Propagation of surface acoustic waves across random gratings

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The propagation of surface acoustic waves of sagittal and of shear horizontal polarization across the grooves of a random grating is investigated theoretically. It is found that the attenuation rate of a sagittally polarized surface acoustic wave in the long-wavelength limit is proportional to $(ka)^4$, where k is the wave vector of the wave and a is the transverse correlation length of the surface roughness. This is in contrast with the $(ka)^5$ dependence of the attenuation rate found in this limit for sagitally polarized surface acoustic waves propagating across a two-dimensional, randomly rough surface. The dominant contribution to the attenuation rate of such a surface acoustic wave comes from its scattering into bulk waves. The attenuation rate of surface acoustic waves of shear horizontal polarization is found to be proportional to $(ka)^5$ in the long-wavelength limit. The surface roughness gives rise to the wave slowing of surface acoustic waves of both polarizations.

I. INTRODUCTION

The propagation of surface acoustic waves across randomly rough surfaces has been studied theoretically by several authors over the past 15 years. The propagation of sagittally polarized surface acoustic waves (Rayleigh waves) across a randomly rough surface was studied first by Urazakov and Fal'kovskii[1] on the basis of Rayleigh's method.[2] In a subsequent paper Maradudin and Mills[3] used a Green's-function method to solve the same problem. Both sets of authors found that the attenuation rate of a Rayleigh wave on a randomly rough surface is proportional to the fifth power of its frequency, in the limit that its wavelength is longer than the transverse correlation length of the surface roughness. However, the work of Maradudin and Mills contained errors. These were corrected in two papers by Eguiluz and Maradudin,[4,5] who used the Rayleigh method[5] and the method of effective boundary conditions[4] in studying the propagation of Rayleigh waves across a randomly rough surface. In addition, these authors obtained the roughness-induced change in the frequency of the Rayleigh wave as it progresses across the surface, a consequence of surface roughness that had not been considered earlier by Urazakov and Fal'kovskii or by Maradudin and Mills. They also showed that the dominant mechanism for the attenuation of a Rayleigh wave on a randomly rough surface is its scattering into bulk elastic waves rather than into other Rayleigh waves. All of these studies of Rayleight waves on a randomly rough surface assumed that the surface is two-dimensionally rough, i.e., that the equation defining it, $x_3=\zeta(x_1,x_2)$, contains a surface profile function $\zeta(x_1,x_2)$ that is a function of both of the coordinates x_1 and x_2 in the plane of the mean surface $x_3=0$.

The propagation of shear horizontal surface acoustic waves across a randomly rough surface was studied first by Bulgakov and Khankina.[6] These authors considered a one-dimensionally rough surface—a random grating—whose defining equation, $x_3=\zeta(x_1)$, contains a surface profile function $\zeta(x_1)$ that is a function of only one of the coordinates in the plane $x_3=0$. Both the attenuation rate of the surface acoustic wave and its frequency shift were obtained in this work. The former was found to be proportional to the fifth power of the frequency of the surface acoustic wave, in the long-wavelength limit. The propagation of a shear horizontal surface acoustic wave across a surface that is two-dimensionally rough was subsequently studied by Hardouin Duparc and Maradudin,[7] and both the attenuation rate and the frequency shift of the wave were obtained in this work.

In this paper we study the propagation of sagittally polarized surface acoustic waves (Rayleigh waves) across the grooves of a random grating. The motivation for carrying out this study is that experiments are currently being carried out in which the attenuation of Rayleigh waves on such surfaces is being measured.[8] It is hoped that the results of the present work will be helpful in interpreting the results of these experiments.

For completeness we also reexamine the propagation of shear horizontal surface acoustic waves across a random grating. Although our final results are in agreement with the earlier results of Bulgakov and Khankina,[6] our approach differs from theirs in that we do not start with the small roughness limit as they do, but instead obtain the homogeneous integral equation for the Fourier coefficient of the displacement field in the medium (within the Rayleigh hypothesis) for an arbitrary surface profile. In addition, we show explicitly that in going to the weak roughness limit it is necessary to retain only the term in the kernel of this integral equation that is linear in the surface profile function $\zeta(x_1)$. The term of second order in this function, which might be thought to contribute to the same extent, is shown in fact to yield a higher-order contribution.

The physical system we consider consists of an isotropic elastic medium, characterized by a mass density ρ and speeds of longitudinal and transverse sound c_l and c_t, respectively, that occupies the region $x_3>\zeta(x_1)$ (Fig.

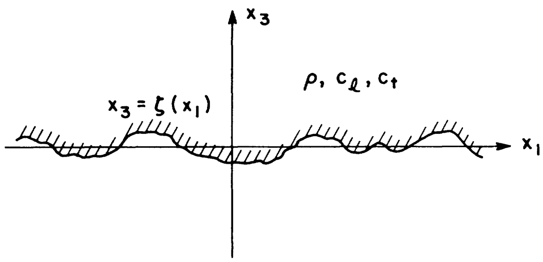

FIG. 1. The physical system studied in this paper.

1). The surface $x_3=\zeta(x_1)$ is assumed to be stress-free. We will be concerned with the propagation of surface acoustic waves of sagittal polarization and of shear horizontal polarization in the x_1 direction along this surface.

The surface profile function $\zeta(x_1)$ for a randomly rough surface is unknown in general. This forces us to characterize it by certain statistical properties. Underlying this characterization is the assumption that there is not a single function $\zeta(x_1)$ but rather an ensemble of realizations of this function. Physical properties associated with a randomly rough surface are to be averaged over this ensemble, and it is assumed that this ensemble average does not differ significantly from the spatial average over a single realization of $\zeta(x_1)$. The probability that $\zeta(x_1)$ has a certain value at the point x_1 is given by a probability distribution function. An explicit form for this distribution function will not be required for what follows; its first two moments suffice for our purposes. In common with most theoretical treatments of surface roughness, ours is based on the assumption that $\zeta(x_1)$ is a stationary stochastic process, and that the first two moments of its probability distribution functions are

$$\langle \zeta(x_1)\rangle = 0 \;, \tag{1.1}$$

$$\langle \zeta(x_1)\zeta(x_1')\rangle = \delta^2 W(|x_1-x_1'|) \;. \tag{1.2}$$

In Eqs. (1.1) and (1.2) the angular brackets $\langle\ \rangle$ denote an average over the ensemble of realizations of the function $\zeta(x_1)$. The quantity δ^2 appearing in Eq. (1.2) is the mean-square departure of the surface from flatness,

$$\langle \zeta^2(x_1)\rangle = \delta^2 \;. \tag{1.3}$$

We will also require the Fourier transform $\hat{\zeta}(Q)$ of $\zeta(x_1)$, which is defined by

$$\zeta(x_1) = \int \frac{dQ}{2\pi}\hat{\zeta}(Q)e^{iQx} \;. \tag{1.4}$$

With the aid of the Fourier inversion formula and Eqs. (1.1) and (1.2) it is readily established that $\hat{\zeta}(Q)$ possesses the following properties:

$$\langle \hat{\zeta}(Q)\rangle = 0 \;, \tag{1.5}$$

$$\langle \hat{\zeta}(Q)\hat{\zeta}(Q')\rangle = 2\pi\delta(Q+Q')\delta^2 g(|Q|) \;, \tag{1.6}$$

where

$$g(|Q|) = \int dx_1\, e^{-iQx_1} W(|x_1|) \tag{1.7}$$

is called the *surface structure factor.*

The general results we obtain in this paper will be given in terms of an arbitrary function $g(|Q|)$. However, in obtaining numerical results we will assume a Gaussian form for $W(|x_1|)$, viz.,

$$W(|x_1|) = \exp(-x_1^2/a^2) \;, \tag{1.8}$$

where the characteristic length a appearing in this expression is called the *transverse correlation length.* The form of $g(|Q|)$ that corresponds to the choice (1.8) is found from Eq. (1.7) to be

$$g(|Q|) = \pi^{1/2} a \exp(-a^2Q^2/4) \;. \tag{1.9}$$

This completes our description of the system underlying the calculations that follow, and we now turn to those calculations.

The outline of the remainder of this paper is as follows. In Sec. II we obtain the dispersion relation for a sagittally polarized surface acoustic wave propagating across the surface $x_3=\zeta(x_1)$. The same calculation for a surface acoustic wave of shear horizontal polarization is carried out in Sec. III. A discussion of the results obtained, in Sec. IV, concludes the paper.

II. SAGITTAL POLARIZATION

In this section we obtain the dispersion relation for sagitally polarized surface acoustic waves propagating in the x_1 direction across the random grating depicted in Fig. 1. The displacement field in this case has the form

$$\mathbf{u}(\mathbf{x};t) = (u_1(x_1,x_3|\omega),\, 0,\, u_3(x_1,x_3|\omega))\exp(-i\omega t) \tag{2.1}$$

in the region $x_3>\zeta(x_1)$. The displacement amplitudes $u_{1,3}(x_1,x_3|\omega)$ satisfy a pair of coupled equations of motion in this region:

$$-\omega^2 u_1 = \left[c_l^2\frac{\partial^2}{\partial x_1^2}+c_t^2\frac{\partial^2}{\partial x_3^2}\right]u_1 + (c_l^2-c_t^2)\frac{\partial^2 u_3}{\partial x_1\partial x_3} \;, \tag{2.2a}$$

$$-\omega^2 u_3 = (c_l^2-c_t^2)\frac{\partial^2 u_1}{\partial x_1\partial x_3} + \left[c_t^2\frac{\partial^2}{\partial x_1^2}+c_l^2\frac{\partial^2}{\partial x_3^2}\right]u_3 \;. \tag{2.2b}$$

The conditions ensuring that the surface $x_3=\zeta(x_1)$ be stress-free take the forms

$$\left[\left[-c_l^2\zeta'(x_1)\frac{\partial}{\partial x_1}+c_t^2\frac{\partial}{\partial x_3}\right]u_1 + \left[-(c_l^2-2c_t^2)\zeta'(x_1)\frac{\partial}{\partial x_3}+c_t^2\frac{\partial}{\partial x_1}\right]u_3\right]_{x_3=\zeta(x_1)} = 0 \;, \tag{2.3a}$$

$$\left[\left[-c_t^2\zeta'(x_1)\frac{\partial}{\partial x_3}+(c_l^2-2c_t^2)\frac{\partial}{\partial x_1}\right]u_1+\left[-c_t^2\zeta'(x_1)\frac{\partial}{\partial x_1}+c_l^2\frac{\partial}{\partial x_3}\right]u_3\right]_{x_3=\zeta(x_1)}=0 . \tag{2.3b}$$

In addition, we require that $u_{1,3}(x_1x_3\mid\omega)$ vanish as $x_3\to\infty$.

The solutions of Eqs. (2.2) in the region $x_3>\zeta(x_1)_{\max}$ that satisfy the boundary condition at infinity can be written as the Fourier integrals

$$u_1(x_1,x_3\mid\omega)=\int\frac{dk}{2\pi}e^{ikx_1}[A_l(k,\omega)e^{-\alpha_l(k,\omega)x_3}+A_t(k,\omega)e^{-\alpha_t(k,\omega)x_3}] , \tag{2.4a}$$

$$u_3(x_1,x_3\mid\omega)=\int\frac{dk}{2\pi}e^{ikx_1}i\left[\frac{\alpha_l(k,\omega)}{k}A_l(k,\omega)e^{-\alpha_l(k,\omega)x_3}+\frac{k}{\alpha_t(k,\omega)}A_t(k,\omega)e^{-\alpha_t(k,\omega)x_3}\right] , \tag{2.4b}$$

where

$$\alpha_{l,t}(k,\omega)=\begin{cases}\left[k^2-\dfrac{\omega^2}{c_{l,t}^2}\right]^{1/2} , & k^2>\dfrac{\omega^2}{c_{l,t}^2} \quad (2.5a)\\[2ex] -i\left[\dfrac{\omega^2}{c_{l,t}^2}-k^2\right]^{1/2} , & k^2<\dfrac{\omega^2}{c_{l,t}^2} . \quad (2.5b)\end{cases}$$

We now invoke the Rayleigh hypothesis[2] and use the solutions (2.4) in satisfying the boundary conditions (2.3) at the rough surface $x_3=\zeta(x_1)$. The result is a pair of coupled homogeneous equations for the amplitude functions $A_l(k,\omega)$ and $A_t(k,\omega)$:

$$\int\frac{dk}{2\pi}e^{ikx_1}\left\{\left[\frac{i\zeta'(x_1)}{k}\left[\frac{\omega^2}{c_t^2}+2\alpha_l^2(k,\omega)\right]+2\alpha_l(k,\omega)\right]A_l(k)e^{-\alpha_l(k,\omega)\zeta(x_1)}\right.$$
$$\left.+\left[2ik\zeta'(x_1)+\frac{\alpha_t^2(k,\omega)+k^2}{\alpha_t(k,\omega)}\right]A_t(k)e^{-\alpha_t(k,\omega)\zeta(x_1)}\right\}=0 , \tag{2.6a}$$

$$\int\frac{dk}{2\pi}e^{ikx_1}\left[\left[2\alpha_l(k,\omega)\zeta'(x_1)-i\frac{k^2+\alpha_t^2(k,\omega)}{k}\right]A_l(k)e^{-\alpha_l(k,\omega)\zeta(x_1)}\right.$$
$$\left.+\left[\frac{k^2+\alpha_t^2(k,\omega)}{\alpha_t(k,\omega)}\zeta'(x_1)-2ik\right]A_t(k)e^{-\alpha_t(k,\omega)\zeta(x_1)}\right]=0 . \tag{2.6b}$$

In writing Eqs. (2.6), to simplify the notation we have stopped indicating explicitly that $A_{l,t}$ are functions of ω as well as of k.

To proceed farther, we introduce the representation

$$e^{-\alpha\zeta(x_1)}=\int\frac{dQ}{2\pi}e^{iQx_1}I(\alpha\mid Q) , \tag{2.7}$$

where

$$I(\alpha\mid Q)=\int dx_1\, e^{-iQx_1}e^{-\alpha\zeta(x_1)} , \tag{2.8}$$

so that

$$\zeta'(x_1)e^{-\alpha\zeta(x_1)}=-\frac{i}{\alpha}\int\frac{dQ}{2\pi}Qe^{iQx_1}I(\alpha\mid Q) . \tag{2.9}$$

When we use Eqs. (2.7) and (2.9) in Eqs. (2.6), the latter become

$$\int\frac{dq}{2\pi}e^{iqx_1}\int\frac{dk}{2\pi}\left[\frac{I(\alpha_l(k,\omega)\mid q-k)}{k\alpha_l(k,\omega)}\left[(q-k)\frac{\omega^2}{c_t^2}+2q\alpha_l^2(k,\omega)\right]A_l(k)+\frac{I(\alpha_t(k,\omega)\mid q-k)}{\alpha_t(k,\omega)}\left[2qk-\frac{\omega^2}{c_t^2}\right]A_t(k)\right]=0 , \tag{2.10a}$$

$$\int\frac{dq}{2\pi}e^{iqx_1}\int\frac{dk}{2\pi}\left[I(\alpha_l(k,\omega)\mid q-k)\frac{2qk-\dfrac{\omega^2}{c_t^2}}{k}A_l(k)+I(\alpha_t(k,\omega)\mid q-k)\frac{2qk^2-(q+k)\dfrac{\omega^2}{c_t^2}}{\alpha_t^2(k,\omega)}A_t(k)\right]=0 . \tag{2.10b}$$

On equating to zero the qth Fourier coefficient on the left-hand side of each of these equations, and then interchanging the roles of the variables q and k, we obtain finally the pair of coupled, homogeneous, integral equations satisfied by the amplitudes $A_l(k)$ and $A_t(k)$:

$$\int \frac{dq}{2\pi}\left[\frac{I(\alpha_l(q,\omega)\,|\,k-q)}{q\alpha_l(q,\omega)}\left[(k-q)\frac{\omega^2}{c_t^2}+2k\alpha_l^2(q,\omega)\right]A_l(q)+\frac{I(\alpha_t(q,\omega)\,|\,k-q)}{\alpha_t(q,\omega)}\left[2kq-\frac{\omega^2}{c_t^2}\right]A_t(q)\right]=0\ , \quad (2.11a)$$

$$\int \frac{dq}{2\pi}\left[\frac{I(\alpha_l(q,\omega)\,|\,k-q)}{q}\left[2kq-\frac{\omega^2}{c_t^2}\right]A_l(q)+\frac{I(\alpha_t(q,\omega)\,|\,k-q)}{\alpha_t^2(q,\omega)}\left[2kq^2-(k+q)\frac{\omega^2}{c_t^2}\right]A_t(q)\right]=0\ . \quad (2.11b)$$

Within the Rayleigh hypothesis these are the exact equations for $A_l(k)$ and $A_t(k)$. The solvability condition for Eqs. (2.11) is the dispersion relation for sagittally polarized surface acoustic waves propagating across the surface defined by the equation $x_3=\zeta(x_1)$.

To obtain this dispersion relation we invoke the small roughness approximation, which is defined by the expansion

$$I(\alpha\,|\,Q)=\int dx_1 e^{-iQx_1}\left[1-\alpha\zeta(x_1)+\frac{\alpha^2}{2}\zeta^2(x_1)-\cdots\right]$$

$$=2\pi\delta(Q)-\alpha\hat{\zeta}(Q)+\frac{\alpha^2}{2}\hat{\zeta}^{(2)}(Q)+\cdots \quad (2.12)$$

where

$$\hat{\zeta}^{(n)}(Q)=\int dx_1\, e^{-iQx_1}\zeta^n(x_1)\ , \quad (2.13a)$$

$$\hat{\zeta}^{(1)}(Q)\equiv\hat{\zeta}(Q)\ . \quad (2.13b)$$

When Eq. (2.12) is used in Eqs. (2.11) the resulting equations can be written in the form

$$\sum_\beta M_{\alpha\beta}^{(0)}(k)A_\beta(k)=\sum_\beta\int\frac{dq}{2\pi}\hat{\zeta}(k-q)M_{\alpha\beta}^{(1)}(k\,|\,q)A_\beta(q)-\frac{1}{2}\sum_\beta\int\frac{dq}{2\pi}\hat{\zeta}^{(2)}(k-q)\times M_{\alpha\beta}^{(2)}(k\,|\,q)A_\beta(q)\ , \quad (2.14)$$

where the indices $\alpha,\beta,\ldots$ assume the values l and t. The matrices $\overleftrightarrow{M}^{(0)}(k)$, $\overleftrightarrow{M}^{(1)}(k\,|\,q)$, and $\overleftrightarrow{M}^{(2)}(k\,|\,q)$ are given by

$$\overleftrightarrow{M}^{(0)}(k)=\begin{bmatrix} 2\alpha_l(k,\omega) & \dfrac{k^2+\alpha_t^2(k,\omega)}{\alpha_t(k,\omega)} \\ \dfrac{k^2+\alpha_t^2(k,\omega)}{k} & 2k \end{bmatrix}, \quad (2.15a)$$

$$\overleftrightarrow{M}^{(1)}(k\,|\,q)=\begin{bmatrix} \left[(k-q)\dfrac{\omega^2}{c_t^2}+2k\alpha_l^2(q,\omega)\right]\dfrac{1}{q} & 2kq-\dfrac{\omega^2}{c_t^2} \\ \left[2kq-\dfrac{\omega^2}{c_t^2}\right]\dfrac{\alpha_l(q,\omega)}{q} & \dfrac{2kq^2-(k+q)\dfrac{\omega^2}{c_t^2}}{\alpha_t(q,\omega)} \end{bmatrix}, \quad (2.15b)$$

$$\overleftrightarrow{M}^{(2)}(k\,|\,q)=\begin{bmatrix} \left[(k-q)\dfrac{\omega^2}{c_t^2}+2k\alpha_l^2(q,\omega)\right]\dfrac{\alpha_l(q,\omega)}{q} & \left[2kq-\dfrac{\omega^2}{c_t^2}\right]\alpha_t(q,\omega) \\ \left[2kq-\dfrac{\omega^2}{c_t^2}\right]\dfrac{\alpha_l^2(q,\omega)}{q} & 2kq^2-(k+q)\dfrac{\omega^2}{c_t^2} \end{bmatrix}. \quad (2.15c)$$

Equations (2.14) [and Eqs. (2.11) as well] are stochastic integral equations because of the presence of the stochastic function $\zeta(x_1)$ in their kernels. The solutions $A_l(k)$ and $A_t(k)$ therefore are stochastic functions also. We will not seek the probability distribution functions of $A_{l,t}(k)$. Instead we will seek their first moments $\langle A_{l,t}(k)\rangle$, which describe the propagation of the mean wave across the random grating. This suffices for obtaining the dispersion relation of sagittally polarized surface acoustic waves across the random grating.

Our task therefore is to extract the equations satisfied by $\langle A_{l,t}(k)\rangle$ from the equations, Eq. (2.14), satisfied by $A_{l,t}(k)$. To accomplish this we introduce the smoothing operator P that averages everything on which it acts over the ensemble of realizations of the surface profile function $\zeta(x_1)$: $Pf\equiv\langle f\rangle$. We also introduce the complementary operator $Q=1-P$ that projects out the fluctuating part of everything on which it acts. We first apply P to both sides of Eq. (2.14):

$$\sum_\beta M^{(0)}_{\alpha\beta}(k)PA_\beta(k)=\sum_\beta \int \frac{dq}{2\pi} P\hat{\zeta}(k-q)M^{(1)}_{\alpha\beta}(k\mid q)[PA_\beta(q)+QA_\beta(q)]$$
$$-\tfrac{1}{2}\sum_\beta \int \frac{dq}{2\pi} P\hat{\zeta}^{(2)}(k-q)M^{(2)}_{\alpha\beta}(k\mid q)[PA_\beta(q)+QA_\beta(q)] \ . \quad (2.16)$$

We have used the identity $A_\beta(q)=PA_\beta(q)+QA_\beta(q)$ in writing this equation. We can simplify Eq. (2.16) by the use of Eq. (1.5) and the result that

$$\langle \hat{\zeta}^{(2)}(k-q)\rangle=\delta^2 2\pi\delta(k-q) \ , \quad (2.17)$$

which follows from Eqs. (2.13a) and (1.3). We obtain

$$\sum_\beta M^{(0)}_{\alpha\beta}(k)PA_\beta(k)=\sum_\beta \int \frac{dq}{2\pi} P\hat{\zeta}(k-q)M^{(1)}_{\alpha\beta}(k\mid q)QA_\beta(q)-\tfrac{1}{2}\delta^2\sum_\beta M^{(2)}_{\alpha\beta}(k\mid k)PA_\beta(k)+o(\delta^2) \ . \quad (2.18)$$

We have neglected the term $P\hat{\zeta}^{(2)}(k-q)QA_\beta(q)$ on the right-hand side of Eq. (2.16) because, as we will see below, $QA_\beta(q)$ is of $O(\zeta)$, and we will work only to $O(\delta^2)$.

We next apply the operator Q to both sides of Eq. (2.14), keeping in mind that we need $QA_\beta(q)$ only to first order in $\zeta(x_1)$ to obtain the right-hand side of Eq. (2.18) to $O(\zeta^2)$:

$$\sum_\beta M^{(0)}_{\alpha\beta}(k)QA_\beta(k)=\sum_\beta \int \frac{dq}{2\pi} Q\hat{\zeta}(k-q)M^{(1)}_{\alpha\beta}(k\mid q)[PA_\beta(q)+QA_\beta(q)]$$
$$\cong\sum_\beta \int \frac{dq}{2\pi} \hat{\zeta}(k-q)M^{(1)}_{\alpha\beta}(k\mid q)PA_\beta(q)+o(\delta) \ . \quad (2.19)$$

When we substitute the solution of Eq. (2.19) into Eq. (2.18), and use Eq. (1.6) we obtain the equation sought;

$$\sum_\beta M^{(0)}_{\alpha\beta}(k)\langle A_\beta(k)\rangle=\delta^2\sum_{\beta,\mu,\nu}\int \frac{dq}{2\pi} g(|k-q|)M^{(1)}_{\alpha\mu}(k\mid q)[\overleftrightarrow{\mathrm{M}}^{(0)-1}(q)]_{\mu\nu}M^{(1)}_{\nu\beta}(q\mid k)\langle A_\beta(k)\rangle$$
$$-\tfrac{1}{2}\delta^2\sum_\beta M^{(2)}_{\alpha\beta}(k\mid k)\langle A_\beta(k)\rangle+o(\delta^2) \ . \quad (2.20)$$

That this is an algebraic equation rather than an integral equation is due to the restoration of infinitesimal translational invariance by the averaging process.

The inverse of the matrix $\overleftrightarrow{\mathrm{M}}^{(0)}(k)$ can be written in the form

$$[\overleftrightarrow{\mathrm{M}}^{(0)-1}(k)]_{\alpha\beta}=\frac{C_{\alpha\beta}(k,\omega)}{D(k,\omega)} \ , \quad (2.21)$$

where

$$D(k,\omega)=4k^2\alpha_l(k,\omega)\alpha_t(k,\omega)-[k^2+\alpha_t^2(k,\omega)]^2 \ , \quad (2.22)$$

and the matrix $\overleftrightarrow{\mathrm{C}}(k,\omega)$ is

$$\overleftrightarrow{\mathrm{C}}(k,\omega)=\begin{bmatrix} 2k^2\alpha_t(k,\omega) & -k[k^2+\alpha_t^2(k,\omega)] \\ -\alpha_t(k,\omega)[k^2+\alpha^2(k,\omega)] & 2k\alpha_l(k,\omega)\alpha_t(k,\omega)\end{bmatrix} \ . \quad (2.23)$$

The vanishing of $D(k,\omega)$ is the dispersion relation for Rayleigh waves on a planar, stress-free surface of a semi-infinite, isotropic elastic medium. Equation (2.20) can therefore be written in the form

$$\sum_\beta m^{(0)}_{\alpha\beta}(k,\omega)\langle A_\beta(k)\rangle=\delta^2\sum_\beta m_{\alpha\beta}(k,\omega)\langle A_\beta(k)\rangle \ , \quad (2.24)$$

where

$$\overleftrightarrow{\mathrm{m}}^{(0)}(k,\omega)=\begin{bmatrix} 2\alpha_l(k,\omega)X_l(k,\omega) & \dfrac{k^2+\alpha_t^2(k,\omega)}{\alpha_t(k,\omega)}X_t(k,\omega) \\ \dfrac{k^2+\alpha_t^2(k,\omega)}{k}X_l(k,\omega) & 2kX_t(k,\omega)\end{bmatrix} \ , \quad (2.25)$$

and

$$X_{l,t}(k,\omega)=1+\tfrac{1}{2}\delta^2\alpha_{l,t}^2(k,\omega) , \quad (2.26)$$

while

$$m_{\alpha\beta}(k,\omega)=\int\frac{dq}{2\pi}\frac{g(|k-q|)}{D(q,\omega)}\sum_{\mu,\nu}M_{\alpha\mu}^{(1)}(k\mid q)C_{\mu\nu}(q,\omega)\times M_{\nu\beta}^{(1)}(q\mid k) . \quad (2.27)$$

The dispersion relation for sagittally polarized surface acoustic waves propagating across a random grating is obtained by equating to zero the determinant of the coefficients in Eq. (2.24). For this purpose we use the result

$$|\overleftrightarrow{m}^{(0)}-\delta^2\overleftrightarrow{m}| = |\overleftrightarrow{m}^{(0)}|[1-\delta^2\mathrm{Tr}(\overleftrightarrow{m}^{(0)-1}\overleftrightarrow{m})+O(\delta^4)] . \quad (2.28)$$

Now we have

$$|\overleftrightarrow{m}^{(0)}(k,\omega)| = \frac{X_l(k,\omega)X_t(k,\omega)}{k\alpha_t(k,\omega)}D(k,\omega) \quad (2.29)$$

while

$$\overleftrightarrow{m}^{(0)-1}(k,\omega)=\frac{1}{D(k,\omega)}\overleftrightarrow{X}^{-1}(k,\omega)\overleftrightarrow{C}(k,\omega) , \quad (2.30)$$

where

$$\overleftrightarrow{X}^{-1}(k,\omega)=\begin{bmatrix} X_l^{-1}(k,\omega) & 0 \\ 0 & X_t^{-1}(k,\omega)\end{bmatrix} . \quad (2.31)$$

Thus the dispersion relation takes the form

$$D(k,\omega)=\delta^2\mathrm{Tr}[\overleftrightarrow{X}^{-1}(k,\omega)\overleftrightarrow{C}(k,\omega)\overleftrightarrow{m}(k,\omega)] . \quad (2.32)$$

Since the right-hand side of this equation is explicitly proportional to δ^2, we can set $X_{l,t}(k,\omega)=1$ in Eq. (2.32) to obtain a dispersion relation correct to this order, viz.,

$$D(k,\omega)=\int\frac{dq}{2\pi}\frac{g(|k-q|)}{D(q,\omega)}\times\sum_{\alpha,\beta}N_{\alpha\beta}(k;q\mid\omega)N_{\beta\alpha}(q;k\mid\omega) , \quad (2.33a)$$

where

$$N_{\alpha\beta}(k;q\mid\omega)=\sum_{\mu}C_{\alpha\mu}(k,\omega)M_{\mu\beta}^{(1)}(k\mid q) . \quad (2.33b)$$

Equation (2.33) is the dispersion relation we would have obtained directly if we had omitted the last term on the right-hand side of Eq. (2.12) at the outset.

We will solve Eq. (2.33) for ω as a function of k by setting

$$\omega_S(k)=\omega_0(k)+\delta^2\Delta(k) , \quad (2.34)$$

where $\omega_0(k)=c_Rk$ is the solution of the equation $D(k,\omega_0(k))=0$, so that c_R is the speed of Rayleigh waves on a planar surface. The subscript S here denotes sagitall. The equation for c_R is obtained by substituting $\omega=c_Rk$ into the equation $D(k,\omega)=0$, and takes the form

$$\xi^6-8\xi^4+8(3-2\lambda^2)\xi^2-16(1-\lambda^2)=0 \quad (2.35)$$

when we introduce the notation

$$\xi=\frac{c_R}{c_t}, \quad \lambda=\frac{c_t}{c_l} . \quad (2.36)$$

ξ is that solution of Eq. (2.35) that satisfies the condition $0<\xi<1$. When we substitute Eq. (2.34) into Eq. (2.33), we find that $\Delta(k)$ is given by

$$\Delta(k)=\frac{1}{\left[\frac{\partial D(k,\omega)}{\partial\omega}\right]_{\omega=\omega_0(k)}}\int\frac{dq}{2\pi}g(|k-q|)\sum_{\alpha,\beta}\left[\frac{N_{\alpha\beta}(k;q\mid\omega)N_{\beta\alpha}(q;k\mid\omega)}{D(q,\omega)}\right]_{\omega=\omega_0(k)} , \quad (2.37)$$

to lowest order in δ^2.

The derivative $\partial D(k,\omega)/\partial\omega$ can be written in the form

$$\frac{\partial D(k,\omega)}{\partial\omega}=-\frac{4\omega}{c_t^2}\left[k^2\frac{\alpha_l^2(k,\omega)+\lambda^2\alpha_t^2(k,\omega)}{\alpha_l(k,\omega)\alpha_t(k,\omega)}-[k^2+\alpha_t^2(k,\omega)]\right] . \quad (2.38)$$

It follows that

$$\left[\frac{\partial D(k,\omega)}{\partial\omega}\right]_{\omega=\omega_0(k)}=-\frac{k^3}{c_t}\frac{4\xi}{(2-\xi^2)^2}[\xi^6-6\xi^4+4(3-2\lambda^2)\xi^2-4(1-\lambda^2)] . \quad (2.39)$$

We can simplify this result by using Eq. (2.35). In this way we finally obtain

$$\left[\frac{\partial D(k,\omega)}{\partial\omega}\right]_{\omega=\omega_0(k)}=-\frac{k^3}{c_t}G(\xi,\lambda) , \quad (2.40a)$$

where

$$G(\xi,\lambda)=\frac{8\xi}{(2-\xi^2)^2}[\xi^4-2(3-2\lambda^2)\xi^2+6(1-\lambda^2)] . \quad (2.40b)$$

Thus, Eq. (2.37) becomes

$$\Delta(k)=-\frac{c_t}{G(\xi,\lambda)k^3}\int\frac{dq}{2\pi}g(|k-q|)\sum_{\alpha,\beta}\frac{N_{\alpha\beta}(k;q\,|\,c_Rk)N_{\beta\alpha}(q;k\,|\,c_Rk)}{D(q,c_Rk)}\ . \tag{2.41}$$

At this point it is convenient to introduce dimensionless variables according to

$$k=\frac{x}{a},\quad q=\frac{x}{a}u\ , \tag{2.42}$$

where k, and therefore x, is assumed to be positive. Equation (2.41) is transformed thereby into

$$\Delta(k)=-\frac{c_t}{2\pi G(\xi,\lambda)}\frac{x^4}{a^4}\int_{-\infty}^{\infty}du\,g((x/a)|1-u|)\times\sum_{\alpha,\beta}\frac{n_{\alpha\beta}^{(0)}(u)\hat{n}_{\beta\alpha}^{(0)}(u)}{d_0(u)}\ . \tag{2.43}$$

Explicit expressions for $d_0(u)$, $n_{\alpha\beta}^{(0)}(u)$, and $\hat{n}_{\alpha\beta}^{(0)}(u)$ are given in the Appendix.

Equation (2.43) gives $\Delta(k)$ for an arbitrary surface structure factor $g(|Q|)$. The numerical calculations of $\Delta(k)$ in this paper will be carried out on the basis of the Gaussian form for this function given by Eq. (1.9). In this case Eq. (2.43) becomes

$$\Delta(k)=-c_Rk\frac{1}{2\sqrt{\pi}\xi G(\xi,\lambda)}\frac{x^3}{a^2}I(x)\ , \tag{2.44a}$$

where

$$I(x)=\int_{-\infty}^{\infty}du\,e^{-x^2(1-u)^2/4}\sum_{\alpha,\beta}\frac{n_{\alpha\beta}^{(0)}(u)\hat{n}_{\beta\alpha}^{(0)}(u)}{d_0(u)}\ . \tag{2.44b}$$

It follows from Eqs. (2.34) and (2.44) that the frequency of a Rayleigh wave on a randomly rough grating is

$$\omega_S(k)=c_Rk\left[1-\frac{1}{2\sqrt{\pi}\xi G(\xi,\lambda)}\frac{\delta^2}{a^2}x^3I(x)\right]\ , \tag{2.45}$$

to lowest nonzero order in δ.

The integral $I(x)$ was evaluated numerically. The infinite range of integration was divided into five intervals: $(-\infty,-\xi)$, $(-\xi,-\lambda\xi)$, $(-\lambda\xi,\lambda\xi)$, $(\lambda\xi,\xi)$, and (ξ,∞). This is because the functions $(u^2-\xi^2)^{1/2}$ and $(u^2-\lambda^2\xi^2)^{1/2}$ have different forms in these intervals. According to Eqs. (2.5) we have, for $|u|>\xi$,

$$\begin{aligned}(u^2-\xi^2)^{1/2}&\to(u^2-\xi^2)^{1/2}\ ,\\ (u^2-\lambda^2\xi^2)^{1/2}&\to(u^2-\lambda^2\xi^2)^{1/2}\ ,\end{aligned} \tag{2.46}$$

for $\lambda\xi<|u|<\xi$,

$$\begin{aligned}(u^2-\xi^2)^{1/2}&\to-i(\xi^2-u^2)^{1/2}\ ,\\ (u^2-\lambda^2\xi^2)^{1/2}&\to(u^2-\lambda^2\xi^2)^{1/2}\ ,\end{aligned} \tag{2.47}$$

and for $|u|<\lambda\xi$,

$$\begin{aligned}(u^2-\xi^2)^{1/2}&\to-i(\xi^2-u^2)^{1/2}\ ,\\ (u^2-\lambda^2\xi^2)^{1/2}&\to-i(\lambda^2\xi^2-u^2)^{1/2}\ .\end{aligned} \tag{2.48}$$

The expressions for $d_0(u)$, $n_{\alpha\beta}^{(0)}(u)$, and $\hat{n}_{\alpha\beta}^{(0)}(u)$ given in the Appendix are those for $|u|>\xi$. They must be modified according to Eqs. (2.46)–(2.48) in the corresponding intervals of u.

In addition, when $|u|>\xi$, $d_0(u)$ has simple zeros at $u=\pm1$. In fact, in this interval we have

$$\frac{1}{d_0(u)}=\frac{N(u\,|\,\xi,\lambda)}{\xi^2}\frac{1}{16(1-\lambda^2)u^6-8(3-2\lambda^2)\xi^2u^4+8\xi^4u^2-\xi^6}\ , \tag{2.49}$$

where

$$N(u\,|\,\xi,\lambda)=4u^2(u^2-\lambda^2\xi^2)^{1/2}(u^2-\xi^2)^{1/2}+(2u^2-\xi^2)^2\ . \tag{2.50}$$

Equation (2.49) can be rewritten as

$$\frac{1}{d_0(u)}=-\frac{N(u\,|\,\xi,\lambda)}{8\xi^2D(u\,|\,\xi,\lambda)}\frac{1}{u^2-1}\ , \tag{2.51}$$

where

$$\begin{aligned}D(u\,|\,\xi,\lambda)=&[\xi^4-2(3-2\lambda^2)\xi^2+6(1-\lambda^2)]\\&-[(3-2\lambda^2)\xi^2-6(1-\lambda^2)](u^2-1)\\&+2(1-\lambda^2)(u^2-1)^2\ .\end{aligned} \tag{2.52}$$

From Eqs. (2.5) it is found that the correct way to deal with the poles at $u=\pm1$ in the expression is to rewrite it as

$$\begin{aligned}\frac{1}{d_0(u)}&=-\frac{N(u\,|\,\xi,\lambda)}{16\xi^2D(u\,|\,\xi,\lambda)}\left[\frac{1}{u-1-i0}-\frac{1}{u+1+i0}\right]\\&=-\frac{N(u\,|\,\xi,\lambda)}{16\xi^2D(u\,|\,\xi,\lambda)}\left[\frac{1}{(u-1)_P}-\frac{1}{(u+1)_P}\right]\\&\quad-\frac{1}{\xi G(\xi,\lambda)}[\delta(u-1)+\delta(u+1)]\ ,\end{aligned} \tag{2.53}$$

where $1/(x)p$ denotes the principal part of $1/x$. This is the form for $d_0^{-1}(u)$ that was used in the region $|u|>\xi$.

The integral $I(x)$ is therefore complex,

$$I(x)=I_1(x)+iI_2(x)\ . \tag{2.54}$$

Its real part is associated with the roughness-induced change in the frequency of the Rayleigh wave. Its imaginary part describes the roughness-induced damping of the Rayleigh wave caused by its scattering into bulk waves (the contribution from the interval $|u|<\xi$) and into other Rayleigh waves (the contribution from the interval $|u|>\xi$).

In Fig. 2 we have plotted $I_1(x)$ and $I_2(x)$ as functions of x for the case where $\lambda^2=\frac{1}{3}$ (the Poisson case), for which $\xi=(2-\frac{2}{3}\sqrt{3})^{1/2}=0.919\,402$.

It is convenient to rewrite Eq. (2.45) in the form

$$\omega_S(k)=c_R k\left[1+\frac{\delta^2}{a^2}\omega_1(x)-i\frac{\delta^2}{a^2}\omega_2(x)\right], \quad (2.55)$$

where

$$\omega_1(x)=-\frac{x^3 I_1(x)}{2\sqrt{\pi}\xi G(\xi,\lambda)}, \quad (2.56a)$$

$$\omega_2(x)=\frac{x^3 I_2(x)}{2\sqrt{\pi}\xi G(\xi,\lambda)} \quad (2.56b)$$

are universal functions of $x=ka$. In Fig. 3 we have plotted $\omega_1(x)$ and $\omega_2(x)$ as functions of x for $\lambda^2=\frac{1}{3}$. In plotting $\omega_2(x)$ we have plotted the contribution associated with the scattering of the Rayleigh wave into other Rayleigh waves, $\omega_2^{(s)}(x)$, as well as the total function. It is clearly evident that the contribution $\omega_2^{(s)}(x)$ is dominated by the contribution associated with the scattering of the Rayleigh wave into bulk waves.

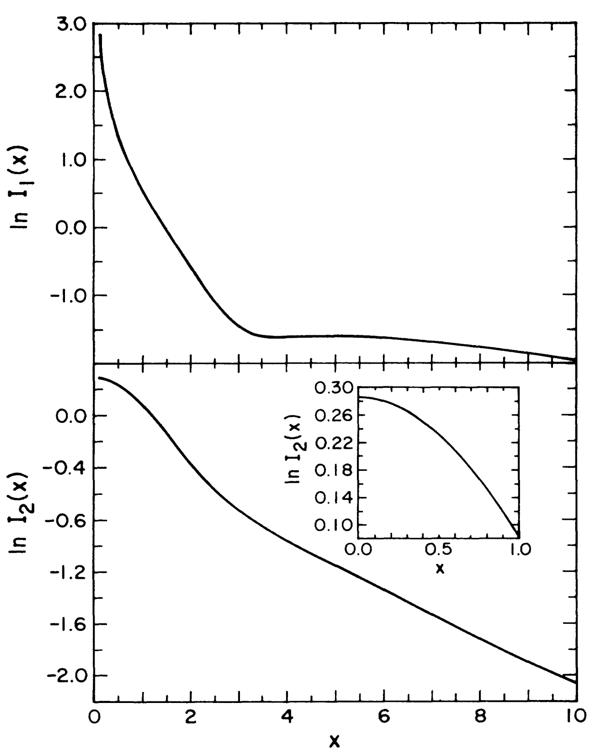

FIG. 2. The functions $I_1(x)$ and $I_2(x)$ defined by Eqs. (2.44b) and (2.54) of the text, for $\lambda^2=\frac{1}{3}$.

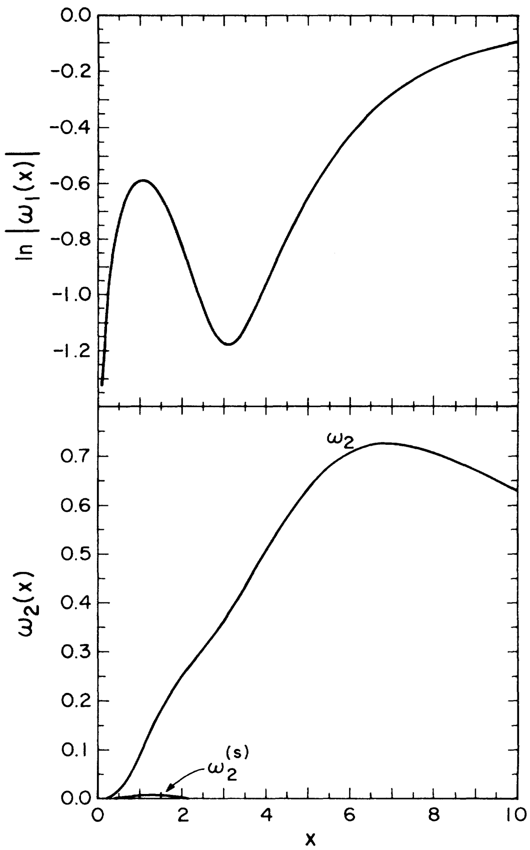

FIG. 3. The functions $\omega_1(x)$ and $\omega_2(x)$, defined by Eqs. (2.55) and (2.56) of the text, for Rayleigh waves on a random grating, when $\lambda^2=\frac{1}{3}$.

III. SHEAR HORIZONTAL POLARIZATION

In this section we consider the propagation of shear horizontal surface acoustic waves across the random grating depicted in Fig. 1. The displacement field in this case has the form

$$\mathbf{u}(\mathbf{x};t)=(0,\,u_2(x_1,x_3\,|\,\omega),\,0)\exp(-i\omega t) \quad (3.1)$$

in the region $x_3>\zeta(x_1)$. The time-independent equation of motion satisfied by $u_2(x_1,x_3\,|\,\omega)$ in this region is

$$-\omega^2 u_2=c_t^2\left[\frac{\partial^2}{\partial x_1^2}+\frac{\partial^2}{\partial x_3^2}\right]u_2 . \quad (3.2)$$

The stress-free boundary condition at the surface, $x_3=\zeta(x_1)$, can be expressed in the form

$$\left[-\zeta'(x_1)\frac{\partial}{\partial x_1}+\frac{\partial}{\partial x_3}\right]u_2\bigg|_{x_3=\zeta(x_1)}=0 . \quad (3.3)$$

In addition, we require that $u_2(x_1,x_3\,|\,\omega)$ vanish as $x_3\to\infty$.

The solution of Eq. (3.2) in the region $x_3>\zeta(x_1)_{\max}$ can be written as the Fourier integral

$$u_2(x_1,x_3\mid\omega)=\int\frac{dk}{2\pi}A(k,\omega)e^{ikx_1-\alpha_t(k,\omega)x_3}\ , \quad (3.4)$$

where $\alpha_t(k,\omega)$ has been defined in Eq. (2.5).

We now invoke the Rayleigh hypothesis and substitute the expression (3.4) into the boundary condition (3.3). The result is

$$\int\frac{dk}{2\pi}A(k,\omega)e^{ikx_1-\alpha_t(k,\omega)\zeta(x_1)}\times[-\zeta'(x_1)ik-\alpha_t(k,\omega)]=0\ . \quad (3.5)$$

With the use of the representations (2.7) and (2.9) Eq. (3.5) becomes

$$\int\frac{dq}{2\pi}e^{iqx_1}\int\frac{dk}{2\pi}I(\alpha_t(k,\omega)\mid q-k)\frac{qk-\frac{\omega^2}{c_t^2}}{\alpha_t(k,\omega)}A(k,\omega)=0\ . \quad (3.6)$$

When we equate to zero the qth Fourier coefficient on the left-hand side of Eq. (3.6), and subsequently interchange the roles of q and k, we obtain the following homogeneous integral equation for the Fourier coefficient $A(k,\omega)$ of the displacement component $u_2(x_1,x_3\mid\omega)$:

$$\int\frac{dq}{2\pi}I(\alpha_t(q,\omega)\mid k-q)\frac{kq-\frac{\omega^2}{c_t^2}}{\alpha_t(q,\omega)}A(q,\omega)=0\ . \quad (3.7)$$

Within the Rayleigh hypothesis this is the exact equation for $A(k,\omega)$. The solvability condition for Eq. (3.7) is the dispersion relation for shear horizontal surface acoustic waves propagating across the surface defined by the equation $x_3=\zeta(x_1)$.

To proceed farther, we invoke the small roughness approximation (2.12) and study the equation

$$\alpha_t(k,\omega)A(k,\omega)=\int\frac{dq}{2\pi}\hat{\zeta}(k-q)\left[kq-\frac{\omega^2}{c_t^2}\right]A(q,\omega)-\tfrac{1}{2}\int\frac{dq}{2\pi}\hat{\zeta}^{(2)}(k-q)\left[kq-\frac{\omega^2}{c_t^2}\right]\times\alpha_t(q,\omega)A(q,\omega)\ . \quad (3.8)$$

Equation (3.8) [and Eq. (3.7) as well] is a stochastic integral equation because of the presence of the stochastic function $\zeta(x_1)$ in its kernel. The solution $A(k,\omega)$ is therefore also a stochastic function. As in our discussion of sagittally polarized surface acoustic waves in the preceding section we will not seek the probability distribution function of $A(k,\omega)$; instead we will limit ourselves to obtaining its first moment $\langle A(k,\omega)\rangle$, which describes the mean wave propagating across the random grating. This suffices for obtaining the dispersion relation of shear horizontal surface acoustic waves on the random grating.

We begin by applying the operators P and Q defined in the preceding section to both sides of Eq. (3.8) in turn, and using the fact that $A(q,\omega)=PA(q,\omega)+QA(q,\omega)$:

$$\alpha_t(k,\omega)PA(k,\omega)=\int\frac{dq}{2\pi}P\hat{\zeta}(k-q)\left[kq-\frac{\omega^2}{c_t^2}\right]QA(q,\omega)-\tfrac{1}{2}\delta^2\alpha_t^3(k,\omega)PA(k,\omega)\ , \quad (3.9a)$$

$$\alpha_t(k,\omega)QA(k,\omega)=\int\frac{dq}{2\pi}Q\hat{\zeta}(k-q)\left[kq-\frac{\omega^2}{c_t^2}\right]PA(q,\omega). \quad (3.9b)$$

In writing Eq. (3.9a) we have used the fact that $\langle\hat{\zeta}(k-q)\rangle=0$, and that $\langle\hat{\zeta}^{(2)}(k-q)\rangle=\delta^2 2\pi\delta(k-q)$. We have also omitted a term in $QA(q,\omega)$ from the right-hand side of Eq. (3.9b) as of second order in $\hat{\zeta}(Q)$, since we need $QA(k,\omega)$ only to first order in $\hat{\zeta}(Q)$ to obtain a right-hand side of Eq. (3.9a) that is of $O(\zeta^2)$. The solution of Eq. (3.9b) is

$$QA(k,\omega)=\frac{1}{\alpha_t(k,\omega)}\times\int\frac{dq}{2\pi}\hat{\zeta}(k-q)\left[kq-\frac{\omega^2}{c_t^2}\right]PA(q,\omega)\ , \quad (3.9c)$$

where we have again used the fact that $\langle\hat{\zeta}(k-q)\rangle=0$. When Eq. (3.9c) is substituted into Eq. (3.11a) we obtain the equation satisfied by $\langle A(k,\omega)\rangle$:

$$\alpha_t(k,\omega)\langle A(k,\omega)\rangle=\delta^2\int\frac{dq}{2\pi}\frac{g(|k-q|)}{\alpha_t(q,\omega)}\times\left[kq-\frac{\omega^2}{c_t^2}\right]^2\langle A(k,\omega)\rangle-\tfrac{1}{2}\delta^2\alpha_t^3(k,\omega)\langle A(k,\omega)\rangle\ , \quad (3.10)$$

where we have used Eq. (1.5). The dispersion relation for shear horizontal surface acoustic waves on a random grating is therefore

$$\alpha_t(k,\omega)=\delta^2\int\frac{dq}{2\pi}\frac{g(|k-q|)}{\alpha_t(q,\omega)}\left[kq-\frac{\omega^2}{c_t^2}\right]^2-\tfrac{1}{2}\delta^2\alpha_t^3(k,\omega)\ . \quad (3.11)$$

However, since $\alpha_t(k,\omega)$ is itself of $O(\delta^2)$ we can drop the second term on the right-hand side of this equation as ultimately of $O(\delta^8)$, to obtain the dispersion relation finally in the form[6]

$$\alpha_t(k,\omega)=\delta^2\int\frac{dq}{2\pi}\frac{g(|k-q|)}{\alpha_t(q,\omega)}\left[kq-\frac{\omega^2}{c_t^2}\right]^2\ . \quad (3.12)$$

Equation (3.12) is the dispersion relation we would have obtained if we had dropped the second term on the right-hand side of Eq. (3.8) from the outset.

When $\delta=0$ (a planar surface), this equation reduces to $\alpha_t(k,\omega)=0$. The solution of this equation, $\omega=c_t k$, is not the frequency of a surface acoustic wave, but instead is the frequency of a surface-skimming bulk transverse acoustic wave. As we will now see, the effect of surface roughness on this wave will be to convert it into a surface acoustic wave.

We will solve Eq. (3.12) for ω as a function of k, by setting

$$\frac{\omega^2}{c_t^2}=k^2-\delta^4\Delta^2(k) , \tag{3.13}$$

from which it follows that

$$\omega_{\mathrm{SH}}(k)=c_t k\left[1-\delta^4\frac{\Delta^2(k)}{2k^2}\right] , \tag{3.14}$$

to lowest nonzero order in δ. The subscript SH here denotes shear horizontal. When Eq. (3.13) is substituted into Eq. (3.12) the equation for $\Delta(k)$ that results can be written as

$$\Delta(k)=\Delta_1(k)+i\Delta_2(k) , \tag{3.15}$$

where

$$\Delta_1(k)=\int_{-\infty}^{-k}dq\frac{g(|k-q|)}{2\pi(q^2-k^2)^{1/2}}(kq-k^2)^2 + \int_k^{\infty}dq\frac{g(|k-q|)}{2\pi(q^2-k^2)^{1/2}}(kq-k^2)^2 , \tag{3.16a}$$

$$\Delta_2(k)=\int_{-k}^{k}dq\frac{g(|k-q|)}{2\pi(k^2-q^2)^{1/2}}(kq-k^2)^2 , \tag{3.16b}$$

to lowest order in δ. If we make the changes of variables $k-q=2ku$ and $k-q=-2ku$ in the first and second integrals on the right-hand side of Eq. (3.16a), respectively, and the change of variable $k-q=2ku$ in the integral on the right-hand side of Eq. (3.16b), we obtain the simpler expressions

$$\Delta_1(k)=\frac{2k^4}{\pi}\left[\int_1^{\infty}du\frac{u^{3/2}g(2ku)}{(u-1)^{1/2}} + \int_0^{\infty}du\frac{u^{3/2}g(2ku)}{(u+1)^{1/2}}\right] , \tag{3.17a}$$

$$\Delta_2(k)=\frac{2k^4}{\pi}\int_0^1 du\frac{u^{3/2}g(2ku)}{(1-u)^{1/2}} . \tag{3.17b}$$

If we assume for $g(|Q|)$ the Gaussian form given by Eq. (1.9), we obtain finally that

$$\Delta_{1,2}(k)=\frac{2x^4}{\sqrt{\pi}a^3}d_{1,2}(x) , \tag{3.18}$$

where

$$d_1(x)=\int_1^{\infty}du\frac{u^{3/2}e^{-x^2u^2}}{(u-1)^{1/2}}+\int_0^{\infty}du\frac{u^{3/2}e^{-x^2u^2}}{(u+1)^{1/2}} =2\int_0^{\infty}d\theta\cosh^4\theta\, e^{-x^2\cosh^4\theta} +2\int_0^{\infty}d\theta\sinh^4\theta\, e^{-x^2\sinh^4\theta} , \tag{3.19a}$$

$$d_2(x)=\int_0^1 du\frac{u^{3/2}e^{-x^2u^2}}{(1-u)^{1/2}} =2\int_0^{\pi/2}d\theta\sin^4\theta\, e^{-x^2\sin^4\theta} , \tag{3.19b}$$

and we have defined $ka=x$.

If we write $\omega_{\mathrm{SH}}(k)$ as

$$\omega_{\mathrm{SH}}(k)=\omega_{\mathrm{SH}}^{(1)}(k)-i\omega_{\mathrm{SH}}^{(2)}(k) , \tag{3.20}$$

we have

$$\omega_{\mathrm{SH}}^{(1)}(k)=c_t k\left[1-\frac{2}{\pi}\frac{\delta^4}{a^4}x^6[d_1^2(x)-d_2^2(x)]\right] , \tag{3.21a}$$

$$\omega_{\mathrm{SH}}^{(2)}(k)=\frac{4}{\pi}\frac{\delta^4}{a^4}\frac{c_t}{a}x^7 d_1(x)d_2(x) . \tag{3.21b}$$

The asymptotic forms of $d_1(x)$ and $d_2(x)$ for small x are readily obtained. We find that

$$d_1(x)\sim\frac{1}{x^2}-\tfrac{3}{4}\ln x+O(1) , \tag{3.22a}$$

$$d_2(x)\sim\frac{3\pi}{8}-\frac{35\pi}{128}x^2+O(x^4) . \tag{3.22b}$$

It follows that in this limit

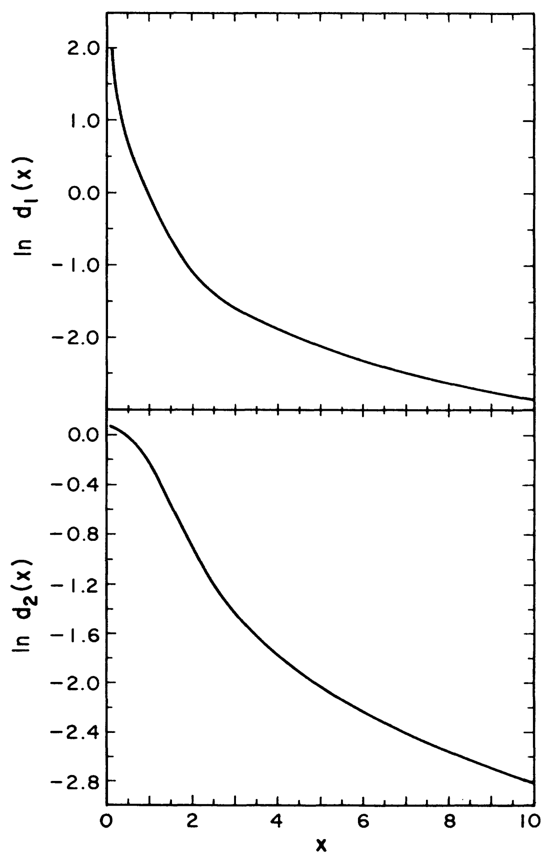

FIG. 4. The functions $d_1(x)$ and $d_2(x)$ defined by Eqs. (3.19) of the text.

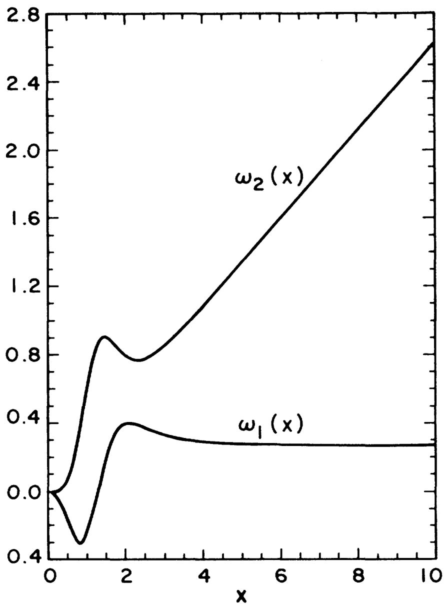

FIG. 5. The functions $\omega_1(x)$ and $\omega_2(x)$, defined by Eqs. (3.24) and (3.25) of the text, for shear horizontal surface acoustic waves on a random grating.

$$\omega_{\mathrm{SH}}^{(1)}(k) \sim c_t k \left[1 - \frac{2}{\pi} \frac{\delta^4}{a^4} x^2 \right] , \tag{3.23a}$$

$$\omega_{\mathrm{SH}}^{(2)}(k) \sim c_t k \left[\frac{3}{2} \frac{\delta^4}{a^4} x^4 \right] . \tag{3.23b}$$

In Fig. 4 we have plotted $d_1(x)$ and $d_2(x)$ as functions of x, while in Fig. 5 we have plotted the universal functions $\omega_1(x)$ and $\omega_2(x)$ defined by

$$\omega_{\mathrm{SH}}(k) = c_t k \left[1 + \frac{\delta^4}{a^4} \omega_1(x) - i \frac{\delta^4}{a^4} \omega_2(x) \right] , \tag{3.24}$$

so that

$$\omega_1(x) = -\frac{2}{\pi} x^6 [d_1^2(x) - d_2^2(x)] , \tag{3.25a}$$

$$\omega_2(x) = \frac{4}{\pi} x^6 d_1(x) d_2(x) . \tag{3.25b}$$

These results will be discussed in the following section.

IV. DISCUSSION

The most striking difference between the results obtained in this paper for the propagation of a Rayleigh wave across a random grating and the results of Refs. 1 and 3–5 for the propagation of a Rayleigh wave across a two-dimensional, randomly rough surface is in the roughness-induced attenuation rate of the Rayleigh wave. In contrast to an attenuation rate that is proportional to $(ka)^5$ in the limit of small ka obtained in the latter case, we find here an attenuation rate that is proportional to $(ka)^4$ in the same limit. This result can be seen from the following evidence. The contribution to $\mathrm{Im} d_0^{-1}(u)$ from the region $|u| > \xi$, given explicitly by Eq. (2.53), has delta-function contributions at $u = \pm 1$. The remainder of the integrand in the definition (2.44b) of $I(x)$ is real for these values of u. It follows that the contribution to $I_2(x) = \mathrm{Im} I(x)$ from this range of integration is nonzero and finite at $x = 0$. Similarly, the results of a numerical integration yield a nonzero, finite value for $I_2(x)$ as $x \to 0$ (Fig. 2). The factors of x, multiplying $I(x)$ in the definition of $\omega_2(x)$, Eq. (2.57), together with Eq. (2.55) yield the above-mentioned proportionality of the attenuation rate to $x^4 = (ka)^4$ in the limit as $k \to 0$.

The physical origin of the difference between the results obtained in the present work and that of Refs. 1 and 3–5 lies in the fact that the frequency dependence of Rayleigh scattering in d dimensions is ω^{d+1}. The ridges and grooves responsible for the scattering of a Rayleigh wave in the present case are two-dimensional, since they are defined by the equation $x_3 = \zeta(x_1)$. The protuberances and indentations responsible for the scattering in Refs. 1 and 3–5 are three dimensional, since they are defined by the equation $x_3 = \zeta(x_1, x_2)$. Thus the Rayleigh scattering law in the present case gives us an ω^3 frequency dependence of the scattering rate in the low-frequency, long-wavelength limit. The remaining factor of ω arises because the penetration depth of the Rayleigh wave into the solid is of the order of its wavelength parallel to the surface.[3]

A similar explanation can be given for the $(ka)^5$ dependence of the attenuation rate of shear horizontal surface acoustic waves on a random grating in the long-wavelength limit, Eq. (3.23b). It just has to be remembered that the penetration depth of this wave is proportional to the square of its wavelength parallel to the surface [$\alpha_t(k,\omega)$ is proportional to k^2 in this case; see Eqs. (2.5), (3.13), (3.18), and (3.22)].

Apart from the difference in the x dependence of the attenuation rates the results obtained in this work and in Ref. 5 for Rayleigh waves are quite similar. The function $\omega_1(x)$ is negative here, as it is in Ref. 5. This means that the Rayleigh wave on a random grating is slower than it is on a planar surface. The function $\omega_1(x)$ also displays structure in its dependence on x that is similar to that found in Ref. 5. The function $\omega_2(x)$ also has a dependence on x that is similar to that found in Ref. 5. It tends to zero (as x^3) when $x \to 0$ and also when $x \to \infty$, and has a maximum at an intermediate value of x that is close to the value of x (≈ 8) at which $\omega_2(x)$ has a maximum in Ref. 5. This behavior is accounted for by the fact that as $x \to 0$ the wavelength of the Rayleigh wave is much larger than the transverse correlation length a and it therefore sees a planar surface on which it is undamped. In the opposite limit of very short wavelengths the Rayleigh wave rides adiabatically over what it sees as a locally flat surface and is again undamped. Finally, as in Ref. 5, the dominant contribution to the attenuation rate of a Rayleigh wave on a random grating comes from the scattering into bulk elastic waves rather than into other Rayleigh waves (Fig. 3).

In connection with our results for shear horizontal surface acoustic waves on a random grating, we note that they coincide with those obtained by Bulgakov and Khankina,[6] except for a factor of 2 difference between the numerical coefficients in $\omega_{\rm SH}^{(2)}(k)$ in our Eq. (3.23) and their Eq. (15). We also note that, since in the absence of surface roughness a shear horizontal surface acoustic wave cannot exist[9] in the approximation maintained here, the attenuation of the roughness-induced shear horizontal surface acoustic waves studied here is due entirely to their scattering into bulk elastic waves. These waves also display the phenomenon of wave slowing $[\omega_1(x)<0]$ for small x, and indeed it is this wave slowing that binds them to the surface $[{\rm Re}\alpha_t(k,\omega)>0]$. It follows that for larger values of x ($x \gtrsim 1$), when $\omega_1(x)$ becomes positive, we no longer have a surface acoustic wave.

We also note that in order to obtain a dispersion relation that is correct to $O(\delta^2)$ in both the sagittal and shear horizontal polarizations, it suffices to expand the integrand in the integral $I(\alpha\,|\,Q)$ defined by Eq. (2.8) only to first order in $\zeta(x_1)$. In each case the term of $O(\zeta^2(x_1))$ which would seem at first glance to contribute to $O(\delta^2)$ has been shown in fact to contribute to the dispersion relation in a higher order in δ^2. A similar result had been obtained earlier for the propagation of Rayleigh waves across a two-dimensional randomly rough surface,[5] and for the propagation of surface polaritons across a classical grating.[10]

We also note that if we substitute the expansions

$$A_{l,t}(k,\omega)=2\pi\sum_{n=-\infty}^{\infty}A_n^{(l,t)}(q,\omega)\delta(k-q_n) \tag{4.1}$$

into Eqs. (2.4) and (2.11), and the expansion

$$A(k,\omega)=2\pi\sum_{n=-\infty}^{\infty}A_n(q,\omega)\delta(k-q_n) \tag{4.2}$$

into Eqs. (3.4) and (3.7), where $q_n\equiv q+(2\pi n/a)$, and represent $\zeta(x_1)$ in the form

$$\zeta(x_1)=\sum_{n=-\infty}^{\infty}\hat{\zeta}(n)e^{i2\pi n x_1/a}, \tag{4.3}$$

the resulting equations for the amplitudes $\{A_n^{(l,t)}(q,\omega)\}$ and $\{A_n(q,\omega)\}$ yield the displacement fields and dispersion curves for Rayleigh waves and for shear horizontal surface acoustic waves propagating across a classical grating of period a. The equations obtained, in fact, coincide with those obtained in Refs. 11 and 12, respectively.

Finally, it should be mentioned that in the case that $\zeta(x_1)$ is a deterministic, but nonperiodic function of x_1, that is sensibly zero for $|x_1|$ greater than some length R, Eqs. (2.11) and (3.7) could provide a convenient starting point for theoretical investigations of acoustic surface shape resonances. These are vibrational modes that are spatially localized in the vicinity of an isolated ridge or groove on an otherwise planar surface of a semi-infinite elastic medium. Although the electrostatic,[13] electromagnetic,[13,14] and magnetostatic[15] versions of these modes have been studied by now, their elastic counterparts are as yet untouched.

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APPENDIX

We record here the expressions for $d_0(u)$, $n_{\alpha\beta}^{(0)}(u)$, and $\hat{n}_{\alpha\beta}^{(0)}(u)$ that enter Eq. (2.43):

$$d_0(u)=4u^2(u^2-\lambda^2\xi^2)^{1/2}(u^2-\xi^2)^{1/2}-(2u^2-\xi^2)^2,$$

$$n_{ll}^{(0)}(u)=2(1-\xi^2)^{1/2}[2u^2+\xi^2(1-2\lambda^2-u)]-(2-\xi^2)(2u-\xi^2)(u^2-\lambda^2\xi^2)^{1/2},$$

$$n_{lt}^{(0)}(u)=2(1-\xi^2)^{1/2}(2u-\xi^2)(u^2-\xi^2)^{1/2}-(2-\xi^2)[2u^2-\xi^2(1+u)],$$

$$n_{tl}^{(0)}(u)=-(2-\xi^2)[(1-2\lambda^2)\xi^2-\xi^2u+2u^2]+2(1-\lambda^2\xi^2)^{1/2}(2u-\xi^2)(u^2-\lambda^2\xi^2)^{1/2},$$

$$n_{tt}^{(0)}(u)=-(2-\xi^2)(2u-\xi^2)(u^2-\xi^2)^{1/2}+2(1-\lambda^2\xi^2)^{1/2}[2u^2-\xi^2(1+u)],$$

$$\hat{n}_{ll}^{(0)}(u)=2u(u^2-\xi^2)^{1/2}\{u[2+(1-2\lambda^2)\xi^2]-\xi^2\}-(2u^2-\xi^2)(2u-\xi^2)(1-\lambda^2\xi^2)^{1/2},$$

$$\hat{n}_{lt}^{(0)}(u)=2u(u^2-\xi^2)^{1/2}(2u-\xi^2)(1-\xi^2)^{1/2}-(2u^2-\xi^2)[2u-\xi^2(1+u)],$$

$$\hat{n}_{tl}^{(0)}(u)=-(2u^2-\xi^2)\{u[2+\xi^2(1-2\lambda^2)]-\xi^2\}+2u(u^2-\lambda^2\xi^2)^{1/2}(2u-\xi^2)(1-\lambda^2\xi^2)^{1/2},$$

$$\hat{n}_{tt}^{(0)}(u)=-(2u^2-\xi^2)(2u-\xi^2)(1-\xi^2)^{1/2}+2u(u^2-\lambda^2\xi^2)^{1/2}[2u-\xi^2(1+u)].$$

[1]E. I. Urazakov and L. A. Fal'kovskii, Zh. Eksp. Teor. Fiz. **63**, 2297 (1972) [Sov. Phys.—JETP **36**, 1214 (1973)].

[2]Lord Rayleigh, Philos. Mag. **14**, 70 (1907); *Theory of Sound*, 2nd ed. (Dover, New York, 1945), Vol. II, p. 89.

[3]A. A. Maradudin and D. L. Mills, Ann. Phys. (N.Y.) **100**, 262 (1976).

[4]A. G. Eguiluz and A. A. Maradudin, Phys. Rev. B **28**, 711 (1983).

[5]A. G. Eguiluz and A. A. Maradudin, Phys. Rev. B **28**, 728 (1983).

[6]A. A. Bulgakov and S. I. Khankina, Solid State Commun. **44**, 55 (1982).
[7]O. Hardouin Duparc and A. A. Maradudin, J. Electron Spectrosc. Relat. Phenom. **30**, 145 (1983).
[8]M. de Billy (private communication).
[9]See, for example, A. A. Maradudin, in *Nonequilibrium Phonon Dynamics,* edited by W. E. Bron (Plenum, New York, 1985), p. 406.
[10]A. A. Maradudin, in *Surface Polaritons,* edited by V. M. Agranovich and D. L. Mills (North-Holland, Amsterdam, 1982), p. 432.
[11]N. E. Glass, R. Loudon, and A. A. Maradudin, Phys. Rev. B **24**, 6843 (1981).
[12]N. E. Glass and A. A. Maradudin, Electron. Lett. **17**, 773 (1981).
[13]See A. A. Maradudin and W. M. Visscher, Z. Phys. B **60**, 215 (1985), and references contained therein.
[14]A. A. Maradudin, in *Electromagnetic Surface Excitations,* edited by R. F. Wallis and G. I. Stegeman (Springer-Verlag, Berlin, 1986), p. 57.
[15]A. A. Maradudin, in *Surface Waves in Plasmas and Solids,* edited by S. Vukovic (World Scientific, Singapore, 1986), p. 267.