Tunneling and activated motion of a string across a potential barrier

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A string in a potential barrier provides a conventional model for dislocations in crystals, charge-density waves in Peierls dielectrics, and the order-parameter phase in linear Josephson junctions. The joint effects of quantum tunneling and thermal activation result in a finite lifetime of the metastable states of the string. With exponential accuracy the decay rates for typical weakly asymmetric potentials and a cubic potential are calculated in the full temperature range. To do this, a new approach is proposed to calculate the decay rate in the crossover regions from the tunneling regime to that of activated tunneling and from the latter to the regime of thermal activation. The preexponential factor is calculated in the full temperature range except for a low-temperature region, where it changes only by a factor close to unity. For two typical potentials and a cubic potential the preexponential factors are plotted as functions of temperature. In the high-temperature regime the Arrhenius law holds with a universal temperature dependence of the preexponential factor, $B \propto T^{-1/2}$.

I. INTRODUCTION

States of physical systems, separated from the neighboring states by high potential barriers, decay via thermal activation and quantum-mechanical tunneling and may have very long lifetimes. At a finite temperature there is no principal difference between a metastable state and the true stable state, and they can be considered on the same basis. Research on the decay rates of metastable states was initiated by Kramers,[1] who developed a theory of the absolute rate of chemical reactions based on the assumption that the thermal dissociation of a molecule is similar to the escape of a Brownian particle from a deep potential well. Kramers's results pertain, first, to the case of strong friction, when the decay rate decreases with friction due to slowing down of the particle motion, second, to the case of intermediate friction, when the particle states in the initial minimum are in thermal equilibrium and the decay rate does not depend on friction, and, third, to the case of extremely weak friction, when the particle distribution is depleted due to escapes of particles across the barrier and the decay rate is proportional to friction.

A great deal of work has been done on an extension of Kramers's results and their generalizations to quantum-mechanical systems and systems with many degrees of freedom. Below we cite briefly the major achievements and recent trends in the field.

For the regime of intermediate to strong friction the problem of the decay rate has been generalized in two directions. Langer considered the activated decays of classical systems with many degrees of freedom. [2] Caldeira and Leggett investigated the effects of friction on the tunneling decays at zero temperature.[3] Tunneling with strong friction at finite temperatures[4] and quantum effects in activated decays with intermediate to strong friction[5,6] have also been investigated.

In the regime of intermediate friction the friction strength does not enter the problem explicitly, the only role of dissipation is to establish thermal equilibrium between the system states in the initial potential well. At zero temperature only quantum-mechanical tunneling is possible. This mechanism for systems with macroscopical degrees of freedom has been considered by Lifshitz and Kagan.[7] The same procedure is responsible for the phase transition in a phonon system,[8] motion of a dislocation in a crystal[9] and charge-density waves in a Peierls dielectric (Rice *et al.*[10]), the decay of a metastable vacuum (Voloshin *et al.*[11]), and also the decay of metastable states of the Josephson junction.[4] For finite temperatures the decay rate is enhanced by the thermal activation.[7,9] At sufficiently high temperatures, the decay rate obeys the Arrhenius law.[1] The decay rates in this regime for systems with many degrees of freedom with allowance for quantum effects have been calculated by Langer.[12] The crossover from the thermal activation to the regime of activation enhanced by tunneling has been elucidated by Affleck.[13]

In the weak-friction regime the distribution of particles in the initial potential well becomes a nonequilibrium one due to the escape of particles across the barrier. The problem of the decay rate in this regime has been exactly solved in the classical[14] and quantum-mechanical[15] cases.

The present paper deals with the decay of a metastable state of a string lying in a deep potential minimum. This model appears to be relevant to dislocations in a crystal, charge-density waves in a Peierls dielectric, the decays of metastable states of the linear Josephson junction, and the problem of a metastable vacuum. We assume the initial distribution to be an equilibrium one, which means that interaction with a thermal bath is sufficiently strong to thermalize the string states, but has only negligible effect on the dynamics of the string. Our aim is to investigate, in detail, the temperature dependence of the penetration coefficient of a string through a potential barrier. We assume the problem to be a semiclassical one and make use of the formalism developed by Langer[12] and Callan and Colemen.[16] Then the solution of the problem proceeds in the following two stages. First, one looks for the extremal trajectory and the classical action, the result being an exponential approximation for the decay rate. Second, a linearized problem near the extremal trajectory is solved, which gives an expression for the preexponential factor.

Therefore, the present paper consists of two main parts. In the first part of the paper the classical action is calculated. To this end one has to find the extremal trajectory for the string motion across the barrier. As the string represents a continual object, its trajectory is governed by a nonlinear partial differential equation subject to certain boundary conditions. This problem is too complicated to be solved in a general case. However, it simplifies greatly for potentials with nearly degenerate minima, when the extremal trajectory describes a long piece of the string going across the barrier into the lower-lying minimum of the potential. In this situation the tunneling of the string has a close similarity to the problem of nucleation at a first-order phase transition, when the small supercooling is caused by a small difference of the specific free energies of the two phases. Then the size of the critical nucleus is large compared to the thickness of its walls, and a thin-wall approximation is justified. Hence, we may take the inner structure of the nucleus wall to be fixed and describe the nucleus only by a spatial configuration of its boundary. A systematic development of this approach enables one to incorporate into the general scheme the effect of interaction between the neighboring boundaries and calculate the classical action in the whole temperature range. The semiclassical approach is also applied for the calculation of an enhancement of the decay rate under effect of a high-frequency field. The result depends substantially on the inner structure of the nucleus wall.

In the second part of the present paper, the preexponential factor is calculated and a complete expression for the decay rate is obtained. The linearized problem remains rather complicated even in the limit of small supercooling, and its solution can be obtained only for tunneling at zero temperature[17-19] and in a high-temperature regime, when the dependence on the string coordinate and imaginary time can be factorized. Fortunately, the values of the preexponential factor at the two edges of the unattainable interval differ only by a numerical factor $2(3^{1/2})/\pi=1.103$. We would conjecture, therefore, that, at low temperatures, the preexponential factor is practically constant. Then our results will provide a full description of a rather nontrivial temperature dependence of the preexponential factor. A complete solution for the decay rate at sufficiently high temperatures is also obtained near the lability point, where the depth of the potential well is small and the shape of the barrier can be approximated by a cubic potential.

II. EXPONENTIAL APPROXIMATION

Consider a string lying in a deep potential barrier. Weak interaction with a thermal bath maintains an equilibrium distribution of the string states, proportional to the Gibbs factor $\exp(-\varepsilon/\tau)$, where ε is the energy and T is the temperature. At large temperatures the string would shift to an adjacent minimum by activated motion of certain segment of the string across the top of the barrier. In a more general case, the quantum-mechanical tunneling through the classically forbidden region contributes substantially to the decays of the metastable states. At a given energy the tunneling penetration coefficient is $\exp[-S(\varepsilon)]$, $S(\varepsilon)$ being the action for the sub-barrier motion. Combining the Gibbs factor and the tunnel factor, one obtains the decay rate at the given energy

$$D(\varepsilon)\propto\exp[-\varepsilon/T-S(\varepsilon)]\ .$$

This expression has a sharp maximum at an energy, determined by the equation

$$\frac{dS}{d\varepsilon}=-\frac{1}{T}\ .$$

Interpretation of this result is quite simple: at the temperature T the imaginary time interval for the subbarrier motion of the string equals $1/T$.

Therefore, in an exponential approximation one can write the decay rate as

$$D=\exp\left[-\int_{-1/2T}^{1/2T}L[dy/d\tau,y]d\tau\right]$$

with the string Lagrangian given by

$$L=\int dx\left[\frac{1}{2}\rho\left[\frac{\partial y}{\partial\tau}\right]^2+\frac{1}{2}\kappa\left[\frac{\partial y}{\partial x}\right]^2+U_0U\left[\frac{y}{a}\right]-FU_0\frac{y}{a}-U_0E\right]\ .$$

Here $U(z)$ is a symmetric function of the order of a unity with minima at $z=\pm1$. It is illustrated in Fig. 1 by the dashed line. The solid line shows the potential

$$V(z)=U(z)-Fz-E\ .$$

The constant shift of energy E provides a zero value

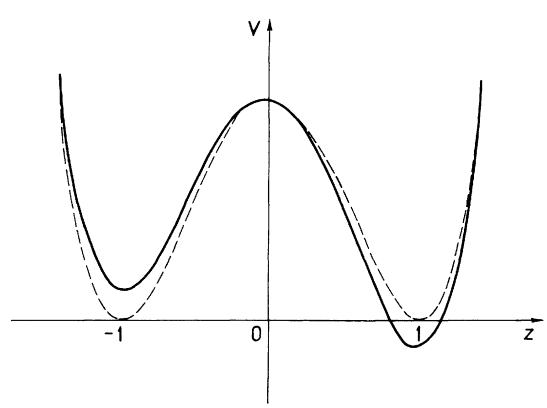

FIG. 1. Metastable potential $V(z)$. The dashed line shows the unbiased potential $U(z)$.

of the potential $V(z)$ at the bottom of the initial (left-hand side) well. The traditionally considered potentials are

$$U(z)=\tfrac{1}{2}(1-z^2)^2$$

and

$$U(z)=\tfrac{1}{2}\cos^2(\pi z/2) .$$

The quantity U_0 determines the scale of the height of the potential barrier; the bias F lifts the degeneracy of the potential minima. The solution of the classical equation of motion in the imaginary time $y(x,\tau)$ is specified by the condition, that $\tau=\pm 1/2T$ are the turning points of the string trajectory. In what follows we shall measure x, τ, and T in units

$$\begin{aligned} x_0&=a(\kappa/U_0)^{1/2} ,\\ \tau_0&=a(\rho/U_0)^{1/2} ,\\ 1/\tau_0&=a^{-1}(U_0/\rho)^{1/2} , \end{aligned} \tag{1}$$

but retain their former notations. Then the decay rate is

$$D=\exp(-A) ,$$

where

$$A=g\int_{-1/2T}^{1/2T}d\tau\int dx\left[\frac{1}{2}\left[\frac{\partial z}{\partial t}\right]^2+\frac{1}{2}\left[\frac{\partial z}{\partial x}\right]^2 +U(z)-Fz-E\right] , \tag{2}$$

and we introduce $z=y/a$ and a semiclassical parameter $g=a^2(\kappa\rho)^{1/2}\gg 1$.

By the order of magnitude the semiclassical action A (in the exponent of D) is a ratio V/ω, where $\omega\sim 1/\tau_0$ and V is proportional to the energy of a kink, $V\sim a(\kappa U_0)^{1/2}$. Variation of the action (2) gives an equation for the classical trajectory $z(x,\tau)$,

$$\frac{\partial^2 z}{\partial\tau^2}+\frac{\partial^2 z}{\partial x^2}-U'(z)+F=0 . \tag{3}$$

The trajectory $z(x,\tau)$ should satisfy the periodic boundary condition

$$z(x,1/2T)=z(x,-1/2T) ,$$

which is equivalent to

$$\left|\frac{\partial z(x,\tau)}{\partial\tau}\right|_{\tau=0,1/2T}=0 . \tag{4}$$

Thus, one encounters a quite complicated problem of solution of a nonlinear partial differential equation in a restricted region. A real progress is only possible for the situation with nearly degenerate minima, in other words, for $F\ll 1$. A systematic expansion in the small parameter F enables one to determine the temperature dependence of A in the whole temperature range. It is worth mentioning that in this way some of the detailed features of the solution structure in the $x\tau$ plane can be clarified.

The organization of the first half of the paper is as follows. We start with a solution at zero temperature, which is circularly symmetric in the $x\tau$ plane. In this case $z=1$ in a circle of a large radius $R\sim F^{-1}$, $z=-1$ in the outer region, and the thickness of the transition region is of the order of unity. This solution describes a circular nucleus with a thin wall. At $T>T_0\sim F$ the circular nucleus hits the boundary of the stripe of the width $1/T$ in the $x\tau$ plane, and at $T>T_0$ the nucleus is bounded by two arcs of the initial circle, which intersect at the points $\tau=1/2T$ and $\tau=-1/2T$. The action A is now a universal function of the ratio T/F, and penetration through the barrier proceeds via the activated tunneling. The curvature of the nucleus boundaries is constant, because in the thin-wall approximation we neglect an interaction between the boundaries. We have derived a comparatively simple equation, which relates the curvature of the boundary to the distance between the neighboring boundaries. This approach is used to investigate in detail the crossover at a temperature T_0 from the pure tunneling regime to that of the activated tunneling. The high-temperature region $T\sim F^{1/2}\gg T_0$ is also investigated. It is shown, that at $T>T_c\sim F^{1/2}$ the string surmounts the barrier due to the classical activation processes. Within the exponential approximation the enhancement of the decay rate under the effect of a high-frequency field is calculated.

A. Tunneling at a zero temperature

At $T=0$ the solution of Eq. (3) is given by a circularly symmetric function, which satisfies trivially the boundary condition (4). In polar coordinates, Eq. (3) is written as

$$\frac{d^2z}{dr^2}-U'(z)=-\frac{1}{r}\frac{dz}{dr}-F,\quad r^2=x^2+\tau^2 . \tag{5}$$

This situation is very similar to the problem of the critical nucleus at a first-order phase transition. The condition $F\ll 1$ corresponds to a small difference of the specific free energies of two phases. Hence, the function $z(r)$ must describe a large nucleus with thin walls. Then the problem can be solved stepwise: first, we find the inner structure of the wall, and then we calculate the radius R of the nucleus. Neglecting the right-hand side of (5), we obtain an equation

$$d^2z_0/dn^2-U'(z_0)=0,\quad n=r-R \tag{6}$$

which describes the structure of the nucleus wall. The right-hand side of (5) will be used below to calculate the radius R. To be specific, we take the line $z(x,\tau)=0$ as a boundary of the nucleus. The wall structure is determined by the first integral of (6),

$$(dz_0/dn)^2=2U(z_0) . \tag{7}$$

Dependence $z_0(\tau)$ is shown schematically in Fig. 2.

To find the nucleus radius we multiply Eq. (5) by dz_0/dr and integrate over r within the interval $(0,\infty)$. Then the contribution of the left-hand side of (5) vanishes, while on the right-hand side one can substitute R for r. The result is

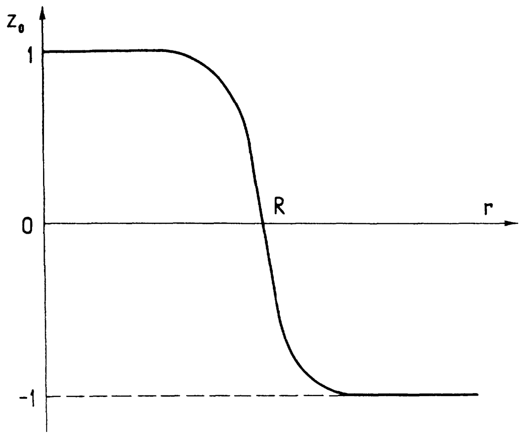

FIG. 2. Radial dependence of z_0 at zero temperature.

$$R=\alpha/F\ , \tag{8}$$

where α is a number of the order of unity,

$$\alpha=\tfrac{1}{2}\int_{-\infty}^{\infty} dx\,(dz_0/dx)^2=\int_0^1 dz\,[2U(z)]^{1/2}\ . \tag{9}$$

Within the same approximation the action (2) is given by[11,16,17]

$$A=2\pi\alpha^2 g/F\ . \tag{10}$$

In contrast to the tunneling of a particle, the barrier penetration by a string is a direct consequence of the fact that the final minimum of the potential is shifted down with respect to the initial one (see Fig. 1). The point is that crossing of the barrier creates two kinks with an energy of the order of unity, which must be compensated by an energy gain $RF\sim 1$ from the replacement of a piece of the string of the length $2R$ into the lower minimum.

B. Activated tunneling regime

It has been shown that in the limit of a low temperature the nucleus is represented by a circle with a radius R. The boundary condition (4) does not affect the nucleus shape as long as the boundary line $\tau=1/2T$ is far from the circle. At a temperature

$$T_0=F/2\alpha\ , \tag{11}$$

this line hits the boundary of the circular nucleus and at higher temperatures the circular solution is incorrect. In this situation the string surmounts the barrier due to both tunneling and activation mechanisms.

To calculate $A(T,F)$ we shall exploit below a thin-wall approximation. The profile of z along a normal to the boundary is determined by an implicit relation

$$n(z)=\int_0^z dy\,[2U(y)]^{-1/2}\ ,$$

which follows from (7). Performing integration over n in the action (2) we arrive to a functional

$$A=2g(\alpha l-FS)\ , \tag{12}$$

which depends only on the circumference l and area S of the nucleus (see Fig. 3). If the nucleus boundary is determined by a function $x=x(\tau)$, the action can be rewritten as[11,17,18]

$$A=4g\int_{-1/2T}^{1/2T} d\tau\{\alpha[1+(dx/d\tau)^2]^{1/2}-Fx\}\ . \tag{13}$$

The condition of extremality of the action (13) shows, that, locally, the nucleus boundary is still an arc of the initial circle of the radius $R=\alpha/F$. This arc is normal to the x axis at $\tau=0$, whereas its full height equals $1/T$ (see Fig. 3). These conditions determine the arc length. The extremal value of the action (13) is achieved at the minimal area of the nucleus, when two symmetric arcs intersect at $\tau=\pm 1/2T$. This gives[20]

$$A(T,F)=\frac{4g\alpha^2}{F}\left\{\sin^{-1}\left[\frac{F}{2\alpha T}\right]+\frac{F}{2\alpha T}\left[1-\left[\frac{F}{2\alpha T}\right]^2\right]^{1/2}\right\}. \tag{14}$$

The two arcs, representing the nucleus boundary, may be thought of as trajectories of a kink and an antikink, formed initially by thermal processes at $\tau=1/2T$ but then separated by quantum tunneling. This interpretation is motivated by the fact that the curvature of the nucleus boundary and its internal structure do not depend on temperature.

In the limiting cases we get

$$A(T,F)=\begin{cases}\dfrac{2\pi\alpha^2 g}{F}\left[1-\dfrac{8(2^{1/2})}{3\pi}\left[1-\dfrac{F}{2\alpha T}\right]^{3/2}\right], & \left[1-\dfrac{F}{2\alpha T}\right]\ll 1 \quad (15)\\[2ex] \dfrac{4\alpha g}{T}-\dfrac{gF^2}{6\alpha T^3}, & F\ll T \quad (16)\end{cases}$$

where the first expression gives a small correction to the tunneling probability from the activated tunneling processes. On the other hand, the leading term of the second expression corresponds to activation, while a small correction comes from the tunneling processes.

The relations (10) and (14) cover the whole temperature range, their matching point being $T_0=F/2\alpha$. The thin-wall approximation implies the small curvature of the nucleus boundary, hence it becomes inadequate near the kink of the boundary (see Fig. 3). The neighborhood

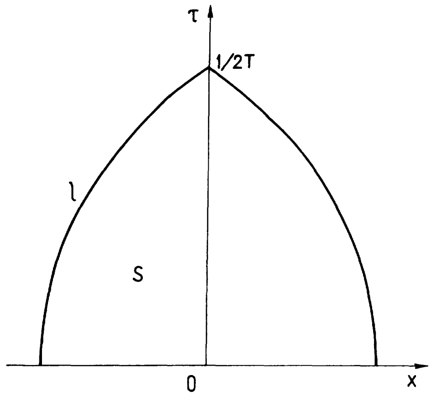

FIG. 3. The nucleus boundary $x(\tau)$, defined by the equation $z(x,\tau)=0$, in the regime of activated tunneling.

of the kink brings a comparatively small contribution into the action, and Eqs. (10) and (14) are correct at a certain distance from $T=T_0$. The calculation of the action in a close neighborhood of this point requires a more sophisticated treatment of the problem of the nucleus shape. The anticipated result is a formula for the action $A(T,F)$, which matches smoothly onto Eqs. (10) and (15) away from $T=T_0$. A similar approach may also be used in the high temperature regime, when the nucleus walls get so close that their interaction becomes substantial and the tunneling correction differs from that given by (16).

C. Interaction of the nucleus walls

In this section we develop a systematic approach to the problem of the nucleus shape, assuming that supercooling of the initial state of the string is small, $F \ll 1$. In this case the nucleus wall is thin compared to an overall nucleus size, and the nucleus is described by the position of its boundary, defined as a solution of the equation $z(x,\tau)=0$. Then our problem is divided into two: (1) calculation of the inner structure of the wall in a region close to the boundary; (2) derivation and solution of an equation for $x(\tau)$, the position of the boundary.

The nucleus boundary may be specified by the radius of its curvature $R(\phi)$, with ϕ being an angle between the normal to the boundary and the τ axis. This angle parametrizes the position of a point on the boundary; displacement normal to the boundary will be denoted by n. After the transformation of Eq. (3) to coordinates (n,ϕ), one obtains

$$\frac{\partial^2 z}{\partial n^2}+\frac{1}{R+n}\frac{\partial z}{\partial n}+\frac{1}{(R+n)^2}\frac{\partial^2 z}{\partial \phi^2}-\frac{1}{(R+n)^3}\frac{dR}{d\phi}\frac{\partial z}{\partial \phi}-U'(z)+F=0 . \tag{17}$$

This equation contains small parameters F and $1/R$. Neglecting these parameters, the solution of Eq. (17) is given by a quadrature of Eq. (7). In the next approximation $z=z_0+z_1$, and a small correction z_1, linear in F and $1/R$, is governed by an equation

$$\frac{\partial^2 z_1}{\partial n^2}-U''(z_0)z_1=-R^{-1}\frac{\partial z_0}{\partial n}-F . \tag{18}$$

We are looking for a solution of (18), which is finite inside the nucleus and grows exponentially in the outer region. Taking $z=-1+\psi$ and solving Eq. (18), at $n \gg 1$ one gets asymptotically

$$\psi=\frac{1}{\beta\gamma^2}\left[\frac{\alpha}{R(\phi)}-F\right]\exp(\gamma n)+\beta\exp(-\gamma n) , \tag{19}$$

where the second term comes from $z_0(n)$. The parameters α has been introduced above,

$$\beta=\exp\left[\int_0^1 dz\left[\frac{\gamma}{[2U(z)]^{1/2}}-\frac{1}{(1-z)}\right]\right] ,$$

$$\gamma=[U''(1)]^{1/2} .$$

With T going down to zero, the boundary $\tau=1/2T$, where condition (4) should be met, goes away to infinity and the nucleus radius is given by Eq. (8), which corresponds to a vanishing coefficient of the growing exponent. The function ψ satisfies a linear equation

$$\nabla^2\psi-\gamma^2\psi=0 ,$$

with boundary conditions (ψ and $\partial y/\partial n$ at the nucleus boundary, $n=0$) specified by R and F, as is clear from (19).

The boundary condition (4) will be satisfied trivially, if we continue $z(x,\tau)$ across the line $\tau=1/2T$ by a mirror reflection against this line, $z(x,\tau)=z(x,1/T-\tau)$. Then the shape of the nucleus is selected by the following condition: the growing and decaying terms of (19) near one boundary match the corresponding terms at the second boundary. A practically important case is that of nearly parallel boundaries. In particular, at high temperatures, $T \gg F$, the nucleus boundaries are nearly parallel to the T axis. Neglecting the corrections of the order of

$$x(\tau)(dx/d\tau)^2 \ll 1 , \tag{20}$$

one can substitute the projection of the normal on the x axis for the distance along the normal from the boundary to the τ axis. Then the matching of the growing and decaying exponents similar to (19) gives an equation for $x(\tau)$,

$$\frac{d^2x}{d^2\tau}+\frac{F}{\alpha}=\frac{\beta^2\gamma^2}{\alpha}\exp(-2\gamma x)\ , \tag{21}$$

where taking into account Eq. (20) we have substituted $-d^2x/d\tau^2$ for R^{-1}. The boundary conditions for Eq. (21) are

$$\left.\frac{dx}{d\tau}\right|_{\tau=0,1/2T}=0\ .$$

Equation (21) holds in the full interval of τ from zero to $1/2T$. With account of Eq. (20), we shall write the validity condition of (21) as an inequality

$$T \gg F[\ln(1/F)]^{1/2}\ . \tag{22}$$

For large distances between the boundaries, Eq. (21) reproduces the nucleus boundary of a constant curvature F/α. This equation also has a solution with parallel boundaries, separated by a distance

$$x_{\parallel}=\frac{1}{2\gamma}\ln\left[\frac{\beta^2\gamma^2}{F}\right]\ . \tag{23}$$

With increasing temperature, $x(0)$ increases and $x(1/2T)$ decreases, their convergence point being $x_{\parallel}$. At temperatures $T \ll [F/\ln(1/F)]^{1/2}$ one can neglect the right-hand side of Eq. (21), obtaining

$$x=\frac{F}{2\alpha}\left[\frac{1}{4T^2}-\tau^2\right],\quad x_{\parallel} \ll x \tag{24}$$

which is just an initial part of a circle. For sufficiently close boundaries, one can neglect the term F/α, then the required solution of (21) is

$$x=\frac{1}{\gamma}\ln\left\{\frac{2T\beta(\alpha\gamma)^{1/2}}{F}\times\cosh\left[\frac{\gamma F}{2\alpha T}\left[\frac{1}{2T}-\tau\right]\right]\right\},\quad x \ll x(0)\ . \tag{25}$$

Expressions (24) and (25) have a common interval of validity, as it follows from the inequality $x_{\parallel} \ll x(0)$. From (24) and (25) one obtains

$$x(0)=\frac{F}{8\alpha T^2}\ ,$$

$$x\left[\frac{1}{2T}\right]=\frac{1}{\gamma}\ln\left[\frac{2T\beta(\alpha\gamma)^{1/2}}{F}\right]\ .$$

It is evident from the given solutions, that the nucleus boundary has no kinks and is directed normally to the lines $\tau=0$ and $\tau=1/2T$. The boundary consists of two smooth curves [see Fig. 4(b)]. The distance between the curves, $2x(1/2T)$, is large compared to unity due to restriction (22).

Thus we have shown that the approach given above proves to be useful in the calculations of the nucleus shape, if the noninteracting boundaries of a nucleus converge at a sufficiently small angle.

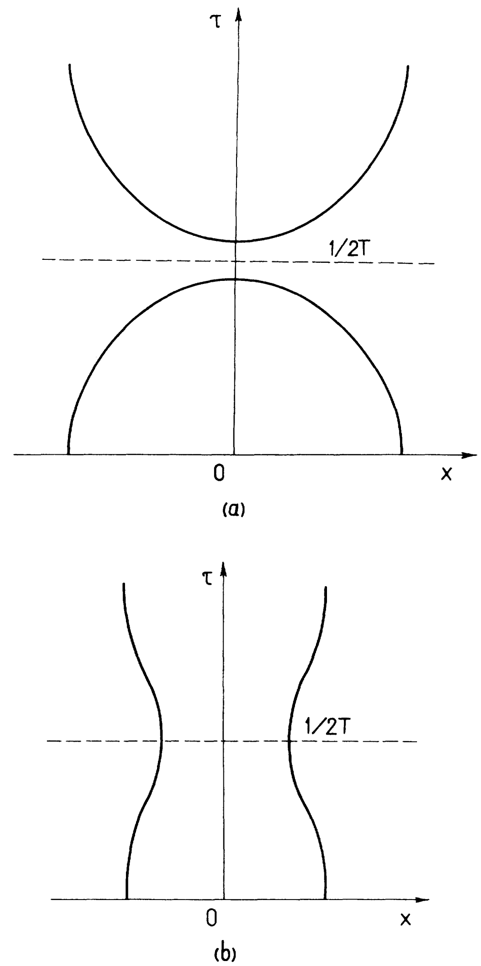

FIG. 4. Interacting nucleus boundaries: (a) at T close to T_0; (b) at T close to T_c.

In a special case of temperatures close to T_0, the noninteracting boundaries intersect at an angle close to π [see Fig. 4(a)]. Then the consideration given above is applicable again, and the equation for $x(\tau)$ is similar to (21):

$$\frac{d^2\tau}{dx^2}+\frac{F}{\alpha}=-\frac{\beta^2\gamma^2}{\alpha}\exp\left[2\gamma\left[\tau-\frac{1}{2T}\right]\right]\ . \tag{26}$$

This equation is correct near the top of the nucleus [see Fig. 4(a)], and its solution describes a curve $x(\tau)$, which goes over into an arc of a circle with τ far way from the point $\tau=1/2T$. Equation (26) is valid in an interval from zero temperature up to that slightly above T_0, when $0<T-T_0 \ll T_0$.

At still larger temperatures, the topology of the nucleus changes from the one shown in Fig. 4(a) to that shown in Fig. 4(b). In the major part of this temperature interval, the distance between the nucleus boundaries at the point $\tau=1/2T$ is of the order of unity and the approach developed above holds no longer. It will be shown below, that, at temperature $T_c \sim F^{1/2}$, the interaction between the nucleus boundaries turns out again to be weak. This region can be investigated with the use of an equation similar to (26).

D. Crossover from tunneling to activated tunneling at temperatures close to T_0

If interaction between the nucleus boundaries is neglected, the action $A(T,F)$ would be given by different functions at $T<T_0$ and at $T>T_0$ [see Eqs. (10) and (15)]. However, the singularity of $A(T,F)$ at $T=T_0$ is smeared out with boundary interactions taken into account.

The energy integral of Eq. (26) is

$$\frac{1}{2}\left[\frac{d\tau}{dx}\right]^2+\frac{\tau F}{\alpha}+\frac{\beta^2\gamma}{2\alpha}\exp\left[2\gamma\left[\tau-\frac{1}{2T}\right]\right]=1 . \tag{27}$$

At some distance from $\tau=1/2T$ Eq. (27) describes an unperturbed boundary. Contribution from the distorted part of a boundary into the action can be easily reconstructed by writing the Lagrangian, corresponding to the energy (27), in a form, that goes over into the integrand of (13) (with substitution of τ for x and vice versa) for τ sufficiently far from $1/2T$. The required Lagrangian is

$$L=4\alpha g\left\{\frac{1}{2}\left[\frac{d\tau}{dx}\right]^2-\frac{\tau F}{\alpha}-\frac{\beta^2\gamma}{2\alpha}\exp\left[2\gamma\left[\tau-\frac{1}{2T}\right]\right]+1\right\} .$$

Making use of Eq. (27), we obtain

$$A(T,F)=\frac{2\pi\alpha^2 g}{F}+\frac{4g}{\gamma}\left[\frac{\alpha F}{\gamma}\right]^{1/2}\chi\left[\frac{4\alpha^2\gamma}{F^2}[T-T_0(F)]\right] , \tag{28}$$

where $T_0(F)$ differs from T_0 by a small correction,

$$T_0(F)=\frac{F}{2\alpha}-\frac{F^2}{4\alpha^2\gamma}\ln\left[\frac{\beta^2\gamma^2}{F}\right] .$$

The function χ is defined by an integral

$$\chi(p)=\int_0^\infty dx\{[x-\exp(p-x)]^{1/2}\Theta(x-x_1)-x^{1/2}\} ,$$

$$p=x_1+\ln x_1$$

with asymptotics

$$\chi(p)=\begin{cases}-(\pi^{1/2}/2)\exp(p), & p\to-\infty\\ -\frac{2}{3}p^{3/2}, & p\to\infty .\end{cases}$$

Thus, the expression (28) matches Eq. (15) for temperatures above T_0 and Eq. (10) at low temperatures. The width of the crossover region is

$$|T-T_0(F)|\sim F^2 .$$

E. Crossover from activated tunneling to pure activation

Above we have followed the crossover from pure tunneling to the activated-tunneling regime and derived expressions (10), (15), and (28) for the action $A(T,F)$. As the temperature rises, the nucleus size keeps diminishing and a relative contribution from the boundary interaction increases in a full analogy to the region around T_0. Then the nucleus shape is described by Eqs. (24) and (25). The Lagrangian for the equation of motion (21) can be derived as previously, which results in the action

$$A=8\alpha g\int_0^{1/2T}d\tau\left[1+\frac{1}{2}\left[\frac{dx}{d\tau}\right]^2-\frac{Fx}{\alpha}-\frac{\beta^2\gamma}{2\alpha}\exp(-2\gamma x)+\frac{F}{2\alpha\gamma}\ln\left[\frac{e\beta^2\gamma^2}{F}\right]\right] .$$

The constant term in the Lagrangian provides a zero value of the minimal potential energy. The final result for the action is

$$A(T,F)=\frac{4\alpha g}{T}-\frac{4g}{\gamma}\left[\frac{\alpha F}{\gamma}\right]^{1/2}\Phi\left[\frac{T}{T_c}\right] , \tag{29}$$

where the function $\Phi(q)$ is determined by the implicit relations

$$\Phi(q)=\frac{\pi Q}{2^{1/2}q}-\int_{x_1}^{x_2}dx[Q-x-\exp(-x-1)]^{1/2} ,$$

$$\frac{\pi 2^{1/2}}{q}=\int_{x_1}^{x_2}dx[Q-x-\exp(-x-1)]^{-1/2} ,$$

and x_1 and x_2 are the roots of the equation

$$Q-x-\exp(-x-1)=0 .$$

The critical temperature is defined as

$$T_c=\frac{1}{\pi}\left[\frac{\gamma F}{2\alpha}\right]^{1/2} ,$$

Asymptotes of the function $\Phi(q)$ are given by

$$\Phi(q)=\begin{cases}\frac{1}{6(2^{1/2})}\left[\frac{\pi}{q}\right]^3, & q\ll 1\\ 3\pi 2^{1/2}(1-q)^2, & (1-q)\ll 1 .\end{cases}$$

The first asymptote matches Eq. (16) in an interval $T_0\ll T\ll T_c$. Close to T_c, when $T_c-T\ll T_c$, and for all temperatures above T_c we obtain

$$A(T,F)=\frac{4\alpha g}{T}-\frac{24\pi g}{\gamma}\left[\frac{\alpha F}{2\gamma}\right]^{1/2}\left[1-\frac{T}{T_c}\right]^2\Theta(T_c-T) ,$$

which shows, that at $T>T_c$ the string crosses the barrier via classical activation processes.

To summarize, the results of calculations in the exponential approximation are given by Eqs. (14), (28), and (29), which determine the action $A(T,F)$ in the whole temperature range.

F. Tunneling under effect of a high-frequency field

Consider a situation, when besides the static field F there is also a small alternating field $F_1\cos(\Omega t)$. The temperature will be taken to be zero. As above, we consider the semiclassical approximation and represent the decay rate in the form

$$D=\exp[-A+A_1\cos(\Omega t)] ,$$

where the correction A_1 to the action has to be small compared to A, but large compared to unity. To calculate A_1 we turn to the approach developed earlier,[21,22] which gives

$$A_1=\tfrac{1}{2}iF_1\int_C dt\int_{-\infty}^{\infty}dx\, z_0(x,t)\cos(\Omega t+\phi) , \tag{30}$$

where z_0 is a solution of Eq. (7) with $n=(x^2-\tau^2)^{1/2}-\alpha/F$. The contour C at zero temperature goes along the imaginary t axis and closes in a remote part of the left half-plane. The integral along the contour C is finite in spite of the divergence of the integral along the imaginary axis. A constant phase shift ϕ in (30) is to be chosen to provide a maximum value of A_1.

Consider the potential $U(z)=\frac{1}{2}(1-z^2)^2$, when

$$z_0(x,t)=-\tanh[(x^2-t^2)^{1/2}-2F/3] .$$

As a function of t this solution has poles at $t=t_k$, which at small x are given by

$$t_k=\pm\frac{2i}{3F}\left[1-\frac{9F^2x^2}{8}\right]-\frac{\pi}{2}(1+2k)\left[1+\frac{9F^2x^2}{8}\right] , \tag{31}$$

where $k=0,1,2,\ldots$. Integration over t in (30) can be performed by residues. In the limit $\Omega\gg F$, the integral is dominated by $x\sim(\Omega F)^{-1/2}$ which justifies the expansion (31). Thus, summation over k gives the final expression

$$A_1=\pi F_1\left[\frac{\pi}{3\Omega F}\right]^{1/2}\exp\left[\frac{2\Omega}{3F}\right]\left[\sin\left[\frac{\pi\Omega}{2}\right]\right]^{-1} . \tag{32}$$

Two features of this result are worth mentioning. The field F_1 is enhanced by a large factor $\exp(2\Omega/F)$, because in crossing the barrier the string spends, in the classically forbidden region, an imaginary time proportional to $1/F$, which greatly surpasses the period of the field oscillations $2\pi/\Omega$. The resonance denominator comes from the string motion in the classically allowed region, where the frequency of small oscillations in the adopted units equals 2.

The important point is that the action A_1 depends crucially on the inner structure of the nucleus wall, since the integral (30) is dominated by the singularities of the function $z_0(x,t)$ in the complex t plane. The exponential factor in (32) survives the reduction of the problem to a one-dimensional one, with the nucleus boundary $x(\tau)$ governed by the Lagrangian (13), but the resonance denominator of Eq. (32) cannot be reproduced by this simplified consideration. A thin-wall approach to solution of a nonstationary problem has been discussed by Maki.[18]

III. THE PREEXPONENTIAL FACTOR

The decay rate of the metastable state of a string has been calculated above in the exponential approximation. To complete the solution we have to calculate the preexponential factor B in the following equation for the decay rate:

$$\frac{D}{L}=B\exp(-A) ,$$

where L is the string length.

At a zero temperature, in the limit $F\ll 1$, the preexponential factor is already known.[18,19] In our notations

$$B=\frac{U_0F}{\pi} . \tag{33}$$

Computation of the preexponential factor B represents a more complicated problem than that of the imaginary action A, so we turn to a general approach developed by Langer[12] and Callan and Coleman.[16] These authors have shown that the preexponential factor can be expressed in terms of the spectrum of the linear excitations near the extremal solution. At a zero temperature this solution is circularly symmetric in the $x\tau$ plane and comparatively simple calculations result in (33). As long as $T<T_0$, the corrections to this result are small. A substantial change of the preexponential factor begins with the distortion of the extremal solution at temperatures above T_0. Since $T_0\sim F$ and $T_c\sim F^{1/2}$, one might expect, that B changes drastically in this interval of temperatures, similar to A, which changes by a factor $F^{1/2}$.

Unfortunately, the direct calculation of B in the interval (T_0,T_c) appears to be impossible due to a complicated shape of the extremal solution (see Fig. 3). However, this situation simplifies drastically at temperatures close to and above T_c, when the extremal trajectory is a function of a single variable, x. In this case the spectrum of excitations and the preexponential factor B can be calculated with the use of factorization of the dependence on the variables x and τ. The important point is that the value of B at T_c, derived in this way, differs from the value of B at a zero temperature only by a factor $2(3^{1/2})/\pi=1.103$, in contrast to what one might expect. Therefore, we conjecture that B is practically constant in the interval $(0,T_c)$. Then the results derived below give a complete solution for $B(T)$ and consequently for the decay rate in the whole range of temperatures.

A. A general expression for the preexponential factor

For temperature close and above T_c the decay rate in the conventional notations is given by[12]

$$D=2\pi\frac{\mathrm{Im}Z}{\mathrm{Re}Z} , \tag{34}$$

where Z is the partition function. We make use of the standard approach to calculate the functional integral for Z. To this end we represent the function $z(x,\tau)$ as a

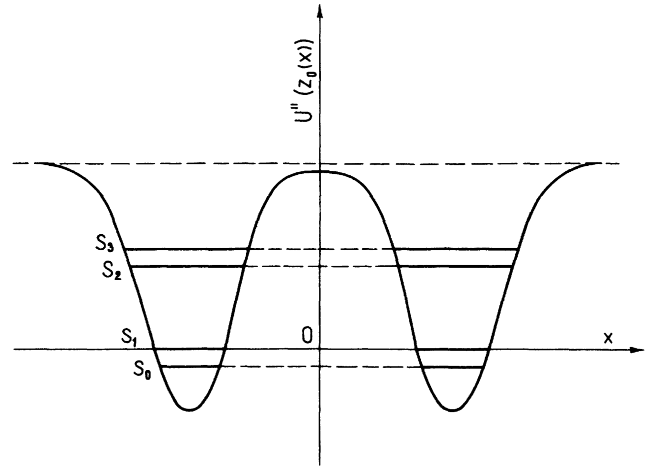

FIG. 5. Potential energy and the bound-state energy levels for the linearized problem.

sum of the saddle-point solution $z_0(x)$, which obeys the equation

$$d^2z_0/dx^2 - U'(z_0) + F = 0 \ ,$$

and a small correction to it, which can be written as an expansion in the periodic functions of τ and the normalized eigenfunctions $z_{mn}(x)$,

$$z(x,\tau) = z_0(x) + T^{1/2}\sum_{m,n} C_{mn} z_{mn}(x) \exp(2\pi i T\tau n) \ . \quad (35)$$

Then the quadratic contribution to the action takes the form

$$\delta A = \frac{g}{2}\sum_{m,n}\lambda_m(n) C_{mn}^2 \ ,$$

where $\lambda_m(n)$ are the eigenvalues of the linearized equation,

$$-d^2 z_{mn}/dx^2 + U''(z_0(x)) z_{mn} - [\lambda_m(n) - (2\pi T n)^2] z_{mn} \ . \quad (36)$$

The potential $U''(z_0(x))$ is pictured schematically in Fig. 5. It consists of the two potential wells with a profile not depending of F as long as $F \ll 1$. The distance between the wells is logarithmically large, as $\ln(1/F)$, and determined by (23). For the solitary well there always exists a solution with a zero energy, which corresponds to the shift mode, and, probably, several higher levels. The interaction between the wells does not shift the zero-energy level, but splits from it a level with a negative energy of the order of F. In a similar way the other levels are split. Denote the eigenvalues of the operator in the left-hand side of (36) by S_m, where $m = 0, 1, 2, \ldots$ (see Fig. 5), $S_0 < 0$, $S_1 = 0$, and $S_m > 0$ for $m \geq 2$.

For the eigenvalues of (36) we obtain

$$\lambda_0(n) = (2\pi)^2(n^2T^2 - T_c^2) \ ,$$

$$\lambda_1(n) = (2\pi T n)^2 \ ,$$

$$\lambda_m(n) = (2\pi T n)^2 + S_m, \quad m = 2, \ldots, M \ .$$

For the continuous spectrum

$$\lambda_q(n) = (2\pi T n)^2 + \gamma^2 + q^2 \ .$$

Thus, at the saddle point there is a negative eigenvalue $\lambda_0(0)$, the zero eigenvalue $\lambda_1(0)$ corresponds to the shift mode, and the others are positive.

For a sufficiently long string the decay probability per unit length of the string is a well-defined quantity. It is convenient to introduce, explicitly, integration over the string length,[4,12,16] writing the functional integral for ImZ in the form

$$\text{Im}Z = \text{Im}\int_{-L/2}^{L/2} dx' \int Dz \exp(-A\{z\})\delta(x' - x_0\{z\}) \ , \quad (37)$$

where L is the length of the string, $x_0\{z\}$ is a position of the minimum of the functional $\mathcal{F}(x' \mid z)$, where

$$\mathcal{F}(x' \mid z) = \int_{-L/2}^{L/2} dx \int_{-1/2T}^{1/2T} d\tau [z(x,\tau) - z_0(x - x')]^2 \ ;$$

$$\left.\frac{d\mathcal{F}}{dx'}\right|_{x'=x_0} = 0 \ .$$

Expanding in x' and making use of Eq. (35) and definition of α [see Eq. (9)], one obtains

$$\frac{d\mathcal{F}}{dx'} = \frac{4\alpha x'}{T} + \left[\frac{8\alpha}{T}\right]^{1/2} C_{10} \ .$$

By virtue of an identity

$$\delta(x' - x_0\{z\}) = \left|\frac{d^2\mathcal{F}}{dx'^2}\right| \delta\left[\frac{d\mathcal{F}}{dx'}\right]$$

one can integrate easily over C_{10}, since the argument of δ function in (37) is linear in C_{10}. Hence, the integral over x' gives the length of the string L. The functional integration in terms of the variables C_{mn} should be performed with the use of the definition

$$\text{Im}Z = L\left[\frac{\alpha}{\pi T}\right]^{1/2}\int_0^\infty \frac{dC_{00}}{(2\pi)^{1/2}} \exp\left[-\frac{g}{2}\,|\lambda_0(0)|\, C_{00}^2\right] \prod_{m,n}\int_{-\infty}^{\infty}\frac{dC_{mn}}{(2\pi)^{1/2}} \exp\left[-\frac{g}{2}\lambda_m(n) C_{mn}^2\right] \ ,$$

where the factors with indices (0,0) and (1,0) are omitted, and integration over C_{00} goes along a half-axis according to the rule of analytical continuation onto the negative values of $\lambda_0(0)$.

We have expressed ImZ in terms of the eigenvalues $\lambda_m(n)$. The functional integral for the ReZ is dominated by a neighborhood of the minimum of the action, where all the eigenvalues of the linearized problem are positive. Substituting the results for ImZ and ReZ into (34), we arrive at a final formula for the decay rate per unit length,

$$\frac{D}{L}=T_c U_0\left[\frac{\alpha}{\pi T g}\right]^{1/2}\left|\frac{\det\left|-\frac{\partial^2}{\partial\tau^2}-\frac{\partial^2}{\partial x^2}+\gamma^2\right|}{\det'\left|-\frac{\partial^2}{\partial r^2}-\frac{\partial^2}{\partial x^2}+U''(z_0(x))\right|}\right|^{1/2}\exp(-A)\ , \tag{38}$$

with det′ omits the zero eigenvalue.

This expression is applicable above but not too close to T_c, since the eigenvalue $\lambda_0(1)$ vanishes with T going to T_c. In Sec. III B we shall calculate the formal expression (38); the vicinity of T_c will be considered separately.

B. The decay rate at temperatures above T_c

To calculate the ratio of the determinants in (38), we write the eigenvalues of the operator in the nominator of (38) as

$$\mu_k(n)=(2\pi T n)^2+\gamma^2+k^2\ .$$

Then the products over n are calculated easily, and the result is

$$\frac{D}{L}=U_0\left[\frac{\alpha T}{2\pi g}\right]^{1/2}\frac{T_c}{\sin(\pi T_c/T)}\exp\left[-\frac{4\alpha g}{T}+\frac{b}{T}\right]$$
$$\times\prod_{m=2}^{M}\left[1-\exp\left[-\frac{S_m^{1/2}}{T}\right]\right]^{-1}\prod_k\left[1-\exp\left[-\frac{(\gamma^2+k^2)^{1/2}}{T}\right]\right]\prod_q\left[1-\exp\left[-\frac{(\gamma^2+q^2)^{1/2}}{T}\right]\right]^{-1},$$

where b is defined by

$$b=-\tfrac{1}{2}\sum_{m=2}^{M}S_m^{1/2}+\tfrac{1}{2}\sum_N[(\gamma^2+k_N^2)]^{1/2}-[(\gamma^2+q_N^2)^{1/2}]\ .$$

The discrete spectra of k_N and q_N are determined by the boundary conditions at the ends of the string. Defining the density of states as $\rho(k)=dN/dk$, one can write

$$\prod_k\left[1-\exp\left[-\frac{(\gamma^2+k^2)^{1/2}}{T}\right]\right]\prod_q\left[1-\exp\left[-\frac{(\gamma^2+q^2)^{1/2}}{T}\right]\right]^{-1}$$
$$=\exp\left\{\int_0^\infty dk[\rho_0(k)-\rho(k)]\ln\left[1-\exp\left[-\frac{(\gamma^2+k^2)^{1/2}}{T}\right]\right]\right\},$$

where $\rho(k)$ is the density of states of the continuous spectrum of equation

$$-d^2\psi/dx^2+U''(z_0(x))\psi=k^2\psi\ ,$$

while $\rho_0(k)$ corresponds to the substitution of $U''(z_0(x))$ by its limiting value γ^2.

Since Eq. (36) just has $M+1$ bound-state eigenvalues, we must have

$$\int_0^\infty dk[\rho_0(k)-\rho(k)]=M+1\ .$$

In a similar way we get

$$b=-\tfrac{1}{2}\sum_{m=2}^{M}S_m^{1/2}+\tfrac{1}{2}\int_0^\infty dk[\rho_0(k)-\rho(k)](\gamma^2+k^2)^{1/2}\ . \tag{39}$$

Finally we obtain

$$\frac{D}{L}=U_0\left[\frac{\alpha T}{2\pi g}\right]^{1/2}\frac{T_c}{\sin(\pi T_c/T)}\exp\left[-\frac{4\alpha g}{T}+\frac{b}{T}\right]$$
$$\times\prod_{m=2}^{M}\left[1-\exp\left[-\frac{S_m^{1/2}}{T}\right]\right]^{-1}\exp\left\{\int_0^\infty dk[\rho_0(k)-\rho(k)]\ln\left[1-\exp\left[-\frac{(\gamma^2+k^2)^{1/2}}{T}\right]\right]\right\}. \tag{40}$$

The dependence on the potential asymmetry F enters this expression only through the parameter T_c. For a vanishing T_c the decay rate obeys the Arhennius law in the whole temperature range. In this case the preexponential factor changes on the temperature scale $T \sim 1$ (in the conventional units $T \sim \hbar\omega$). At low temperatures $B \propto T^{3/2}$ and at high temperatures $B \propto T^{-1/2}$. Hence, in a general case, the preexponential factor has a maximum at $T \sim 1$. When T_c is small but finite, the preexponential factor B also has a minimum at $T = 1.710\ T_c$, where $B = 0.541 U_0(\alpha/g)^{1/2} T_c^{3/2}$. The maximal value of B is of the order of $U_0 g^{1/2}$ and surpasses its minimal value by a factor $T_c^{-3/2}$. The dependence $B(T)$ for typical potentials $U(z)$ will be calculated later.

C. The decay rate at temperature close to T_c

Equation (40) shows that the preexponential factor tends to infinity when T tends to T_c. This divergence is caused by the vanishing of the eigenvalue $\lambda_0(1)$ at $T = T_c$. Hence, the quadratic terms dominate no longer in the action in a proximity of T_c, and one should retain the cubic and quartic terms in the expansion of the action in C_{01}.[4] Integration with respect to all other C_{mn}'s results in the following expression:

$$\frac{D}{L} = 4U_0 T_c^3 (2\pi\alpha g T_c)^{1/2} \exp\left[-\frac{4\alpha g}{T} + \frac{b}{T}\right] \times \int_{-\infty}^{\infty} \frac{dC_{01}}{(2\pi)^{1/2}} \frac{dC_{0-1}}{(2\pi)^{1/2}} \times \exp\left[-2\pi^2 g (T^2 - T_c^2) |C_{01}|^2 - \frac{\pi^2 g T_c^3 \gamma^2}{6\alpha} |C_{01}|^4\right].$$

Performing the remained integrations, we arrive at a final result

$$\frac{D}{L} = 4(3^{1/2}) \frac{\alpha}{\gamma} U_0 T_c^2 \exp\left[-\frac{4\alpha g}{T} + \frac{b}{T}\right] \times K\left[\frac{2\pi(6\alpha g T_c)^{1/2}}{\gamma}\left[1 - \frac{T_c}{T}\right]\right], \qquad (41)$$

where

$$K(x) = \frac{\exp(x^2)}{\pi^{1/2}} \int_x^{\infty} dy \exp(-y^2) \simeq \begin{cases} \exp(x^2), & x \to -\infty \\ (2\pi^{1/2} x)^{-1}, & x \to \infty . \end{cases}$$

The typical scale of the temperature variations in (41),

$$|T_c - T| \sim (T_c/g)^{1/2},$$

is much smaller than T_c, therefore, we have substituted T_c for T elsewhere. This assumption holds when $gT_c \gg 1$. Hence, our consideration is justified for F in an interval $g^{-2} \ll F \ll 1$. The restriction of F from below relates to the criterion of the semiclassical approximation, which implies that contributions to the action should be large compared to the Planck constant. Equations (40) and (41) match in a region

$$(gT_c)^{-1/2} \ll (T - T_c)/T_c \ll 1 .$$

At temperatures well below T_c the preexponential factor becomes a constant, $B = 2(3^{1/2})\pi^{-2} U_0 F$, which surpasses the value of B at a zero temperature by a factor $2(3^{1/2})/\pi = 1.103$ only. Hence, we conjecture that in the interval $(0, T_c)$, the preexponential factor is weakly dependent on temperature. The level of this plateau exceeds above the determined minimal value of B by a factor $F^{1/4} g^{1/2}$. The ratio of the plateau value of B to its maximal value at $T \sim 1$ is of the order $F g^{1/2}$ and in the frame of our approach can be large as well as small compared to unity.

For temperatures

$$(gT_c)^{-1/2} \ll |1 - T/T_c| \ll 1 ,$$

Eqs. (40) and (41) can be substantially simplified with the results

$$\frac{D}{L} = U_0 \exp\left[-\frac{4\alpha g}{T} + \frac{b}{T}\right] \times \begin{cases} \dfrac{2(3^{1/2})F}{\pi} \exp\left[\dfrac{24\pi g}{\gamma}\left[\dfrac{\alpha F}{2\gamma}\right]^{1/2}\left[1 - \dfrac{T}{T_c}\right]^2\right], & T < T_c \\ \dfrac{1}{2\pi^3}\left[\dfrac{\gamma^3 F^3}{2\alpha g^3}\right]^{1/4}\left[1 - \dfrac{T_c}{T}\right]^{-1}, & T > T_c . \end{cases}$$

For still higher temperatures

$$\frac{D}{L} = U_0 \exp\left[-\frac{4\alpha g}{T} + \frac{b}{T}\right] \times \begin{cases} \dfrac{T}{\pi}\left[\dfrac{\alpha T}{2\pi g}\right]^{1/2}, & T_c \ll T \ll 1 \\ \left[\dfrac{\alpha}{2\pi g T}\right]^{1/2} \dfrac{1}{\pi(S_2 \cdots S_M)^{1/2}} \exp\left[\frac{1}{2}\int_0^{\infty} dk [\rho_0(k) - \rho(k)] \ln(\gamma^2 + k^2)\right], & 1 \ll T . \end{cases}$$

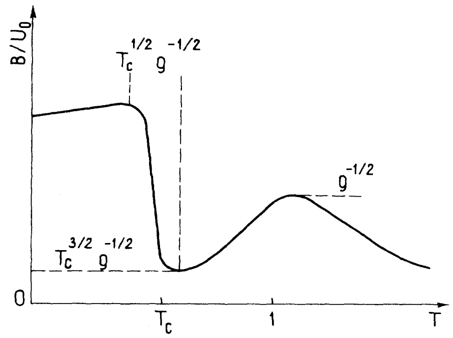

FIG. 6. Schematic temperature dependence of the preexponential factor.

In a broad interval of temperatures, the string motion across a barrier is either crucially or at least substantially dependent on the quantum effects. To make this point quite clear, we shall recall that in the conventional system of units the Planck constant enters our parameters in the following way:

$$U_0 \propto 1/\hbar, \quad T \propto 1/\hbar, \quad g \propto 1/\hbar \, .$$

Hence, the surmounting of a barrier by a string proceeds in a purely classical manner only in the limit of high temperatures. The dependence of the preexponential factor B on temperature is shown schematically in Fig. 6.

It should be pointed out that the integral for a correction b to the activation energy is divergent at large k, and, therefore, should be cut off at k of the order of inverse lattice spacing or some other fundamental length. Experimentally, only combination $4\alpha g - b$ makes sense, which manifests itself as the activation energy.

D. Results for typical potentials

The general expression for D derived above depends on parameters α and γ, eigenvalues S_n, and the variation of the density of states $\rho_0(k)-\rho(k)$. This result can be represented in a more transparent form

$$\frac{D}{L} = \frac{U_0}{g^{1/2}} \frac{\pi T_c/T}{\sin(\pi T_c/T)} \beta(T) \exp\left[-\frac{4\alpha g}{T} + \frac{b}{T}\right],$$

where the factor, dependent on the ratio (T/T_c), is identical to that for the one-particle decay rate.[13] For $T \gg T_c$, this factor tends to unity and the temperature dependence of the preexponential factor is determined by a function $\beta(T)$, which changes within the temperature scale $T \sim 1$. We shall consider two typical potentials.

1. $U(z)=\frac{1}{2}(1-z^2)^2$

In this case $\alpha=\frac{2}{3}$, $\gamma=2$, $M=3$, and $S_2=S_3=3$ (the energy splitting S_3-S_2 is of the order of $F \ll 1$ and can be well neglected for $T \sim 1$),

$$\rho_0 - \rho = \frac{4}{\pi}\left[\frac{1}{1+k^2} + \frac{2}{4+k^2}\right].$$

Thus, we obtain

$$\beta(T) = \frac{1}{3^{1/2}}\left[\frac{T}{\pi}\right]^{3/2}\left[1-\exp\left[-\frac{3^{1/2}}{T}\right]\right]^{-2} \exp\left\{\frac{4}{\pi}\int_0^\infty dk \left[\frac{1}{1+k^2}+\frac{2}{4+k^2}\right] \ln\left[1-\exp\left[-\frac{(4+k^2)^{1/2}}{T}\right]\right]\right\}. \tag{42}$$

with asymptotes

$$B(T) = \frac{1}{3^{1/2}}\left[\frac{T}{\pi}\right]^{3/2}, \quad T \ll 1$$

$$\beta(T) = \frac{16}{\pi}\left[\frac{3}{\pi T}\right]^{1/2}, \quad T \gg 1 \, .$$

2. $U(z)=\frac{1}{2}\cos^2(\pi z/2)$

In this case $\alpha=2/\pi$, $\gamma=\pi/2$, $M=1$,

$$\rho_0 - \rho = \frac{2}{\pi^2/4+k^2},$$

and we obtain

$$\beta(T) = \frac{T^{3/2}}{\pi^2}\exp\left\{2\int_0^\infty \frac{dk}{\pi^2/4+k^2} \ln\left[1-\exp\left[-\frac{(\pi^2/4+k^2)^{1/2}}{T}\right]\right]\right\} \tag{43}$$

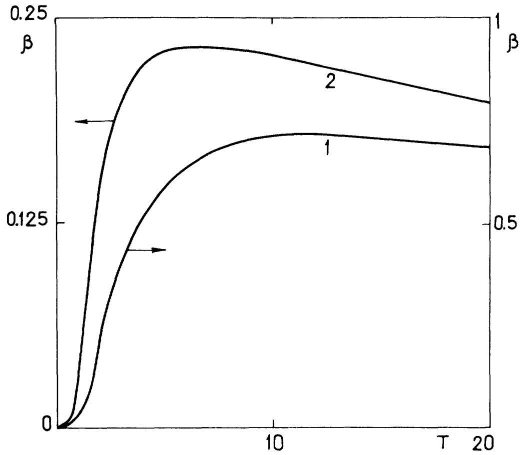

FIG. 7. The preexponential factors $\beta(T)$ for two typical potentials: (1) $U(z)=\frac{1}{2}(1-z^2)^2$; (2) $U(z)=\frac{1}{2}\cos^2(\pi z/2)$.

with asymptotes

$$\beta(T)=\frac{T^{3/2}}{\pi^2}, \quad T \ll 1$$

$$\beta(T)=T^{-1/2}, \quad T \gg 1 .$$

Plots of the functions $\beta(T)$ are given in Fig. 7. Their common feature is a rapid increase at $T\sim 1$ and a flat maximum at a comparatively large temperature $T\sim 10-20$, followed by a rather slow decay.

E. Motion of a string at a critical bias

Above we have considered tunneling and activation for small linear bias of an originally symmetric potential $F\ll 1$. An analytical solution can also be obtained in another special case of F close to the critical value of F_c, which corresponds to the vanishing of the barrier. Then the shape of a barrier is described by a cubic potential, obtained from a general expression

$$U_0V(y/a)=U_0[U(y/a)-Fy/a-E]$$

by expanding it out near the inflection point y_c, determined by the equation

$$U''(y_c/a)=0 .$$

The critical value of the bias F is given by the first derivative of the potential $U(y/a)$ at this point,

$$F_c=U'(y_c/a) .$$

We introduce a new parameter C by a relation

$$C^2=\tfrac{1}{2}\,|\,U'''(y_c/a)\,| .$$

Substituting

$$(y-y_c)/a=C^{-1}(F_c-F)^{1/2}(3z-1)$$

reduces the approximate potential to a standard form

$$U_0V(y/a)=2\tilde{U}_0z^2(1-z) ,$$

where

$$\tilde{U}_0=U_0(9/2C)(F_c-F)^{3/2} .$$

Introducing new scales of coordinate and time and inverse temperature

$$\tilde{x}_0=a(2\kappa/CU_0)^{1/2}(F_c-F)^{-1/4}, \quad \tilde{\tau}_0=\tilde{x}_0(\rho/\kappa)^{1/2}$$

we can write the action A as

$$A=\tilde{g}\int_{-1/2T}^{1/2T}d\tau\int dx\left[\frac{1}{2}\left[\frac{\partial z}{\partial\tau}\right]^2+\frac{1}{2}\left[\frac{\partial z}{\partial x}\right]^2+2z^2(1-z)\right], \qquad (44)$$

where the semiclassical parameter

$$\tilde{g}=a^2(\kappa\rho)^{1/2}(9/C^2)(F_c-F)$$

should exceed greatly unit.

The Lagrangian for the action (44) does not contain small parameters. Therefore, we restrict ourselves to the case of sufficiently high temperatures, $T>T_c$ (T_c will be calculated below). The extremal trajectory depends only on x and obeys an equation

$$\frac{1}{2}\left[\frac{dz_0}{dx}\right]^2=2z_0^2(1-z_0) ,$$

with a solution

$$z_0(x)=1/\cosh^2(x) .$$

The equation, similar to (36), can be written as

$$-d^2z_{mn}/dx^2+[4-12z_0(x)]z_{mn}=[\lambda_m(n)-(2\pi nT)^2]z_{mn} . \qquad (45)$$

The critical temperature T_c is determined by the vanishing condition of the eigenvalue $\lambda_0(1)$,

$$T_c=5^{1/2}/2\pi . \qquad (46)$$

The reflectionless potential of Eq. (45) has three discrete levels, $S_0=-5$, $S_1=0$, and $S_2=3$. Variation of the density of states is given by

$$\rho_0(k)-\rho(k)=\frac{2}{\pi}\left[\frac{1}{1+k^2}+\frac{2}{4+k^2}+\frac{3}{9+k^2}\right] . \qquad (47)$$

Substitution of (46) and (47) along with $M=2$, $\alpha=\frac{4}{15}$, and $\gamma=2$ into Eq. (40) gives the final result for the decay rate,

$$\frac{D}{L}=\frac{\tilde{U}_0}{\tilde{g}^{1/2}}f\left[\frac{T}{T_c}\right]\exp\left[-\frac{16\tilde{g}}{15T}+\frac{b}{T}\right],$$

$$T-T_c \gg \langle T_c/\tilde{g}\rangle^{1/2}$$

where b is defined by (39),

$$f(x)=\frac{(\frac{5}{9})^{1/4}x^{1/2}}{2\pi^2\sin(\pi/x)}\frac{1}{1-\exp[-2\pi(\frac{3}{5})^{1/2}/x]}$$
$$\times\exp\left[\frac{2}{\pi}\int_0^\infty dk\left[\frac{1}{1+k^2}+\frac{2}{4+k^2}+\frac{3}{9+k^2}\right]\ln\left\{1-\exp\left[-\frac{2\pi}{x}\left[\frac{4+k^2}{5}\right]^{1/2}\right]\right\}\right].$$

The function $f(T/T_c)$ is plotted in Fig. 8. In the limiting cases

$$f(x)\propto(x-1)^{-1},\quad x-1\ll 1\ .$$
$$f(x)=\frac{8(5^{1/4})}{\pi x^{1/2}},\quad x\gg 1\ .$$

The critical temperature T_c can be rewritten in the initial notations,

$$T_c=\frac{1}{2\pi a}\left[\frac{5CU_0}{2\rho}\right]^{1/2}(F_c-F)^{1/4}\ .$$

We have investigated the string motion in a cubic potential, which provides a conventional model for the dislocation motion at stress close to the Peierls stress. Our results agree with those obtained earlier in classical[23] and quantum[24] regimes. Comparison of Fig. 8 with Fig. 7 shows that slow temperature dependence of the preexponential factor B is a common feature of all investigated potentials.

IV. CONCLUSION

The problem of a string crossing a potential barrier is more complicated than the corresponding one-particle problem, because the extremal trajectory obeys a partial differential equation. The situation simplifies for potentials with nearly degenerate minima, when the size of the nucleus is much larger than its wall. Making use of a small parameter $F\ll 1$ (F describes a linear bias of an originally symmetric potential), we have calculated the decay rate in an exponential approximation in the full temperature range. The action A in the exponent of the decay rate has different functional forms in different temperature ranges,

$$A=\frac{\pi\alpha g}{T_0},\quad T<T_0\sim F$$
$$A=\frac{2\alpha g}{T_0}\left\{\sin^{-1}\left[\frac{T_0}{T}\right]+\frac{T_0}{T}\left[1-\left[\frac{T_0}{T}\right]^2\right]^{1/2}\right\},\quad T_0<T<T_c$$
$$A=\frac{4\alpha g}{T},\quad T>T_c\sim F^{1/2}\ .$$

We have developed a novel approach to calculate A in the crossover regions close to T_0 and T_c.

The preexponential factor B depends weakly on temperature up to $T=T_c$, when it exhibits a steep decrease. With a further rise of temperature the preexponential factor increases, reaches a flat maximum and then decays rather slowly, tending to an asymptotics $B\propto T^{-1/2}$. This high-temperature behavior of the preexponential factor is universal and does not depend on the particular form of the metastable potential. The Arhennius law for a string has the form

$$D\propto T^{-1/2}\exp(-E_b/T)$$

and differs from the one-particle case, when the preexponential factor does not depend on temperature.[1]

No effects of dissipation manifest themselves in our considerations. Therefore, the results obtained hold in an intermediate range of dissipation when the friction strength η, is small compared to the typical frequency τ_0^{-1}, but the energy loss per an oscillation is large compared to temperature, $g\eta\gg T$. These inequalities define an interval of the friction strength, where our results are applicable,

$$1\gg\eta\tau_0\gg T\tau_0/g\ .$$

In the overdamped regime the decay rate of a metastable state of a string has been calculated by Büttiker and Landauer in the classical case[25] (see also Ref. 26), and by Hida and Eckern in the quantum case,[27] but with the use of a variational procedure. In the underdamped regime the motion of a string in the classically allowed region becomes relevant, and one has to derive and solve either the Fokker-Planck equation, or an equation for the Wigner transform of the quantum density matrix. In this regime, bounces of the string back from the final

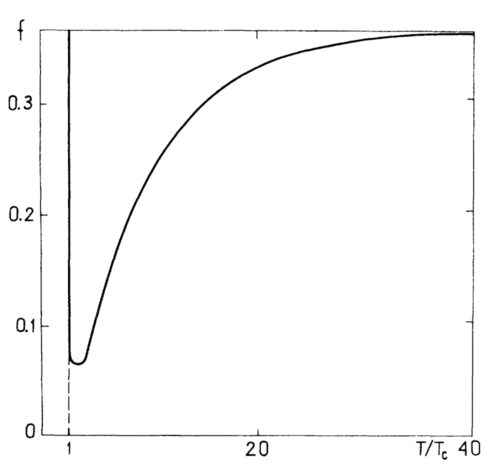

FIG. 8. Temperature dependence of the preexponential factor for a cubic potential.

into the initial well become probable. These effects have been investigated for the Brownian particle in a double-well potential[14] and in a tilted periodic potential.[28] The quantum effects for a particle in a tilted periodic potential have also been considered.[29] It seems, however, that a much more elaborate approach is required to extend these results to the problem of a string in a tilted periodic potential.

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