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Noise rise in parametric amplifiers

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Using methods from dynamical-systems theory, we reexamine the anomalous noise rise that has plagued Josephson-junction parametric amplifiers, devices potentially important for astrophysical measurements in the microwave regime. Our theory explains the puzzling gain-dependent noise temperature, and leads to new predictions which can be readily tested by experiments. Our explanation is based on a general theory of noise sensitivity near *simple* bifurcations: Thus, the same phenomenon should occur in other parametric amplifiers, e.g., those employing semiconductor lasers.

This Brief Report takes a new look at an old problem, namely, the observed "noise rise" in superconducting Josephson-junction parametric amplifiers. The importance of these devices lies in their potential use in observational radio astronomy; making amplifiers that operate at such high frequencies ($\sim 1-100$ Ghz) is a technologically difficult task. Although these devices have achieved appreciable gain (>15 dB),[1-6] they have been plagued by a noise problem[5-7] with a characteristic signature previously unseen in other kinds of parametric amplifiers. Typically, one expects the noise temperature T—proportional to the ratio of broad-band noise amplification G_N to signal amplification G_S—to be a constant for a given device, independent of parameter settings. Surprisingly, however, the Josephson devices show an increase of T with increasing G_S: That is, the greater the signal gain, the worse the signal-to-noise ratio.

Attempts to understand the noise rise have proven difficult,[8-14] and it remains an open problem. The purpose of this Brief Report is to discuss a mechanism that leads to such a noise rise. Our theory exploits recently developed insights linking parametric amplification to the properties of nonlinear dynamical systems near the onset of simple bifurcations.[15-18] The notion that the noise rise might be the result of deterministic chaos[12,13] is incompatible with these insights, since large G_S is not achieved in the chaotic regime. Our theory has more in common with the picture of (random) noise-induced hopping between coexisting attractors.[14] This theory generates a gain-dependent noise temperature like that observed in experiments, and leads to a number of new predictions that can be directly tested by future experiments.

Previous theoretical work devoted to Josephson-junction parametric amplifiers has proven very successful in understanding the noise-free performance of these devices.[1,19,20] Typically, this work proceeds from a direct analysis of the governing circuit equation, e.g.,

$$\ddot{\phi}+\beta\dot{\phi}+\sin\phi=A+B\cos(\omega_p t)+C\cos(\omega_s t+\theta) \ , \quad (1)$$

where ϕ is the phase difference between macroscopic wave functions across the junction, β^{-1} represents the Q of the circuit, and A,B,C are the current drive amplitudes at zero, pump, and signal frequencies, respectively. High gain can be achieved in two "modes," either the three-photon mode $\omega_s=\frac{1}{2}\omega_p+\Delta$, or the four-photon mode $\omega_s=\omega_p+\Delta$, with detuning $\Delta\ll\omega_p$.

The new idea behind our approach is to synthesize two related facts: (i) The high-gain limit of parametric amplifiers coincides with the onset of *bifurcations* of the dynamics of the system, and (ii) it is precisely near such instabilities that the effective dimension of the dynamics is drastically reduced, so that the behavior achieves a certain "universality."[16] Though not recognized as a general dynamical phenomenon, previous researchers showed that the high-gain limit of Eq. (1) corresponds to the onset of a period-doubling bifurcation in the three-photon mode,[3,20] and a cusp (or degenerate saddle-node) bifurcation in the four-photon mode.[3,7]

The key step of our analysis is the observation that the important dynamics is captured by a simple *reduced equation,*

$$\dot{x}=\mu x-x^3+\epsilon\cos(\delta t)+\xi(t) \ , \quad (2)$$

where x represents the system's output, μ varies in proportion to the external control parameter (e.g., $\mu\propto A-A^*$, where A^* is the bifurcation point in the absence of signal and noise), ϵ is proportional to the input signal C, δ is proportional to the detuning Δ, and ξ is a random noise term. The remainder of this report is broken into two parts: We first discuss the precise sense in which the scalar Eq. (2) describes the full phase-space dynamics, then we present the results of a study of this reduced equation, including specific predictions for future experiments.

We now discuss the origin of Eq. (2), beginning with the full dynamical system equation (1). To fix ideas, we focus on the three-photon mode: thus, the signal-free ($C=0$) system is near the onset of a period-doubling bifurcation. [The four-photon case requires a slight modification of Eq. (2), but much of the analysis presented here applies in that case also. Moreover, a similar analysis applies to the case of a symmetry-breaking bifurcation, which is expected to be an additional mode of operation of the unbiased Josephson parametric amplifier, as yet unobserved. The details for all cases will be presented elsewhere.[21]] Consider the geometry of the dynamics in the three-dimensional phase space $(\phi,\dot{\phi},t)$, Fig. 1. The signal-free periodic orbit $\mathbf{\Phi}_0$ is cut by a two-dimensional Poincaré section P. In the presence of the signal $(C>0)$—or any perturbation, for that matter—one finds that successive iterates are confined very nearly to a *one-dimensional curve*: This is the *center manifold.* Said another way, directions perpendicular to the center manifold are fast relaxing, and play an unimportant role in the asymptotic dynamics. *This is true regardless of the original dimension of phase space.*

Figure 1(b) plots a typical noise-free ($\xi=0$) time series near the period doubling, giving the x value of successive intersections of the phase-space trajectory with P, for $\mu<0$. Two features are especially of note: (i) Successive x_n oscillate between the positive and negative branches; these correspond to two branches of the *same* attractor, and (ii) each branch follows a smooth, slowly varying curve having the detuning frequency $\Delta=\omega_s-\omega_p/2$. Going back to the continuous-time trajectory $\boldsymbol{\phi}(t)$, it follows that we can make the decomposition

$$\boldsymbol{\phi}(t)=\mathbf{\Phi}_0(t)+x(t)\mathbf{\Phi}_1(t) \ , \tag{3}$$

where $\mathbf{\Phi}_0(t)$ has frequency ω_p and $\mathbf{\Phi}_1(t)$ has frequency $\omega_p/2$, such that $\mathbf{\Phi}_1(t+2\pi/\omega_p)=-\mathbf{\Phi}_1(t)$. The reduced scalar $x(t)$ evolves according to Eq. (2), and gives the magnitude of the displacement along the center manifold. In the absence of signal and noise, x relaxes to 0 for $\mu<0$, and to $\pm\mu^{1/2}$ for $\mu>0$. The latter corresponds to the appearance of a period-doubled frequency in the spectrum of the full variable $\boldsymbol{\phi}$. Figure 1(c) illustrates a typical time series one would obtain by measuring, say, one of the components of $\boldsymbol{\phi}$: The quantity $x(t)$ is then the slowly varying envelope function.

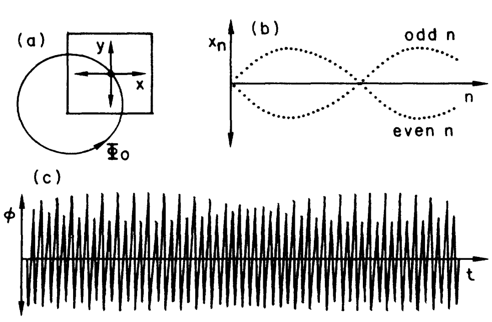

FIG. 1. Phase-space dynamics for Eq. (1), $\mu<0$. (a) Periodic orbit $\mathbf{\Phi}_0$ is cut by a two-dimensional Poincaré section. (b) Displacement along the center manifold of successive iterates x_n of the return map. (c) Corresponding continuous-time output.

The essential point is that Eq. (2) properly describes the reduced dynamics. In order to incorporate the effect of noise, we have studied Eq. (2) with the white-noise term ξ present. From Eq. (3), we can make the connection between $x(t)$ and the power spectrum for $\boldsymbol{\phi}(t)$.

An analysis of Eq. (2) yields several interesting results. An important result previously established for the noise-free ($\xi=0$) case is that the presence of the signal *shifts* the bifurcation point from the signal-free value $\mu=0$ to a new value greater than zero. This suppression of period doubling has been observed in a number of systems,[17,22,23] including numerical integration of Eq. (1).[23] In our present analysis we will focus on the opposite regime where the system is primarily noise driven and the signal is relatively small. Here the noise washes out the sharp transition to period doubling. Nonetheless, the dynamics may be separated into the three regimes: $\mu<0$, $0<\mu<\mu'$, and $\mu>\mu'$, where μ' is defined to be the point of maximum signal gain for a given detuning.

For $\mu<0$, both input signal and noise are amplified with very nearly the same dependence on the control parameter μ. Consequently, there is no gain-dependent noise rise. In fact, for sufficiently negative μ, this regime can be understood by studying linearized theories for signal and noise, and as pointed out early on by Feldman and Levinsen,[10] such linearized analyses cannot generate a noise rise.

The situation is different for $0<\mu<\mu'$: Here, nonlinear effects are crucial, as the system crosses over to a switching type of behavior. This switching (or hopping) occurs between $+\mu^{1/2}$ and $-\mu^{1/2}$, the two stable equilibria of the unperturbed system $\dot{x}=\mu x-x^3$. *Thus, the noise induces hopping between the two closely spaced branches of the orbit* $\boldsymbol{\phi}(t)$. The resulting Poincaré dynamics resembles a two-level telegraph process, a fact which can be exploited to yield analytic results.[21] The switching produces a Lorentzian noise bump in the spectrum centered at zero frequency (thus at $\omega_p/2$ for the spectrum of $\boldsymbol{\phi}$), which rapidly diminishes in width and increases in height as μ increases. *In this regime, both G_S and T increase, and one observes a noise rise.* The results of analog simulations of Eq. (2) are shown in Fig. 2. Power spectra reproduce the kind of noise rise seen in experiments on real junctions.[5-7] For comparison, Fig. 3 shows results of analog simulations on the full dynamical equation (1), also displaying a noise rise.

Finally, for $\mu>\mu'$, G_S now diminishes, while the noise temperature continues to rise; this regime has no practical interest.

Our reduced dynamical picture has been studied analytically for μ not too close to zero (i.e., away from criticality), both for negative μ (linearized analysis) and positive μ (switching analysis). From these calculations and from extensive simulations of Eq. (2), we can make a number of predictions:[21] (i) For $\mu<0$, T is essentially constant, even as G_S increases. (ii) For $\mu>0$ (and nonzero detuning), there exists a μ of maximum signal

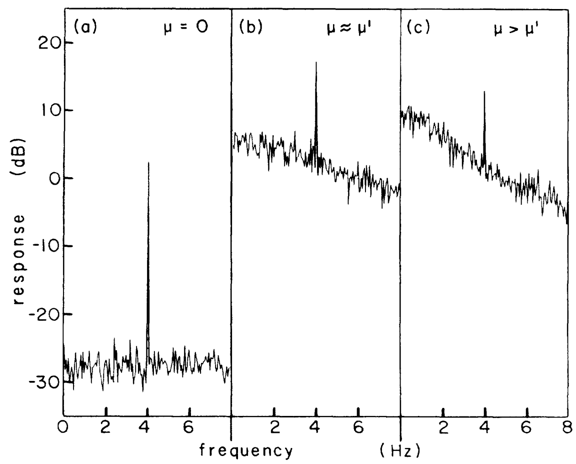

FIG. 2. Analog simulation of Eq. (2). (a) $\mu=0$. (b) $\mu=\mu'$, noise bump starting to move inside detuning frequency. Note diminished signal-to-noise ratio. (c) $\mu>\mu'$, signal gain falls off, noise rise continues.

data.[6] Note that the curves double back after passing μ' (maximum gain point for that δ).

All formulas quoted here follow from analytic considerations rather than fitting to simulation data; we expect these to be most accurate when μ is not too close to zero.

We point out two limitations of the theory: We assume that the system operates in the vicinity of a single bifurcation with no other instabilities or degeneracies nearby, and we assume that noise and signal are small enough that higher-order nonlinearities [not included in Eq. (2)] may be neglected.

This picture for the origin of the noise-rise in Josephson junction parametric amplifiers also leads us to a number of qualitative conclusions. First, since the noise-rise hinges on the smallness of the detuning Δ, it has been observed only in these devices because of their unusually high operating frequencies. It follows that other high-frequency parametric amplifiers should exhibit this phenomenon, e.g., modulated semiconductor laser systems. Second, we conclude (along with others[6,11,14]) that gain, μ', while the noise gain (at the signal frequency)

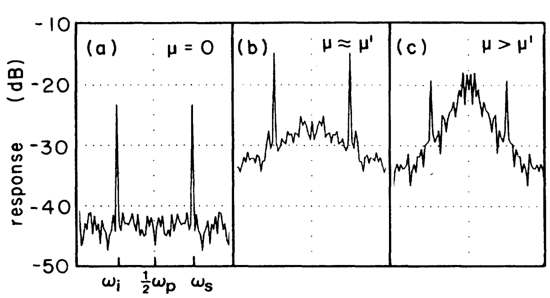

FIG. 3. Analog simulation of Eq. (1) with $\Delta=5\times10^{-4}\omega_p$. Idler frequency $\omega_i=\omega_p/2-\omega_s$. Effective detuning δ is greater than in Fig. 2 so there is less noise rise. $\omega_p/2$ here corresponds to zero frequency in Fig. 2.

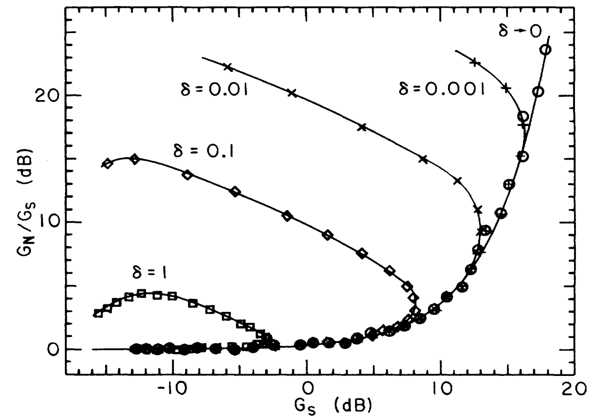

FIG. 4. Digital simulation of Eq. (2), showing ratio G_N/G_S vs signal gain G_S for several values of detuning δ. The parameter $\mu<0$ where the curves are flat and $\mu=\mu'$ where G_S is maximum.

also reaches a maximum which is beyond μ'. Moreover, μ' increases either with increasing noise input or decreasing detuning (see Fig. 4). (iii) For small detuning and negative μ we find $G_S \propto G_N \propto \mu^{-2}$, while for positive μ the behavior is $G_S \propto \ln G_N + \text{const} \propto \mu^2$ (these are power gains). Furthermore, G_S and G_N increase monotonically with μ including in the crossover region near $\mu=0$. (iv) The bandwidth w over which the gain indicated in (iii) may be achieved is proportional to $|\mu|$ for negative μ and to $\mu\exp(-k\mu^2)$ for positive μ, where k is a positive constant. Thus we cross over from a region of constant gain-bandwidth product $G^{1/2}w$ to one which rapidly diminishes and may explain the unexpectedly small values for this product previously reported for the Josephson devices.[5,6] (v) In Fig. 4 we show how G_N/G_S (proportional to the noise temperature) varies with G_S for several values of the detuning. These are not linearly proportional (as has been previously suggested[9,10]) although there is a region of near unity slope which may account for the experimental determinstic chaos is not the origin of the noise rise. Rather, the underlying dynamics is akin to the picture of noise-induced hopping between coexisting attractors,[14] but with a twist: The system actually hops between two closely spaced branches of the same attractor. Finally, our results hold independent of the physical details of the systems. Consequently, they apply equally well to recently proposed modifications of the basic design, e.g., based on low inductance SQUID's,[24] even though this changes the full governing equation (1) of the dynamical system. These dynamical considerations may well prove useful in optimizing the performance of such devices.

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