Transmission coefficient of an electron through a saddle-point potential in a magnetic field

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We study the motion of a charged particle in two dimensions in a quadratic saddle-point potential $V_{SP}(x,y)=U_y y^2-U_x x^2+V_0$, in the presence of a perpendicular magnetic field. A simple, analytic expression is obtained for the transmission coefficient through the saddle point. We also calculate the transmission coefficient in the Wentzel-Kramers-Brillouin approximation, and find that this agrees well with the exact result, provided that the distance of the closest approach of the classical trajectory of the electron to the saddle point is not too small. Our analysis makes use of the fact that the Hamiltonian for this system can be expressed as a sum of two commuting Hamiltonians, one involving only the cyclotron coordinates, and the other involving only the guiding-center coordinates. The former has the form of a one-dimensional particle in a confining harmonic potential and describes the oscillations of the electron about the guiding-center position. The latter has the form of a one-dimensional particle in an inverted harmonic potential.

I. INTRODUCTION

In recent years there has been a substantial interest in the use of semiclassical techniques to describe the quantum mechanics of electrons in strong magnetic fields.[1-4] In particular, there have been applications of these techniques to gain insight into the integral quantized Hall effect and the associated localization problem for a two-dimensional electron in a random potential and a strong magnetic field.[1] There have also been applications to the cooperative motion of electrons in the fractional quantized Hall effect[2] and to the three-dimensional motion of electrons in a disordered potential and a strong magnetic field.[3]

One of the simplest situations to consider is the motion of an electron in two dimensions when there is a uniform magnetic field B perpendicular to the plane, and a weak potential $V(\mathbf{r})$ that varies slowly on the scale of the magnetic length $l_0=|eB/\hbar c|^{-1/2}$. According to the semiclassical analysis of the situation, the motion is described by a circular motion of radius $\approx l_0$ at the cyclotron frequency $\omega_c=eB/mc$, where m is the electron mass, about a guiding center $\mathbf{R}(t)$ which drifts slowly along an equipotential contour of $V(\mathbf{r})$, with drift velocity

$$\mathbf{v}_D=\frac{c\nabla V(\mathbf{r})\times\mathbf{B}}{eB^2} .$$

The equipotential contours of $V(\mathbf{r})$ are a family of curves, which in most instances are closed on themselves. The allowed quantum-mechanical orbits, in the semiclassical approximation, are a discrete set which satisfy the quantization condition for the action around the orbit.[5]

The semiclassical approximation is an accurate and unambiguous description of the electronic motion, provided that the orbit in question is never close to a saddle point of the potential. In the vicinity of a saddle point, the semiclassical analysis breaks down, as one must take into account the finite amplitude for tunneling across the saddle point, from one classical trajectory to another (see Fig. 1).

The purpose of this article is to study this quantum amplitude in the simplest possible case, a purely quadratic potential, of the form

$$V_{SP}(x,y)=-U_x x^2+U_y y^2+V_0 . \tag{1.1}$$

We find that there is a simple analytic expression for the transmission probability in this case,

$$T=\frac{1}{1+\exp(-\pi\epsilon)} , \tag{1.2}$$

where T is the transmission probability in the x direction, and ϵ is a dimensionless measure of the energy of the guiding-center motion relative to the potential V_0 at

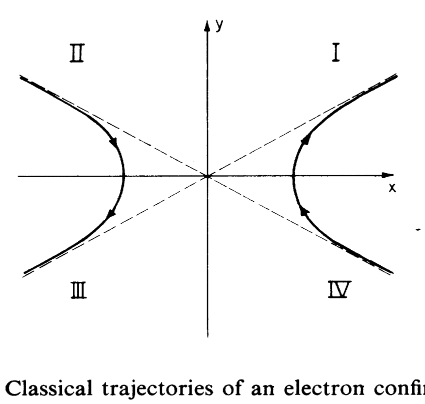

FIG. 1. Classical trajectories of an electron confined to two dimensions in a perpendicular magnetic field and the saddle-point potential given in Eq. (1.1) for energies below the saddle-point energy. The trajectories lie on contours of constant potential. The quantum-mechanical wave functions of the electrons are only significant within several magnetic lengths of such contours for strong magnetic fields. Dashed lines denote the asymptotes of the potential.

the saddle point. Specifically, we have

$$\epsilon \equiv (E_G - V_0)/E_1 \ , \tag{1.3}$$

$$E_1 = \tfrac{1}{2}\left\{ \left[\left[\frac{U_y + U_x}{m} \right]^2 + \omega_c^2 \left[\frac{\omega_c^2}{4} + \frac{U_y - U_x}{m} \right] \right]^{1/2} - \frac{\omega_c^2}{2} - \frac{U_y - U_x}{m} \right\}^{1/2} , \tag{1.4}$$

and E_G is the energy of the motion of the guiding center. If the electron is placed in a pure state of quantum number n for its oscillations about the guiding center, then

$$E_G = E - (n + \tfrac{1}{2})E_2 \ , \tag{1.5}$$

where E is the total energy of the electron, n is a nonnegative integer, and E_2 is the oscillator frequency,

$$E_2 = \left\{ \left[\left[\frac{U_y + U_x}{m} \right]^2 + \omega_c^2 \left[\frac{\omega_c^2}{4} + \frac{U_y - U_x}{m} \right] \right]^{1/2} + \frac{\omega_c^2}{2} + \frac{U_y - U_x}{m} \right\}^{1/2} . \tag{1.6}$$

(We have set $\hbar = 1$ in all of these formulas.) In the limit where U_x and U_y are small compared to $m\omega_c^2$ these formulas simplify further to

$$E_2 \approx \omega_c \ , \tag{1.7}$$

$$E_1 \approx \frac{(U_x U_y)^{1/2}}{m\omega_c} \ , \tag{1.8}$$

while the quantum number n specifies the Landau level for the electron motion.

Our definition of the transmission coefficient T is such that it corresponds to quantum tunneling through the saddle point for energies $E_G < V_0$, while it corresponds to the classical motion around the saddle point for "positive" energies $E_G > V_0$. Note that transmission in the x direction is equivalent to reflection in the y direction, as is evident in Fig. 1. In the limit of $|\epsilon| \gg 1$, we have

$$T \approx e^{-\pi|\epsilon|} \ , \tag{1.9a}$$

for $\epsilon < 0$, and

$$T \approx 1 - e^{-\pi\epsilon} \tag{1.9b}$$

for $\epsilon > 0$.

As we shall see below, these asymptotic formulas may also be derived from a semiclassical WKB (Wentzel-Kramers-Brillouin) approximation, which one would expect to be valid for $|\epsilon| \gg 1$. If the cyclotron frequency is large compared to $(U_x/m)^{1/2}$ and $(U_y/m)^{1/2}$, the condition $|\epsilon| \gg 1$ is equivalent to the requirement that the spatial separation x_0 between the saddle point and the closest point on the classical trajectory is large compared to the magnetic length l_0. The asymptotic form of the tunneling probability then has a simple Gaussian dependence on the parameter x_0, which is given for the case $\epsilon < 0$ by

$$T \approx \exp\left[-\pi \left[\frac{U_x}{U_y} \right]^{1/2} \left[\frac{x_0}{l_0} \right]^2 \right] . \tag{1.10}$$

The results of the present paper have implications for the splitting of energy levels in a double well potential which we plan to discuss in a subsequent work.

This article is organized as follows. In Sec. II we calculate the exact transmission coefficient T of an electron in the potential $V_{SP}(x,y)$ and an arbitrary uniform perpendicular magnetic field. In Sec. III we show how the transmission coefficient may be calculated in the strong magnetic field limit by an alternative complex coordinate technique, which we illustrate for the case $V_{SP}(x,y) = U_x(y^2 - x^2)$. We conclude with a summary in Sec. IV.

II. EXACT CALCULATION OF THE TRANSMISSION COEFFICIENT

In this section we will find an exact expression for the transmission coefficient T of an electron confined to two dimensions in a perpendicular magnetic field B and in the potential $V_{SP}(x,y) = U_y y^2 - U_x x^2 + V_0$. In the symmetric gauge, the vector potential is given by $\mathbf{A} = (B/2)(-y,x,0)$. The Hamiltonian for this system is

$$H = \frac{1}{2m}\left[\frac{1}{i}\nabla + \frac{e}{c}\mathbf{A} \right]^2 + V_{SP}(x,y) \ .$$

We can express this in the form

$$H = \Omega(a_1^\dagger a_1 + a_2^\dagger a_2 + 1) + \frac{\omega_c}{2i}(a_1^\dagger a_2 - a_2^\dagger a_1) + \gamma[(a_2 + a_2^\dagger)^2 - (a_1 + a_1^\dagger)^2] + V_0 \ ,$$

where

$$\Omega = (\tfrac{1}{4}\omega_c^2 + 2U_-/m)^{1/2} \ ,$$

$U_- = \frac{1}{2}(U_y - U_x)$, $\gamma = U_+/2m\Omega$, $U_+ = \frac{1}{2}(U_y + U_x)$, and the operators a_1 and a_2 are given by

$$a_1 = \frac{1}{\sqrt{2}}\left[\sqrt{m\Omega}\,x + \frac{1}{\sqrt{m\Omega}}\frac{\partial}{\partial x} \right] ,$$

$$a_2 = \frac{1}{\sqrt{2}}\left[\sqrt{m\Omega}\,y + \frac{1}{\sqrt{m\Omega}}\frac{\partial}{\partial y} \right] ,$$

so that $[a_1, a_1^\dagger] = [a_2, a_2^\dagger] = 1$ and $[a_1, a_2] = [a_1, a_2^\dagger] = 0$.

We introduce a Bogoliubov transformation to decouple the Hamiltonian into a sum of two commuting Hamiltonians. Setting

$$\begin{bmatrix} a_1 \\ a_2 \end{bmatrix} = \begin{bmatrix} i\cos\phi & \sin\phi \\ -\sin\phi & -i\cos\phi \end{bmatrix} \begin{bmatrix} b_1 \\ b_2 \end{bmatrix}$$

with $\tan(2\phi) = -\omega_c/4\gamma$, we find $H = H_1 + H_2$, where

$$H_1=(b_1^\dagger \;\; b_1)\begin{pmatrix} \frac{1}{2}\Omega-\left[\gamma^2+\left(\frac{\omega_c}{4}\right)^2\right]^{1/2} & \gamma \\ \gamma & \frac{1}{2}\Omega-\left[\gamma^2+\left(\frac{\omega_c}{4}\right)^2\right]^{1/2}\end{pmatrix}\begin{pmatrix} b_1 \\ b_1^\dagger\end{pmatrix},$$

$$H_2=(b_2^\dagger \;\; b_2)\begin{pmatrix} \frac{1}{2}\Omega+\left[\gamma^2+\left(\frac{\omega_c}{4}\right)^2\right]^{1/2} & -\gamma \\ -\gamma & \frac{1}{2}\Omega+\left[\gamma^2+\left(\frac{\omega_c}{4}\right)^2\right]^{1/2}\end{pmatrix}\begin{pmatrix} b_2 \\ b_2^\dagger\end{pmatrix}+V_0 .$$

Our form of the Bogoliubov transformation guarantees $[b_1,b_2]=[b_1,b_2^\dagger]=0$, and $[b_1,b_1^\dagger]=[b_2,b_2^\dagger]=1$. We can diagonalize H_2 with a second Bogoliubov transformation of the form

$$\begin{pmatrix} b_2 \\ b_2^\dagger\end{pmatrix}=\begin{pmatrix} \cosh\theta_2 & \sinh\theta_2 \\ \sinh\theta_2 & \cosh\theta_2\end{pmatrix}\begin{pmatrix} c_2 \\ c_2^\dagger\end{pmatrix},$$

with

$$\tanh(2\theta_2)=\frac{\gamma}{\frac{1}{2}\Omega+\left[\gamma^2+\left(\frac{\omega_c}{4}\right)^2\right]^{1/2}} .$$

For this transformation, $[c_2,c_2^\dagger]=1$, and we have

$$H_2=E_2c_2^\dagger c_2+\tfrac{1}{2}E_2+V_0 ,$$

where

$$E_2=2\left[\left\{\tfrac{1}{2}\Omega+\left[\gamma^2+\left(\frac{\omega_c}{4}\right)^2\right]^{1/2}\right\}^2-\gamma^2\right]^{1/2} . \tag{2.1}$$

We see that H_2 has the form of a harmonic oscillator Hamiltonian. Since $[H_1,H_2]=0$, we know that the eigenfunctions of the full Hamiltonian can simultaneously diagonalize H and H_2, so it is possible for us to diagonalize the Hamiltonian within a single oscillator level.

Unlike H_2, H_1 cannot be written in the form $E_1c_1^\dagger c_1+\frac{1}{2}E_1$ if we wish to preserve the commutation relation $[c_1,c_1^\dagger]=1$. A convenient form for H_1 turns out to be

$$H_1=E_1(c_1^2+c_1^{\dagger 2}) . \tag{2.2}$$

If we write

$$\begin{pmatrix} b_1 \\ b_1^\dagger\end{pmatrix}=\begin{pmatrix} \cosh\theta_1 & \sinh\theta_1 \\ \sinh\theta_1 & \cosh\theta_1\end{pmatrix}\begin{pmatrix} c_1 \\ c_1^\dagger\end{pmatrix},$$

with

$$\tanh(2\theta_1)=\frac{\frac{1}{2}\Omega-\left[\gamma^2+\left(\frac{\omega_c}{4}\right)^2\right]^{1/2}}{-\gamma},$$

we obtain exactly the form in Eq. (2.2) for H_1, with

$$E_1=\left[\gamma^2-\left\{\tfrac{1}{2}\Omega-\left[\gamma^2+\left(\frac{\omega_c}{4}\right)^2\right]^{1/2}\right\}^2\right]^{1/2} . \tag{2.3}$$

Our full Hamiltonian may thus be written

$$H=E_1(c_1^2+c_1^{\dagger 2})+E_2c_2^\dagger c_2+\tfrac{1}{2}E_2+V_0 ,$$

with $[c_1,c_2]=[c_1,c_2^\dagger]=0$, and $[c_1,c_1^\dagger]=[c_2,c_2^\dagger]=1$. We now need a method by which we can extract the transmission coefficient from this form.

To do this, it is convenient to define the following operators:

$$X=\frac{1}{\sqrt{2}i}(c_1^\dagger-c_1) , \tag{2.4a}$$

$$P=\frac{1}{\sqrt{2}}(c_1+c_1^\dagger) , \tag{2.4b}$$

$$s=\frac{1}{\sqrt{2}}(c_2+c_2^\dagger) , \tag{2.4c}$$

$$p=\frac{1}{\sqrt{2}i}(c_2-c_2^\dagger) . \tag{2.4d}$$

In terms of these operators, the Hamiltonians H_1 and H_2 may be written as

$$H_1=E_1(P^2-X^2) ,$$
$$H_2=\tfrac{1}{2}E_2(p^2+s^2)+V_0 ,$$

and we note $[X,P]=i$, $[s,p]=i$, and $[s,X]=[s,P]=[p,X]=[p,P]=0$. The full Hamiltonian is the sum of two commuting Hamiltonians, the first (H_1) being equivalent to that of a one-dimensional particle in an inverted harmonic potential, and the other (H_2) representing a one-dimensional particle in a confining harmonic potential. Physically, we associate the coordinate s with the cyclotron motion of the electron, and the coordinate X with the guiding-center motion, in rescaled units [see Eqs. (2.7)]. One can show that if $U_x,U_y\rightarrow 0$, the quantity $E_1\rightarrow 0$, so that X and P become conserved quantities when the external potential vanishes; this is consistent with the interpretation of X as a guiding-center coordinate.

If we choose a representation in which X and s are diagonal, we may write any wave function of this system in the form $\psi(X,s)$, where X and s are real numbers. Since H_2-V_0 is the Hamiltonian of a one-dimensional harmonic oscillator, we may choose the form of $\psi(X,s)$ to be $\psi_n(s)\phi(X)$, where $H_2\psi_n(s)=[(n+\frac{1}{2})E_2+V_0]\psi_n(s)$, and we take $\psi_n(s)$ to be normalized to unity $(\int_{-\infty}^{\infty}|\psi_n(s)|^2ds=1)$. This represents a wave function that lies purely in the nth oscillator level; we note that in the limit $\omega_c\to\infty$, the nth oscillator level becomes equivalent to the nth Landau level. Schrödinger's equation for wave functions of this form may be written as

$$H_1\phi(X)\equiv E_1(P^2-X^2)\phi(X)=(E_G-V_0)\phi(X)\ , \qquad (2.5)$$

where $E_G=E-(n+\frac{1}{2})E_2$ is the guiding-center energy of the electron.

We can characterize the Hamiltonian H_1 by a transmission coefficient which we denote by T_{1D}. This quantity is defined in the following fashion. Let us construct a wave packet $|\phi_1\rangle$ that at time $t=0$ is centered well to the left of the origin ($\langle X\rangle<0$) with an average momentum directed to the right ($\langle P\rangle>0$). We assign to this wave packet a mean energy $\langle H_1\rangle=E_G-V_0$ and an energy variance ΔE which we may choose to be as small as we like. After a long time t, the wave packet has scattered off of the potential maximum at the origin, after which we have two wave packets of unequal amplitude, one a reflected wave packet located to the left of the origin and traveling to the left ($\langle X\rangle<0,\langle P\rangle<0$), and the other a transmitted wave packet, located to the right of the origin and traveling to the right ($\langle X\rangle>0,\langle P\rangle>0$). If we wait long enough, the two wave packets become well separated in space, so that it becomes possible for us to construct a state $|\phi_2\rangle$ that coincides with the wave packet on the right, and has no significant overlap with the wave packet on the left. If we normalize our states as $\langle\phi_1|\phi_1\rangle=\langle\phi_2|\phi_2\rangle=1$, we can write the transmission coefficient in the form

$$T_{1D}(E_0)=\lim_{\Delta E\to 0}|\langle\phi_2|e^{-iH_1t}|\phi_1\rangle|^2\ . \qquad (2.6)$$

To relate this to the transmission coefficient of the two-dimensional system, we need to examine the form of the wave functions $|\psi_1\rangle=\psi_n(s)|\phi_1\rangle$ and $|\psi_2\rangle=\psi_n(s)|\phi_2\rangle$ in real space; these represent wave packets that lie purely in the nth oscillator level. To do this, we need to know the relationship of the operators X, P, s, and p to the coordinates x and y. One can show from Eq. (2.4) and from the definitions of the operators c_1, $c_1^\dagger$, c_2, and $c_2^\dagger$ in terms of the operators a_1, $a_1^\dagger$, a_2, and $a_2^\dagger$ that

$$x=\frac{1}{\sqrt{m\Omega}}(\alpha_1X-\beta_2s)\ , \qquad (2.7a)$$

$$y=\frac{1}{\sqrt{m\Omega}}(\beta_1P+\alpha_2p)\ , \qquad (2.7b)$$

where

$$\alpha_i=\cos\phi(\cosh\theta_i-\sinh\theta_i)\ , \qquad (2.8a)$$

$$\beta_i=-\sin\phi(\cosh\theta_i+\sinh\theta_i)\ , \qquad (2.8b)$$

for $i=1,2$. It is easy to show α_i, $\beta_i>0$ for both values of i.

Let us suppose that for the initial wave packet $|\phi_1\rangle$, $\langle X^2\rangle\approx\langle P^2\rangle$. (For a fixed value of E_G, we can always guarantee this by placing the center of the wave packet sufficiently far to the left.) Since $\psi_n(s)$ is a harmonic oscillator wave function, we must have $\langle p\rangle=\langle x\rangle=0$. We thus have for $|\psi_1\rangle$, $\langle x\rangle=(1/\sqrt{m\Omega})\alpha_1\langle X\rangle$ and $\langle y\rangle=(1/\sqrt{m\Omega})\beta_1\langle P\rangle$. In real space, then, the wave function $|\psi_1\rangle$ is centered in the upper left quadrant of Fig. 1. By similar reasoning, one can show that the wave function $|\psi_2\rangle$ is centered in the upper right quadrant of Fig. 1.

We note that in the limit of strong magnetic fields $(\omega_c\gg U_x/m,U_y/m)$, $\beta_1/\alpha_1\approx(U_x/U_y)^{1/2}$, so that $U_x\langle x\rangle^2\approx U_y\langle y\rangle^2$ for both $|\psi_1\rangle$ and $|\psi_2\rangle$. This means that if the wave packets are far enough from the origin, they must be centered near the asymptotes of the saddle-point potential $V_{\rm SP}(x,y)$. We can see that the semiclassical picture of the eigenfunctions of an electron in a strong magnetic field and a slowly varying potential is consistent with this by the following argument. If the energy variance of the wave packets is small, and the harmonic oscillator state n has been specified, then when the wave packets $|\psi_1\rangle$ and $|\psi_2\rangle$ are expanded in eigenfunctions of the full Hamiltonian H, the weight for a particular eigenfunction will be significant only if its guiding-center energy is close to the value E_G. According to the semiclassical picture, far from the saddle point, such wave functions only have a significant amplitude near the contours satisfying $V_{\rm SP}(x,y)=E_G$. It follows then that the wave packets $|\psi_1\rangle$ and $|\psi_2\rangle$ must be localized somewhere along these contours. Since any contour of constant $V_{\rm SP}$ approaches the asymptotes defined by $U_xx^2=U_yy^2$ for values of x and y sufficiently far from the origin, we see that the semiclassical picture does indeed predict that the wave packets $|\psi_1\rangle$ and $|\psi_2\rangle$ should be localized near the asymptotes of $V_{\rm SP}$ if they are far enough from the saddle point.

To calculate the transmission coefficient for the full two-dimensional system, we use $|\psi_1\rangle$ as an incoming wave packet with average guiding-center energy E_G. The wave packet $|\psi_2\rangle$ is located in the first quadrant, so that if $E_G<V_0$, we associate this with the transmitted wave packet of the two-dimensional scattering event, since the electron had to cross a classically forbidden region to travel from the upper left quadrant to the upper right quadrant. Conversely, for "positive" guiding-center energies $E_G-V_0>0$, the classical trajectory connects the upper left and right quadrants, so that one might identify $|\psi_2\rangle$ as a reflected wave packet in this case. We adopt a convention, however, in which $|\psi_2\rangle$ is called the transmitted wave packet for all energies, so that if we have an incoming electron wave packet located in the upper left quadrant, the transmission coefficient is the probability that the electron scatters into the upper right quadrant. With this convention, the transmission coefficient may be written as

$$T(E_G)=\lim_{\Delta E\to 0}|\langle\psi_2|e^{-iHt}|\psi_1\rangle|^2\ . \qquad (2.9)$$

Using $|\psi_i\rangle=\psi_n(s)|\phi_i\rangle$, $H=H_1+H_2$, and $[H_1,H_2]=0$, one can easily show

$$T(E_G)=T_{1D}(E_G-V_0)\ . \tag{2.10}$$

Thus if we can calculate the transmission coefficient associated with H_1, we can use Eq. (2.10) to find the transmission of the full two-dimensional system.

The transition amplitude of a one-dimensional particle passing through an inverted parabolic barrier has been studied by Connor[6] in the context of resonance tunneling reactions. Although his results are closely related to ours, he does not explicitly solve for the transmission coefficient of this system. We therefore include a few details of the calculation below.

To find $T_{1D}(E_0-V_0)$, we explicitly solve $H_1\phi(X)=(E_G-V_0)\phi(X)$. Writing $P=(1/i)(d/dX)$ and $\epsilon=(E_G-V_0)/E_1$, this equation becomes

$$\left[\frac{d^2}{dX^2}+X^2+\epsilon\right]\phi(X)=0\ . \tag{2.11}$$

Equation (2.11) is discussed in detail by Morse and Feshbach.[7] The solutions are called parabolic cylindrical functions. For every value of ϵ, there is an even and an odd solution, which we denote respectively as $\phi_e(X)$ and $\phi_0(X)$. These may be expressed in terms of confluent hypergeometric functions $F(a\,|\,b\,|\,u)$ as

$$\phi_e(X)=e^{-iX^2/2}F(\tfrac{1}{4}+\tfrac{1}{4}i\epsilon\,|\,\tfrac{1}{2}\,|\,iX^2)\ , \tag{2.12a}$$

$$\phi_0(X)=Xe^{-iX^2/2}F(\tfrac{3}{4}+\tfrac{1}{4}i\epsilon\,|\,\tfrac{3}{2}\,|\,iX^2)\ . \tag{2.12b}$$

We now examine the solutions to Eq. (2.11) for large values of $|X|$. Writing $u=|u|e^{i\theta}$, one can show[7]

$$F(a\,|\,b\,|\,u)\to|u|^{a-b}e^{i(a-b)\theta}e^{u}\frac{\Gamma(b)}{\Gamma(a)}+|u|^{-a}e^{ia(\pi-\theta)}\frac{\Gamma(b)}{\Gamma(b-a)} \tag{2.13}$$

for large $|u|$ and $0<\theta<\pi$, where Γ is the gamma function. Substituting into Eqs. (2.12), we find

$$\phi_e(X)\to e^{-iX^2/2}\left[\frac{\Gamma(\frac{1}{2})}{\Gamma(\frac{1}{4}+\frac{1}{4}i\epsilon)}\exp\left[-i(\tfrac{1}{4}-\tfrac{1}{4}i\epsilon)\frac{\pi}{2}\right]|X|^{-2[(1/4)-(1/4)i\epsilon]}e^{iX^2}+\frac{\Gamma(\frac{1}{2})}{\Gamma(\frac{1}{4}-\frac{1}{4}i\epsilon)}\exp\left[i(\tfrac{1}{4}+\tfrac{1}{4}i\epsilon)\frac{\pi}{2}\right]|X|^{-2[(1/4)+(1/4)i\epsilon]}\right]\ , \tag{2.14a}$$

$$\phi_0(X)\to Xe^{-iX^2/2}\left[\frac{\Gamma(\frac{3}{2})}{\Gamma(\frac{3}{4}+\frac{1}{4}i\epsilon)}\exp\left[-i(\tfrac{3}{4}-\tfrac{1}{4}i\epsilon)\frac{\pi}{2}\right]|X|^{-2[(3/4)-(1/4)i\epsilon]}e^{iX^2}+\frac{\Gamma(\frac{3}{2})}{\Gamma(\frac{3}{4}-\frac{1}{4}i\epsilon)}\exp\left[i(\tfrac{3}{4}+\tfrac{1}{4}i\epsilon)\frac{\pi}{2}\right]|X|^{-2[(3/4)+(1/4)i\epsilon]}\right]\ . \tag{2.14b}$$

We see that ϕ_0 and ϕ_e each has one term that is proportional to $\exp(iX^2/2)$ and another that is proportional to $\exp(-iX^2/2)$ for large values of $|X|$. We wish to associate one of these terms with the incoming current and the other with the outgoing current. Noting that $Pe^{\pm iX^2/2}=\pm Xe^{\pm iX^2/2}$, we see that the current associated with the term proportional to $e^{iX^2/2}$ is directed away from the origin, and the current associated with the term proportional to $e^{-iX^2/2}$ is directed toward the origin. We thus associate the former with the outgoing current, and the latter with the incoming current.

We proceed by forming an eigenstate of the form $\phi(X)=A\phi_e(X)+B\phi_0(X)$, where the coefficients A and B are chosen such that for large positive values of X the coefficient of the $\exp(-iX^2/2)$ term vanishes. The physical picture of this situation is that well to the right of $X=0$ there is only an outgoing current, while on the left side of the origin there is both an incoming and an outgoing current. For large values of $|X|$ we denote the asymptotic forms of the wave function associated with the incoming and outgoing current as $\phi_{\rm in}(u)$ and $\phi_{\rm out}(u)$, respectively. The transmission coefficient may be written as

$$T_{1D}=\lim_{X\to\infty}\frac{|\phi_{\rm out}(X)|^2}{|\phi_{\rm in}(-X)|^2}\ . \tag{2.15}$$

In accordance with the above discussion, we now choose A and B to satisfy the equation

$$A\frac{\Gamma(\frac{1}{2})}{\Gamma(\frac{1}{4}-\frac{1}{4}i\epsilon)}e^{i\pi/8}+B\frac{\Gamma(\frac{3}{2})}{\Gamma(\frac{3}{4}-\frac{1}{4}i\epsilon)}e^{i3\pi/8}=0\ . \tag{2.16}$$

For large values of $X>0$, one finds $\phi(X)\to\phi_{\rm out}(X)$, with

$$\phi_{\rm out}(X)=|X|^{-2[(1/4)+(1/4)i\epsilon]}e^{-\pi\epsilon/8}e^{iX^2/2}\times\left[A\frac{\Gamma(\frac{1}{2})}{\Gamma(\frac{1}{4}+\frac{1}{4}i\epsilon)}e^{-i\pi/8}+B\frac{\Gamma(\frac{3}{2})}{\Gamma(\frac{3}{4}+\frac{1}{4}i\epsilon)}e^{-i3\pi/8}\right]\ . \tag{2.17}$$

For large negative X, one finds

$$\phi_{\rm in}(X) = |X|^{-2[(1/4)+(1/4)i\epsilon]} e^{-\pi\epsilon/8} e^{-iX^2/2} \times \left[A \frac{\Gamma(\frac{1}{2})}{\Gamma(\frac{1}{4}-\frac{1}{4}i\epsilon)} e^{i\pi/8} - B \frac{\Gamma(\frac{3}{2})}{\Gamma(\frac{3}{4}-\frac{1}{4}i\epsilon)} e^{i3\pi/8} \right] . \tag{2.18}$$

Substituting Eq. (2.16) into Eqs. (2.17) and (2.18), and using Eq. (2.15), we find

$$T_{1D} = \frac{1}{4} \left| \frac{\Gamma(\frac{1}{4}-\frac{1}{4}i\epsilon)}{\Gamma(\frac{1}{4}+\frac{1}{4}i\epsilon)} e^{i\pi/4} - \frac{\Gamma(\frac{3}{4}-\frac{1}{4}i\epsilon)}{\Gamma(\frac{3}{4}+\frac{1}{4}i\epsilon)} e^{-i\pi/4} \right|^2 . \tag{2.19}$$

This form can be greatly simplified using the identities

$$\Gamma(x+iy) = \Gamma^*(x-iy) ,$$

and

$$\Gamma(\tfrac{1}{4}+iy)\Gamma(\tfrac{3}{4}-iy) = \frac{\pi\sqrt{2}}{\cosh(\pi y)+i\sinh(\pi y)} ,$$

where x and y are arbitrary real numbers.[8] We find

$$T(E_G) = T_{1D}(E_G - V_0) = \frac{1}{1+\exp(-\pi\epsilon)} . \tag{2.20}$$

In the strong-field limit, one can show $E_2 \approx \omega_c$ and $E_1 \approx l_0^2 (U_x U_y)^{1/2}$, so that

$$\epsilon \approx \frac{E - \frac{\omega_c}{2} - V_0}{l_0^2 (U_x U_y)^{1/2}} . \tag{2.21}$$

If we wish to parametrize the energy in terms of the distance of closest approach of the electron to the origin, we write $E_G - V_0 = -U_x x_0^2$. In the strong-field limit, we then have

$$\epsilon \approx -\left[\frac{U_x}{U_y} \right]^{1/2} \frac{x_0^2}{l_0^2} .$$

Notice that this implicitly assumes that the classical motion of the electron is confined either to the left or the right of the saddle point; this is always the case if $\epsilon < 0$. For $x_0 \gg l_0$, we have

$$T \approx \exp\left[-\pi \left[\frac{U_x}{U_y} \right]^{1/2} \frac{x_0^2}{l_0^2} \right] . \tag{2.22}$$

This result may also be obtained for $\epsilon < 0$ by applying the WKB approximation to the Hamiltonian H_1. Specifically, we find[9]

$$T_{\rm WKB} = \frac{16\theta^2}{(4\theta^2+1)^2} , \tag{2.23}$$

where

$$\theta = \exp\left[\int_{-X_0}^{X_0} (X_0^2 - X^2)^{1/2} dX \right] \equiv \exp\left[\frac{\pi X_0^2}{2} \right] ,$$

and $X_0^2 = -\epsilon$. Note that with the relation $\langle x \rangle = (1/\sqrt{m\Omega})\alpha_1 \langle X \rangle$, one can show in the strong-field limit that the distance of the turning points from the origin in the one-dimensional problem (X_0) corresponds to the distance of closest approach of the electron to the saddle point (x_0) in the two-dimensional problem. We see that in the limit $|\epsilon| \gg 1$, $T_{\rm WKB}$ is approximately $\exp(-\pi|\epsilon|)$. It follows that in terms of x_0, $T_{\rm WKB}$ has precisely the same form as T in Eq. (2.22) for $x_0/l_0 \gg 1$ in the strong magnetic field limit.

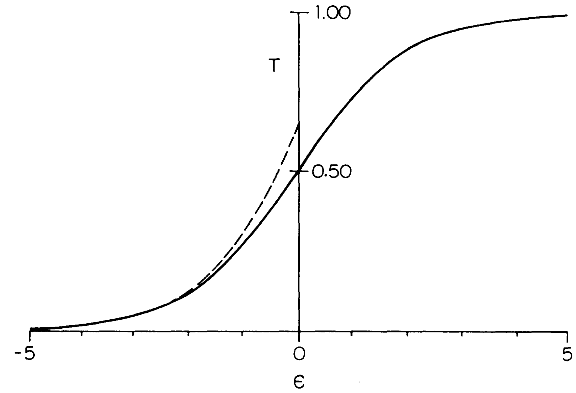

FIG. 2. Transmission coefficient of an electron through a saddle-point potential in a strong magnetic field as a function of the dimensionless parameter ϵ, defined by Eqs. (1.3)–(1.6). Dashed line shows WKB approximation of the transmission coefficient, solid line shows exact result.

In Fig. 2 we plot $T_{\rm WKB}$ and the exact transmission coefficient T in the strong-field limit as a function of the parameter ε. For $-\varepsilon \gg 1$, $T_{\rm WKB}$ and T agree quite well. As $\varepsilon \to 0$, we find $T_{\rm WKB} = \frac{16}{25}$. In contrast, the exact transmission coefficient equals $\frac{1}{2}$ for $\varepsilon = 0$. (One can show by symmetry considerations that this must be the case.) That the WKB approximation breaks down for small values of $|\varepsilon|$ is not surprising: this represents the situation in which the classical turning points of the one-dimensional problem are close to one another; the WKB approximation often does poorly when this occurs.[8] Although the approximations in the WKB method do poorly for small values of x_0, we expect that for large values of x_0 it should give us a good representation of the exact transmission coefficient. This gives us a limit in which we check the correctness of our exact evaluation of T. As we see in Fig. 2, the agreement in this limit is quite good.

III. COMPLEX COORDINATE METHOD

In this section we describe an alternate approach to the saddle-point problem which uses the complex coordinate method appropriate to an electron in the lowest Landau level.[10] This technique allows one to estimate the transmission coefficient in the strong magnetic field limit without working through the Bogoliubov transformations described in Sec. II. For simplicity, we will only consider the case $V_{\rm SP} = U_x(y^2 - x^2)$. Writing $z = x - iy$, and defining

$$b = \frac{1}{\sqrt{2}} \left[2l_0 \frac{\partial}{\partial z^*} + \frac{1}{2} \frac{z}{l_0} \right] ,$$

the Hamiltonian may be written as

$$H=\omega_c(b^\dagger b+\tfrac{1}{2})-\frac{U_x}{2}(z^2+z^{*2})+V_0 .$$

The operator b is the Landau-level lowering operator. We introduce our first approximation by projecting this Hamiltonian into the lowest Landau level. To do this, we consider only wave functions of the form $\psi(z,z^*)=f(z)\exp(-zz^*/4l_0^2)$, where f is an analytic function of z, which is the most general form for a function in the lowest Landau level. We may then set $b=0$ and replace z^* with $2l_0^2(\partial/\partial z)$ in the Hamiltonian.[10] The projected Hamiltonian now acts only on the function $f(z)$. With this procedure, Schrödinger's equation becomes

$$\left[-\frac{U_x}{2}\left[4l_0^4\frac{\partial^2}{\partial z^2}+z^2\right]+V_0-E+\tfrac{1}{2}\omega_c\right]f(z)=0 . \quad (3.1)$$

The form of this equation is similar to that of Eq. (2.11), Schrödinger's equation for a one-dimensional particle in an inverted harmonic oscillator potential. The equations, however, are not identical. We will find, in particular, that the classical turning points for Eq. (3.1) are not equivalent to the points of closest approach of the classical trajectories to the origin, as they are for Eq. (2.11). The coefficient of the $\partial^2/\partial z^2$ term and the turning point separation, however, combine in such a way that the action integral for the present calculation is precisely the same as that of Sec. II. Thus in the strong-field limit, the transmission coefficient we calculate below is the same as the result of Sec. II.

The solution to Eq. (3.1) may be expressed in terms of parabolic cylindrical functions, as discussed in Sec. II. For our present calculation, however, we will present the solutions in the WKB approximation. In this context, the WKB approximation may be written in the following manner. Write the function $f(z)$ as

$$f(z)=e^{\phi(z)} , \quad (3.2)$$

and expand $\phi(z)$ in powers of l_0^2,

$$\phi(z)=\frac{\phi_{-1}(z)}{l_0^2}+\phi_0(z)+\phi_1(z)l_0^2+\cdots . \quad (3.3)$$

In the limit $B\to\infty$, $l_0\to 0$, so that for strong fields it is only necessary to keep the first two terms in the expansion. Substituting Eqs. (3.2) and (3.3) into Eq. (3.1), one finds for the lowest-order contribution to $\phi(z)$,

$$\phi_{-1}(z)=\pm\tfrac{1}{2}\int_{z_0}^{z}(a^2-\xi^2)^{1/2}d\xi ,$$

where z_0 is an arbitrary constant, and

$$a^2\equiv -2(E-V_0-\tfrac{1}{2}\omega_c)/U_x .$$

In the following analysis, we only consider energies for which a^2 is positive. For the zeroth-order term, one finds

$$\phi_0(z)=-\tfrac{1}{2}\ln\frac{d\phi_{-1}(z)}{dz} .$$

The WKB approximation thus gives two linearly independent solutions, which may be written as

$$f^{\pm}(z)=\left[\frac{2}{(a^2-\xi^2)^{1/2}}\right]^{1/2}\times\exp\left[\pm\frac{1}{2l_0^2}\int_{z_0}^{z}(a^2-\xi^2)^{1/2}d\xi\right] .$$

The above approximation breaks down near $z=\pm a$. This is not surprising because if z were a real variable, these would represent the classical turning points of a one-dimensional particle in an inverted harmonic potential. We note that these turning points are related to the distance of closest approach of the electron to the origin by $a=\sqrt{2}x_0$.

To gain some insight into what the wave functions look like, it is useful to examine $f^{\pm}(z)$ in the large $|z|$ limit. In doing this, we must define our branch cuts for the function $(a^2-\xi^2)^{1/2}$; we will choose them in such a way that $f^+(z)\sim\exp(iz^2/4l_0^2)$ for $\mathrm{Re}z\gg a$, and $f^+(z)\sim\exp(-iz^2/4l_0^2)$ for $\mathrm{Re}z\ll -a$. If we stay with this convention, it follows that $f^-(z)\sim\exp(-iz^2/4l_0^2)$ for $\mathrm{Re}z\gg a$, and $f^-(z)\sim\exp(iz^2/4l_0^2)$ for $\mathrm{Re}z\ll -a$. If we now take $\psi(z,z^*)=f^+(z)\exp(-|z|^2/4l_0^2)$ and write $z=re^{-i\theta}$, we see that for large r along the directions $\theta=\pi/4$ and $\theta=3\pi/4$, the wave-function amplitude falls off only as $1/\sqrt{r}$, rather than as a Gaussian, as it does in other directions. Similarly, for

$$\psi(z,z^*)=f^-(z)\exp(-|z|^2/4l_0^2) ,$$

the wave function falls off as $1/\sqrt{r}$ along $\theta=5\pi/4$ and $\theta=7\pi/4$. If we draw rays from the origin along these directions, the resulting lines coincide precisely with the asymptotes of the saddle-point potential $V_0+U_x(y^2-x^2)$. This means that for large r an eigenfunction of the Hamiltonian has its largest amplitude near some or all of the asymptotes of the potential; along a direction perpendicular to these asymptotes, the wave function falls off as a Gaussian with a characteristic length scale of l_0. Referring to Fig. 1, and noting that the current density can only be significant where the wave-function amplitude is significant, it is clear that one can identify the solutions $f^+(z)$ with an outgoing current (i.e., with a current that at large distances is directed away from the origin) for $\mathrm{Re}z>a$, and with an incoming current for $\mathrm{Re}z<-a$. Similarly, the solutions $f^-(z)$ may be identified with an incoming current for $\mathrm{Re}z>a$, and with an outgoing current for $\mathrm{Re}z<-a$.

We now write the most general form of the wave function for a given parameter a^2 as

$$\psi(z,z^*)=f(z)\exp(-|z|^2/4l_0^2)$$

with $f(z)$ given by the following. For $\mathrm{Re}z<-a$,

$$f(z)=\left[A\exp\left[\frac{i}{2l_0^2}\int_{-a}^{z}(-a^2+\zeta^2)^{1/2}d\zeta\right)+B\exp\left[\frac{-i}{2l_0^2}\int_{-a}^{z}(-a^2+\zeta^2)^{1/2}d\zeta\right]\right]\times\left[\frac{2}{(-a^2+\zeta^2)^{1/2}}\right]^{1/2};$$

for $-a < \mathrm{Re}z < a$,

$$f(z)=\left[C\exp\left[\frac{1}{2l_0^2}\int_{-a}^{z}(a^2-\zeta^2)^{1/2}d\zeta\right]+D\exp\left[\frac{-1}{2l_0^2}\int_{-a}^{z}(a^2-\zeta^2)^{1/2}a\zeta\right]\right]\times\left[\frac{2}{(a^2-\zeta^2)^{1/2}}\right]^{1/2};$$

and for $\mathrm{Re}z > a$,

$$f(z)=\left[F\exp\left[\frac{i}{2l_0^2}\int_{a}^{z}(-a^2+\zeta^2)^{1/2}d\zeta\right]+G\exp\left[\frac{-i}{2l_0^2}\int_{a}^{z}(-a^2+\zeta^2)^{1/2}d\zeta\right]\right]\times\left[\frac{2}{(-a^2+\zeta^2)^{1/2}}\right]^{1/2}.$$

Once we have specified the coefficients A and B, then the remaining coefficients C, D, F, and G are determined by the WKB matching formulas. If we restrict our attention to values of z that are on the real line ($y=0$), these matching conditions become precisely those of a one-dimensional WKB problem, for which we can carry over standard results. Setting $G=0$, so that there is no incoming current from the right, we follow Merzbacher[9] to find

$$T_{\mathrm{WKB}}=\frac{|F|^2}{|A|^2}=\frac{16\theta^2}{(4\theta^2+1)^2}, \tag{3.4}$$

where

$$\theta=\exp\left[\frac{1}{2l_0^2}\int_{-a}^{a}(a^2-x^2)^{1/2}dx\right]\equiv\exp\left[\frac{\pi a^2}{4l_0^2}\right].$$

Substituting $a^2=2x_0^2$, we find $\theta=\exp(\pi x_0^2/2l_0^2)$. This is precisely the result we found for the WKB approximation to the transmission coefficient in the strong-field limit in Sec. II for $U_x=U_y$ and $x_0/l_0 \gg 1$.

IV. SUMMARY

We have calculated the transmission coefficient T for an electron in an arbitrary magnetic field and a saddle-point $V_{\mathrm{SP}}(x,y)=U_y y^2-U_x x^2+V_0$. The result agrees well with the WKB approximation for this quantity when the classical turning points are not too close to one another. We derive the WKB approximation to T, for the special case $U_x=U_y$, in the strong-field limit with a complex coordinate technique.

In our exact calculation, we have expressed the Hamiltonian as a sum of two Hamiltonians, one involving only the cyclotron coordinates s and p, the other involving only the guiding-center coordinates X and P. The former Hamiltonian is that of a one-dimensional particle in a confining harmonic potential, so that an eigenfunction of the Hamiltonian has an oscillator index n. In the limit that the magnetic field $B\to\infty$, a wave function in the nth oscillator level becomes equivalent to a wave function in the nth Landau level. The guiding-center Hamiltonian is that of a one-dimensional particle in an inverted harmonic potential. For any fixed energy, a wave packet sufficiently far from the saddle point will be localized near one of the asymptotes of V_{SP}. The motion of the wave packet is consistent with the semiclassical picture of the eigenfunctions of an electron in a strong magnetic field and a slowly varying potential. The probability that the particle in two dimensions will tunnel through the saddle-point barrier is equivalent to the probability that the one-dimensional particle will be transmitted through the inverted harmonic-oscillator potential.

Note added in proof. Professor Mark Azbel has kindly pointed out that problems mathematically similar to the current one were studied in the 1950's, in the context of the semiclassical electron orbits in a metal with a complicated Fermi surface, where the magnetic length l_0 is much larger than the size of the crystal unit cell. See, for example, G. E. Zil'berman, Zh. Eksp. Teor. Fiz. **33**, 387 (1958) [Sov. Phys.—JETP **6**, 299 (1958)]; M. Ya. Azbel, *ibid.* **39**, 1272 (1961) [**12**, 891 (1961)] and references therein.

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