PHYSICAL REVIEW B VOLUME 36, NUMBER 1 1 JULY 1987

Analytic solutions for bunched two-fluxon states in Josephson transmission lines

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(Received 26 February 1987)

Analytic solutions for bunched two-fluxon states in Josephson transmission lines in the presence of surface losses are described. The theory is based on assuming a triangular current-phase relation. Multiple solutions are found when the velocity is larger than some critical values. The stability of these solutions is also discussed using numerical simulations with a finite-difference method.

Analytic solutions for a single fluxon in a long Josephson junction with surface impedance losses (β term) were recently obtained by assuming a triangular current-phase relation.[1] The results provide much significant insight on single-fluxon properties. As for bunched fluxons, much attention has also been paid to them. It is known from phase-plane analyses that, if the junction is infinitely long and if $\beta=0$, there are no bunched states of permanent profile.[2] In the presence of β, bunched states were shown by Scott and Johnson in their pioneering work by means of simulations using a hybrid computer.[3] Numerical experiments using a finite difference method have also shown that the bunching may take place in junctions with β on the open or periodic boundary condition.[4,5] By such numerical methods, however, it is hard to justify whether bunched line shapes have permanently a fixed shape and a constant velocity even if the computation time is very long.

In this paper we present an analytical method for bunched two-fluxon states. The theory is based on assuming a triangular current-phase relation and extending the single-fluxon analysis[1] to the bunched two-fluxon states. If solutions are found by this theory, they give decisive evidence of the bunching.

Fluxon dynamics in the long Josephson junction with uniform bias current η is assumed to be described by[2]

$$-\phi_{xx}+\phi_{tt}+f(\phi)=\eta-\alpha\phi_t+\beta\phi_{xxt}. \tag{1}$$

Here $f(\phi)$ is the current-phase relation and normally $f(\phi)=\sin\phi$ is assumed. The term $\beta\phi_{xxt}$ describes the surface losses, and $\alpha\phi_t$ describes the shunt losses. The normalizations may be found in Ref. 5.

Here we are interested in traveling-wave solutions of the form $\phi=\phi(\xi)$ with $\xi=x-ut$. Equation (1) is transformed into an ordinary differential equation (ODE).[1,2] As found in Ref. 1, we assume the following triangular current-phase relation in order to solve the ODE analytically:

$$f(\phi)=\begin{cases} k(\phi-2n\pi), & \text{for } -\pi/2+2n\pi\leqq\phi<\pi/2+2n\pi,\\ -k(\phi-\pi-2n\pi), & \text{for } \pi/2+2n\pi\leqq\phi<3\pi/2+2n\pi. \end{cases} \tag{2}$$

In Fig. 1 a schematic drawing of two solitons is shown. As seen in the figure, region I corresponds to the leading edge of the first fluxon, region III to the trailing edge of the first fluxon and the leading edge of the second one, and region V to the trailing edge of the second. The joining between regions i and $i+1$ happens at $\xi=\xi_i$, where $i=$I–IV, as indicated in the figure. In each region a linear ODE of the third order is solved:

$$\phi_i=\begin{cases} \sum_{j=1}^{3} a_{ij}e^{q_j\xi}+\pi(i-1)+\eta/k & (i=1,3,5),\\ \sum_{j=1}^{3} b_{ij}e^{r_j\xi}+\pi(i-1)-\eta/k & (i=2,4), \end{cases} \tag{3}$$

where ϕ_i indicates the solution in the region i ($i=$I–V), and q_j and r_j ($j=1$–3) are the roots of the following polynomial P_+ and P_-, respectively:[1]

$$P_\pm(\xi)\equiv\xi^3+p_2\xi^2+p_1\xi\pm p_0=0\ , \tag{4}$$

where $p_2=-(1-u^2)/\beta u$, $p_1=-\alpha/\beta$, and $p_0=k/\beta u$. Because of the form of $P_\pm$, for P_+, q_1 is always a negative real root and q_2 and q_3 are either two positive real roots or a couple of complex-conjugate roots with positive real part. For P_-, r_3 is always a positive root, and r_1 and r_2 are either two negative roots or a couple of complex-conjugate roots with negative real part. Since the solutions must satisfy the boundary conditions $\phi_1\rightarrow\eta/k$ for $\xi\rightarrow\infty$ in region I and $\phi_5\rightarrow\eta/k+4\pi$ for $\xi\rightarrow-\infty$ in region V, we get $a_{12}=a_{13}=a_{51}=0$. The remaining coefficients in Eq. (3) together with ξ_i ($i=1$–4) and η are determined by coupled equations obtained by joining ϕ, ϕ_ξ and $\phi_{\xi\xi}$ at $\phi(\xi_1)=\pi/2$, $\phi(\xi_2)=3\pi/2$, $\phi(\xi_3)=5\pi/2$ and $\phi(\xi_4)=7\pi/2$. After laborious rearranging treatment of these equations, we get appropriate expressions for numerical calculations. Since the results have systematic forms, the theory is extendable to n-fluxon states ($n\geqq3$).

The procedure is as follows. First, with α and β as parameters and u as the independent variable, the roots q_1,

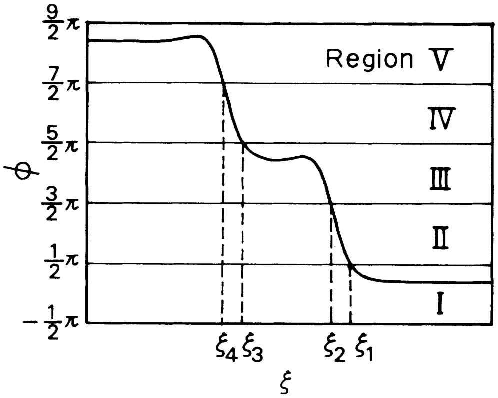

FIG. 1. Schematic drawing of a bunched two-fluxon state.

q_2, q_3 and r_1, r_2, r_3 to Eq. (4) are determined. Next, ξ_{12}, ξ_{23}, and ξ_{34} are determined from

$$\sum_{i=1}^{3}[Q(\xi_{23})AR(\xi_{12})B]_{i1}=1 \ , \tag{5}$$

$$\sum_{i=1}^{3}[R(\xi_{34})BQ(\xi_{23})AR(\xi_{12})B]_{i1}=\sum_{i=1}^{3}[R(\xi_{12})B]_{i1} \ , \tag{6}$$

$$[AR(\xi_{34})BQ(\xi_{23})AR(\xi_{12})B]_{11}=0 \ , \tag{7}$$

where $\xi_{i,i+1}$ $(\equiv \xi_i-\xi_{i+1})$ is the length of region $i+1$. The matrices Q, R, A, and B have 3×3 dimensions and $[\ldots]_{ij}$ is the ij component of the corresponding matrix. The matrices Q and R are diagonal and

$$[Q(\xi)]_{ij}=e^{-q_i\xi}\delta_{ij} \ , \tag{8}$$

$$[R(\xi)]_{ij}=e^{-r_i\xi}\delta_{ij} \ . \tag{9}$$

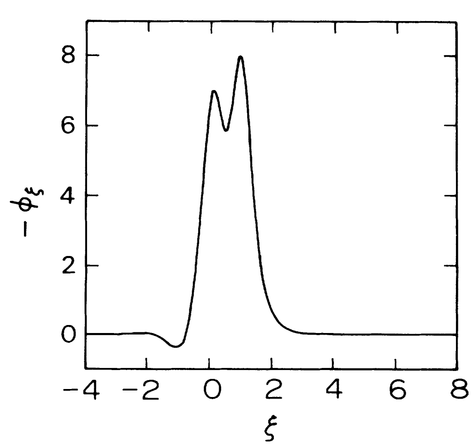

FIG. 2. Line shape of a two-fluxon bunch with $\alpha=\beta=0.02$, and $u=0.98$ giving $\eta=0.353$.

The matrices A and B are a function of both q_i and r_i.

$$[A]_{ij}=\frac{r_j^2q_i^2-(p_1r_j-p_0)q_i-p_0r_j}{q_i(2q_i^3+p_2q_i^2-p_0)} \ , \tag{10}$$

$$[B]_{ij}=\frac{q_j^2r_i^2-(p_1q_j+p_0)r_i+p_0q_j}{r_i(2r_i^3+p_2r_i^2+p_0)} \ . \tag{11}$$

Next, the bias η is given by

$$\eta=\frac{\pi k}{2}\frac{\sum_{i=1}^{3}B_{i1}e^{-r_i\xi_{12}}-1}{\sum_{i=1}^{3}B_{i1}e^{-r_i\xi_{12}}+1} \ . \tag{12}$$

Finally the coefficients in Eq. (3) are calculated by

$$a_{11}=\left[\frac{\pi}{2}-\frac{\eta}{k}\right]e^{-q_1\xi_1} \ , \tag{13}$$

$$b_{2i}=\left[\frac{\pi}{2}-\frac{\eta}{k}\right][R(\xi_1)B]_{i1} \ \ (i=1-3) \ , \tag{14}$$

$$a_{3i}=\left[\frac{\pi}{2}-\frac{\eta}{k}\right][Q(\xi_2)AR(\xi_{12})B]_{i1} \ \ (i=1-3) \ , \tag{15}$$

$$b_{4i}=\left[\frac{\pi}{2}-\frac{\eta}{k}\right][R(\xi_3)BQ(\xi_{23})AR(\xi_{12})B]_{i1} \ \ (i=1-3) \ , \tag{16}$$

$$a_{52}=\sum_{i=1}^{3}\frac{r_i(r_i-q_3)}{q_2(q_2-q_3)}b_{4i} \ , \tag{17}$$

$$a_{53}=\sum_{i=1}^{3}\frac{r_i(r_i-q_2)}{q_3(q_3-q_2)}b_{4i} \ . \tag{18}$$

For Eqs. (17) and (18), $\xi_4=0$ is assumed. This is allowed because the solution has the translational symmetry on ξ axis. In general, we get multiple sets of these roots to Eqs. (5)-(7). The following conditions are required to obtain the roots: (i) All roots must be positive. (ii) The obtained line shapes must be consistent with the assumption; meaning that, as shown in Fig. 1, ϕ_i in region i cannot exceed the range of their own.

A mathematical routine, DNOLBR, is used to get the roots of Eqs. (5)-(7).[6] For the initial values of ξ_{12} and ξ_{34}, the length of the core region of the one-fluxon solution of the same velocity is used, and the initial value of ξ_{23} is changed as a parameter.

The result is summarized as follows: (i) We found the solution of bunched two-fluxon states at high-velocity regions. The range of u where the bunching takes place is included in the range of u where the single-fluxon solutions have an oscillatory tail.[1] So, the relation $\beta>4\alpha^3/27k^2$, being the necessary condition[1] for the oscillation of the single fluxon, is also the necessary condition for the bunching. (ii) At a fixed u, we found multiple solutions depending on the initial value of ξ_{23} to solve Eqs. (5)-(7). The solution with a smaller bias η has a smaller distance between two fluxons.

Figure 2 shows the line shape $(-\phi_\xi)$ of the solution

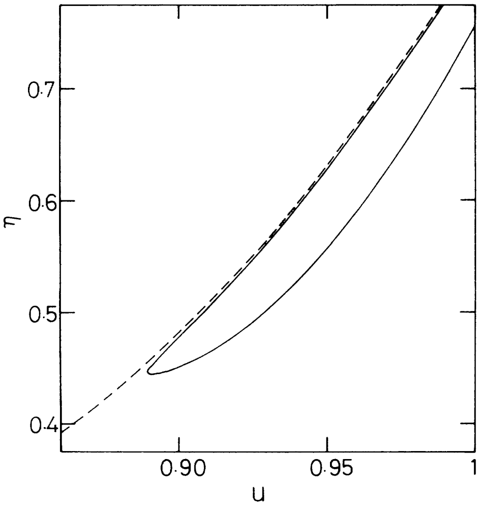

FIG. 3. Bias vs velocity curves of bunched two-fluxon states (solid curves) and one-fluxon states (dashed curve), when $\alpha=\beta=0.1$. The lower (higher) branch of the solid curves corresponds to the first- (second-) nearest bunching state.

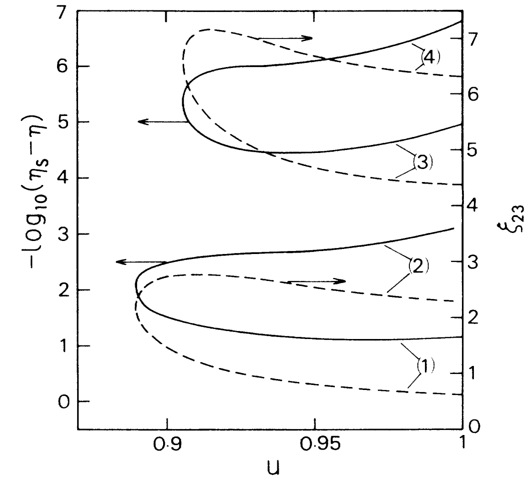

FIG. 4. $-\log_{10}(\eta_s-\eta)$ and ξ_{23} as a function of u with $\alpha=\beta=0.1$. The curves denoted by (i) represent the ith nearest bunching state, where $i=1-4$.

when two fluxons are bunched in the nearest way on the condition of $\alpha=\beta=0.02$ and $u=0.98$ giving $\eta=0.353$. In the present paper, k is chosen to be $8/\pi^2$, under which the analytic results of single-fluxon cases are very close to the numerical ones for the original $\sin\phi$ system.[7]

In Fig. 3, the lower (higher) branch of the solid curves shows the relation of η vs u where two fluxons take the first- (second-) nearest configuration on the condition of $\alpha=\beta=0.1$, and the dashed curve shows the relation of the single-fluxon solutions. Although not shown in Fig. 3, there are *higher-mode* curves in the small gap between the higher branch of the solid curves and the dashed curve, where the higher mode means such a state that two fluxons take the nth nearest configuration ($n\geqq 3$). In order to see this clearly, $-\log_{10}(\eta_s-\eta)$ is computed. Here η_s and η are used, respectively, as the biases giving the single-fluxon states and the bunched two-fluxon states. In Fig. 4, the third- and fourth-nearest bunching modes are shown by the solid curves (3) and (4), respectively, together with the first nearest mode [curve (1)] and the second one [curve (2)]. The length of region III, ξ_{23}, corresponding roughly to the distance between two fluxons is also shown by dashed curves denoted similarly by (i). At some critical value of u the first- and second-nearest bunching modes are merged together, and below this value the solution cannot be found. Similarly, the third and fourth modes are merged at some critical value.

To investigate stability of the solutions, numerical simulations of Eq. (1) are performed by means of a finite difference method. The same current-phase relation as Eq. (2) and the 4π periodic boundary condition are imposed. The periodic length l is chosen to be much larger than the fluxon size ($l=16$). The first method for checking the stability is to observe how each mode responds to the abrupt changes of the bias (i.e., $\eta=\eta_1$ for $t=0$ and $\eta=\eta_2$ for $t>0$). Analytic solutions obtained by this theory are used for the initial conditions at $t=0$. The results after giving η_2 are summarized as follows: (i) The first- and third-nearest bunching states approach the solutions given by η_2 on the same modes, respectively. (ii) The second-nearest bunching mode shifts to the first or the third mode. These two possibilities are governed by the value of η_2. (iii) The fourth mode does not remain in its own mode, but it is difficult to see the final state in spite of lengthy computation ($t\sim 10^4$), because the distance between the two fluxons is very large, i.e., the interaction is very weak. The second method is to put some disturbance in the bias term. We found that the first and third modes are stable after adding a sinusoidal bias disturbance. From the above we conclude that the second and the fourth modes are unstable, and we strongly believe that the first and the third modes are stable.

Part of this work was done during a stay in Physics Laboratory I, The Technical University of Denmark. The author wishes to thank N. F. Pedersen for many valuable discussions. Discussions with O. H. Olsen and M. R. Samuelsen are also acknowledged.

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