Late times

In the presence of thermal fluctuations, the domains will continue to increase their size, but (in two dimensions) they eventually reach a point beyond which it is energetically unfavorable to expand. The local concentrations of random fields which have guided the growth of the domains do not allow any further reduction in their interfacial energy.[5–7] The system is then in a disordered equilibrium state, which is observable for the stronger fields in Fig. 2. In contrast to the intermediate-time regime, the temperature now has a disordering effect. This crossover in the temperature dependence is displayed in Fig. 10, which shows the domain growth and equilibration for several different temperatures, at a single field strength. From these two

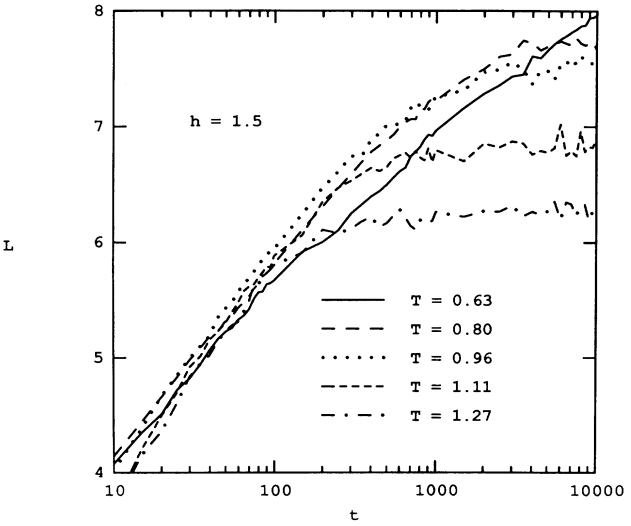

FIG. 10. The domain size L as a function of time at $h=1.5$, for several different temperatures T. The crossover to the equilibrium state is shown, where the temperature has a disordering, rather than ordering, effect.

figures it can be seen that the equilibrium domain size $L_{\rm eq}$ and the equilibration time $t_{\rm eq}$ are strongly decreasing functions of both temperature and field strength.

Continuum theories of the equilibrium RFIM (Ref. 6) predict that the equilibrium domain size in two dimensions has an exponential dependence on the field strength,

$$L_{\rm eq} \sim \exp(\text{const}/h^2) \ . \tag{3.8}$$

The theory of Grant and Gunton,[15] Eq. (1.3), even though it is not supposed to be applicable in the late-time regime, makes a similar but more detailed prediction for the *maximum* domain size,

$$\begin{aligned} t_{\rm eq} &\propto \exp[(A-BT)/h^2] \ , \\ L_{\rm eq} &\propto h t_{\rm eq}^{1/2} \ , \end{aligned} \tag{3.9}$$

where A and B are constants. For the lattice-based RFIM considered here, the appropriate analysis is that of Binder,[7] who reproduced the result (3.8) when thermal fluctuations could be considered irrelevant, $e^{-\text{const}/T} \lesssim h$, but predicted that otherwise the h dependence would be modified; in particular,

$$L_{\rm eq} \propto \exp(\text{const} \times T^{1/3} e^{2/3T}/h^{4/3}) \ . \tag{3.10}$$

The argument in (3.10) is a decreasing function of temperature for all $T<2$.

These theories generally assume that the random-field strength is small in some sense, whereas the equilibrium state seen in Figs. 2 and 10 is only visible for relatively large fields. A reasonable comparison can still be made, however, using the available data. The logarithm of the domain size $L_{\rm eq}$ is shown in Fig. 11(a) for several different temperatures in the range $0.6 \lesssim T \lesssim 1.3$. For small enough field strengths, $h \lesssim 2$, a good fit can be made to the field dependence of (3.10): The lines in Fig. 11(a) all have slope $-\frac{4}{3}$. For $h \gtrsim 2$ the data have a larger slope, with a value $\gtrsim 2$. This crossover is consistent with Binder's general discussion of thermal effects, although it occurs at a larger field strength than might be expected.

The temperature dependence of $\ln L_{\rm eq}$ is shown in Fig. 11(b). The data have only a qualitative similarity to the temperature prefactor in (3.10), being much better fit (for the smaller field strengths) by a decreasing linear function, as in (3.9),

$$h^{4/3} \ln L_{\rm eq} = (4.0 \pm 0.1) - (0.7 \pm 0.1) T \ . \tag{3.11}$$

The magnitude of (3.11) is consistent with that of a transfer matrix analysis[8] of the equilibrium domain size at a much lower temperature $T=0.01$, where it was

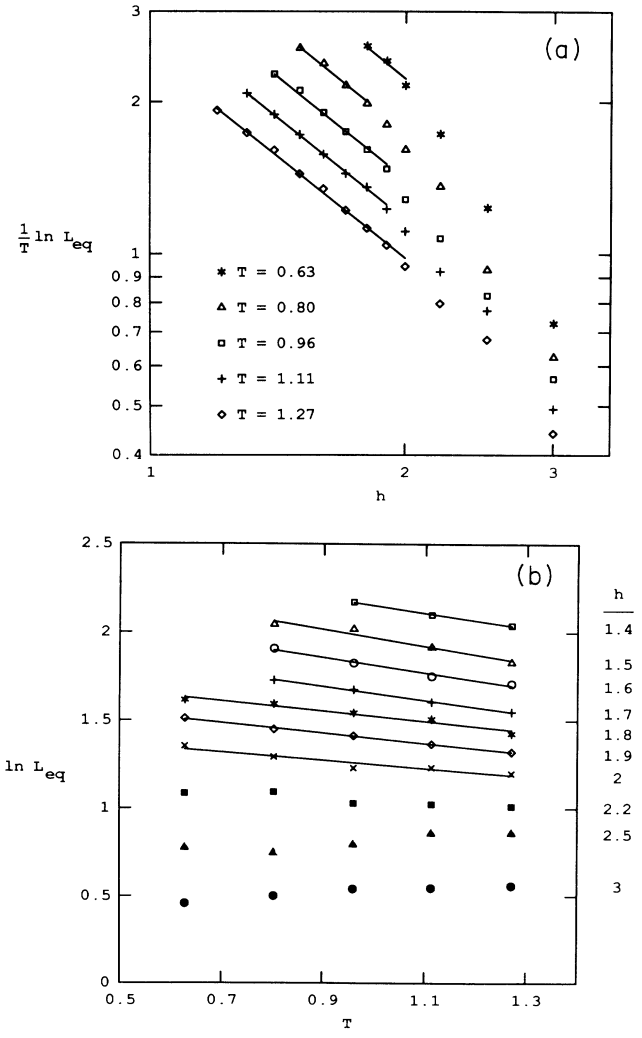

FIG. 11. The logarithm of the equilibrium domain size $L_{\rm eq}$: (a) *As a function of field strength h.* The data are scaled by temperature to make them more easily visible. For each of the temperatures indicated, the solid line is a best fit assuming a slope of $-\frac{4}{3}$. (b) *As a function of temperature T.* Best-fit straight lines are shown for several different field strengths.

found that $L_{eq} \approx \exp(3.4/h^2)$. For the larger field strengths, the data are approximately independent of temperature, which is indicative of its subordinate role in the disorder.

The equilibration time t_{eq} can also be extracted, defined here as the time after which the domain size is constant within the error. While it is not as difficult to pinpoint as the melting time t_0 in the preceding section, the data are still very noisy, as pictured in Fig. 12. It is obvious, however, that t_{eq} is strongly field dependent, and it is natural to expect that it has a form such as (3.9), but with an $h^{-4/3}$ dependence. Marginal fits can indeed be found, as indicated by the solid lines; again, the coefficient of $h^{-4/3}$ is, approximately, a linearly decreasing function of temperature. The equilibration time does not appear to have a simple relation to the equilibrium domain size, however, as suggested by (3.9).

If these equilibrium results are extrapolated to small values of h, it is easy to see that very large domain sizes and extremely long equilibration times result, as in experimental systems. At $h=0.5$ and $T=1$, for example, $L_{eq}=5000$ lattice constants and $t_{eq} \sim 10^{16}$ MCS. To reach this final state, one run on a Sun-3 computer would take 10^{11} years; even a Cray computer would not be much help here. But even though both L_{eq} and t_{eq} are relatively large, the latter dwarfs the former; an increase in L_{eq} by one lattice constant would require $\sim 10^{12}$ MCS. On a linear scale this growth would be effectively unobservable.

An exponential dependence of L_{eq} on field strength, as found here, is a hallmark of the lower critical dimension of the RFIM.[5-7] These results, therefore, provide additional numerical support for a lower critical dimension of two. They are also the first evidence for Binder's modification of the field dependence of L_{eq}, $h^2 \rightarrow h^{4/3}$.

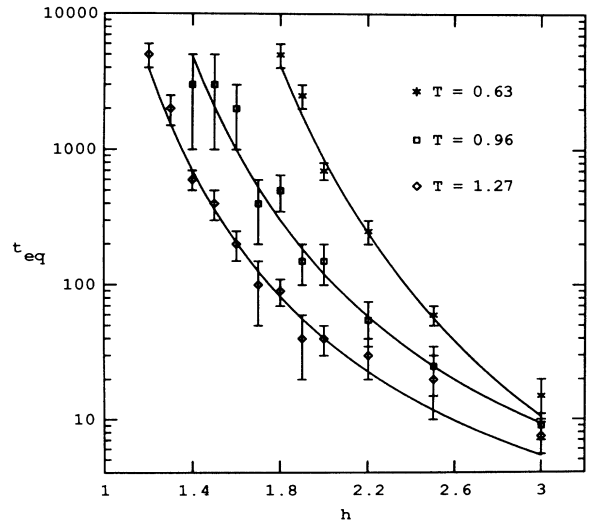

FIG. 12. The equilibration time t_{eq}, as a function of field strength h for several different temperatures. The solid lines are fits to Eq. (3.9) using an exponent of $\frac{4}{3}$ instead of 2.

IV. SUMMARY AND CONCLUSIONS

This paper has attempted to provide a comprehensive view of the growth and equilibration of the RFIM by bringing together the relevant theories and giving them a firmer foundation through numerical simulation. As seen in the preceding sections, the RFIM exhibits a wide variety of behavior, each of which occurs in a restricted range of time, temperature, and random-field strength. A relatively large body of data is necessary, therefore, to get an overall picture and identify the appropriate regions on which to focus. It is otherwise difficult to make definite statements.

The RFIM's relaxation naturally breaks into several distinct time regimes. The boundaries between these regimes are not particularly well defined, but in general they are decreasing functions of field strength. So, even though slow equilibration is an inherent feature of the RFIM, the later-time regimes can still be observed by considering larger field strengths. In each case, however, there is also a maximum field strength beyond which the appropriate theory will no longer apply. For example, the initial growth involves the formation of domains, and it is approximately independent of both temperature and field strength for $h \lesssim 1$. Then curvature-driven growth sets in, which is weakly consistent with a modified form of the Grant-Gunton theory[15] if the field strength is not too large, $h \lesssim 0.7$. This is followed by a slow, thermally produced growth, the first stage of which is only partially described by the theory of Grinstein and Fernandez, for field strengths in the range $0.2 \lesssim h \lesssim 1.5$.[12] The second stage of the growth shows a much closer correspondence to their theory, and to that of Villain,[11] in the range $0.4 \lesssim h \lesssim 2$. Ultimately, a crossover to the final equilibrium state occurs, with the equilibrium domain size consistent with the field dependence predicted by Binder[7] for $1.1 \lesssim h \lesssim 2$.

There is a similar limitation on the ranges of temperature that can be considered. At very low temperatures, the Bruinsma-Aeppli freezing effect[10] occurs, delaying the onset of the equilibration process. Moderately high temperatures, $T \gtrsim 0.3h$, must therefore be used. At the other extreme, the RFIM's behavior can be expected to be very different near the critical temperature T_c^0, and this region has been deliberately avoided.

Most of the theories considered here were derived assuming small values of temperature and field strength. It is not unexpected, however, that they would be applicable in a larger region of the phase diagram. Although each of these theories has been verified in only a restricted range of the parameters, the lower limits are due to the prohibitively long relaxation times involved. While extrapolations must always be performed cautiously, it seems reasonable, therefore, that they will continue to be valid at lower temperatures and field strengths as well. In contrast, the upper limits typically result from a breakdown in the theories; the data are simply no longer consistent with their predictions.

Within these confines, the degree of support for the theories has varied from very close agreement with the

Villain[11] and large-domain Grinstein-Fernandez theories[12] to only qualitative agreement with the theories of Bruinsma and Aeppli[10] and of Nattermann.[13] Often, one aspect of a theory matched the data while another part did not, as with the field and temperature dependence, respectively, of the Grinstein-Fernandez description of small-domain growth, and of Binder's[7] expression for the equilibrium domain size. Again, this may be due to an unsuitable range of parameters, and better agreement may be found, for example, at lower field strengths. Finally, while the data were inconsistent with some predictions, such as the value of the random-field exponent in the Grant-Gunton theory of curvature-driven growth,[15] the differences were often qualitatively understandable in terms of the lattice or field distribution used.

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[1]There have been numerous reviews of theoretical and experimental results; see, for example, G. Grinstein, J. Appl. Phys. **55**, 2371 (1984); P.-z. Wong, J. W. Cable, and P. Dimon, *ibid.* **55**, 2377 (1984); D. P. Belanger, A. R. King, and V. Jaccarino, *ibid.* **55**, 2383 (1984); R. J. Birgeneau, R. A. Cowley, G. Shirane, and H. Yoshizawa, J. Stat. Phys. **34**, 817 (1984); Y. Imry, *ibid.* **34**, 849 (1984).

[2]C. Ro, G. S. Grest, C. M. Soukoulis, and K. Levin, Phys. Rev. B **31**, 1682 (1985); G. S. Grest, C. M. Soukoulis, and K. Levin, *ibid.* **33**, 7659 (1986).

[3]J. Z. Imbrie, Phys. Rev. Lett. **53**, 1747 (1984); Commun. Math. Phys. **98**, 145 (1985); A. T. Ogielski, Phys. Rev. Lett. **57**, 1251 (1986).

[4]G. Grinstein and D. Mukamel, Phys. Rev. B **27**, 4503 (1983).

[5]Y. Imry and S.-k. Ma, Phys. Rev. Lett. **35**, 1399 (1975).

[6]G. Grinstein and S.-k. Ma, Phys. Rev. Lett. **49**, 685 (1982); Phys. Rev. B **28**, 2588 (1983); J. Villain, J. Phys. (Paris) Lett. **43**, L551 (1982); J. Villain, B. Semeria, F. Lancon, and L. Billard, J. Phys. C **16**, 6153 (1983); T. Nattermann, *ibid.* **16**, 6407 (1983).

[7]K. Binder, Z. Phys. B **50**, 343 (1983).

[8]J. F. Fernandez and E. Pytte, Phys. Rev. B **31**, 2886 (1985).

[9]I. B. Ferreira, A. R. King, V. Jaccarino, J. L. Cardy, and H. J. Guggenheim, Phys. Rev. B **28**, 5192 (1983).

[10]R. Bruinsma and G. Aeppli, Phys. Rev. Lett. **52**, 1547 (1984).

[11]J. Villain, Phys. Rev. Lett. **52**, 1543 (1984).

[12]G. Grinstein and J. F. Fernandez, Phys. Rev. B **29**, 6389 (1984).

[13]T. Nattermann, Phys. Status Solidi B **129**, 153 (1985).

[14]D. Andelman and J.-F. Joanny, Phys. Rev. B **32**, 4818 (1985).

[15]M. Grant and J. D. Gunton, Phys. Rev. B **29**, 1521 (1984); **29**, 6266 (1984).

[16]I. M. Lifschitz, Zh. Eksp. Teor. Fiz. **42**, 1354 (1962) [Sov. Phys.—JETP **15**, 939 (1962)]; S. M. Allen and J. W. Cahn, Acta Metall. **27**, 1085 (1979).

[17]S. Fishman and A. Aharony, J. Phys. C **12**, L729 (1979); J. L. Cardy, Phys. Rev. B **29**, 505 (1984).

[18]H. Yoshizawa, R. A. Cowley, G. Shirane, R. J. Birgeneau, H. J. Guggenheim, and H. Ikeda, Phys. Rev. Lett. **48**, 438 (1982); R. J. Birgeneau, H. Yoshizawa, R. A. Cowley, G. Shirane, and H. Ikeda, Phys. Rev. B **28**, 1438 (1983).

[19]R. A. Cowley, H. Yoshizawa, G. Shirane, and R. J. Birgeneau, Z. Phys. B **58**, 15 (1984).

[20]M. Hagen, R. A. Cowley, S. K. Satija, H. Yoshizawa, G. Shirane, R. J. Birgeneau, and H. J. Guggenheim, Phys. Rev. B **28**, 2602 (1983); R. J. Birgeneau, Y. Shapira, G. Shirane, R. A. Cowley, and H. Yoshizawa, Physica **137B**, 83 (1986).

[21]H. Ikeda, J. Phys. C **16**, L1033 (1983); P.-z. Wong and J. W. Cable, Phys. Rev. B **28**, 5361 (1983); Solid State Commun. **51**, 545 (1984); D. P. Belanger, A. R. King, and V. Jaccarino, Phys. Rev. Lett. **54**, 577 (1985).

[22]M. Grant and J. D. Gunton, Phys. Rev. B **35**, 4922 (1987).

[23]J. L. Cambier and M. Nauenberg, Phys. Rev. B **34**, 7998 (1986).

[24]E. Pytte and J. F. Fernandez, Phys. Rev. B **31**, 616 (1985).

[25]E. T. Gawlinski, K. Kaski, M. Grant, and J. D. Gunton, Phys. Rev. Lett. **53**, 2266 (1984); E. T. Gawlinski, S. Kumar, M. Grant, J. D. Gunton, and K. Kaski, Phys. Rev. B **32**, 1575 (1985).

[26]D. Chowdhury and D. Stauffer, Z. Phys. B **60**, 249 (1985).

[27]A. Sadiq and K. Binder, J. Stat. Phys. **35**, 517 (1984).

[28]E. T. Gawlinski, M. Grant, J. D. Gunton, and K. Kaski, Phys. Rev. B **31**, 281 (1985).

[29]R. B. Pearson, J. Richardson, and D. Toussaint, J. Comput. Phys. **51**, 241 (1983); A. Hoogland, J. Spaa, B. Selman, and A. Compagner, *ibid.* **51**, 250 (1983); J. H. Condon and A. T. Ogielski, Rev. Sci. Instrum. **56**, 1691 (1985).

[30]T. Ohta, D. Jasnow, and K. Kawasaki, Phys. Rev. Lett. **49**, 1223 (1982); J. D. Gunton and M. Droz, *Introduction to the Theory of Metastable and Unstable States*, Vol. 183 of *Lecture Notes in Physics* (Springer, Berlin, 1983).

[31]More precisely, $\nu_h = 2/(2-\eta)$, where η is the standard critical exponent; see Ref. 11.

[32]S. R. Anderson and G. F. Mazenko, Phys. Rev. B **33**, 2007 (1986).