Effect of interplane coupling on the quantized Hall effect in multilayer structures

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The effect of electron-electron interaction on the particle-hole excitations for a fully filled band of coupled layers is studied. For coupled layers, there exists a continuum of particle-hole excitations with a plasmonlike collective excitation which lies at the bottom of the continuum for small transverse momentum transfer q. As q is increased, this excitation rises and is split off the top of the continuum. Eventually it comes down and merges with the continuum again when q reaches $2.2l^{-1}$. This collective excitation has similarities to the so-called magnetoexciton (plasmon) studied in two-dimensional (2D) systems. Due to the long-range nature of the Coulomb interaction, in addition to the cyclotron energy, the excitation energies depend on the angle of propagation ϑ with respect to the xy plane as $\cos^2\vartheta$. The response function is also investigated. Over a substantial region of phase space, it can be very well reproduced by a single-mode approximation. The instability toward the formation of a Wigner solid in multilayer structures is investigated by treating the interlayer coupling in the mean-field approximation. It is found that this tendency is much stronger in the fractional case than in the integral case, chiefly because the density-response function is much larger in the latter situation. The effect of the interplane hopping and charge transfer between layers is discussed.

I. INTRODUCTION

The quantized Hall effect has received much interest lately,[1] partly stimulated by experimental discoveries that were originally carried out on systems that consist essentially of a single layer of electrons such as metal-oxide-semiconductor field-effect transistors (MOSFET's). Due to the progress of the semiconductor technology, it is now possible to manufacture devices that consist of coupled layers[2] such as the $Ga_{1-x}AsAl_x$ or the Hg-Cd-Te heterojunctions. At the same time there are other systems such as the graphite intercalation compounds that consist of coupled layers of electrons in which the quantized Hall effect may also occur.[3] In this paper we investigate the possible effect of the interplane coupling on the integral and the fractional quantized Hall effects.

The effect of electron-electron interaction on the particle-hole excitations for a fully filled band of coupled layers is studied. For coupled layers, there exists a continuum of particle-hole excitations with a plasmonlike collective excitation which lies at the bottom of the continuum for small transverse momentum transfer q. As q is increased, this excitation rises and is split off the top of the continuum. Eventually it comes down and merges with the continuum again when q reaches $2.2l^{-1}$. This collective excitation has similarities to the so-called magnetoexciton (plasmon) studied in 2D systems. Due to the long-range nature of the Coulomb interaction, in addition to the cyclotron energy, the excitation energies depend on the angle of propagation ϑ with respect to the xy plane as $\cos^2\vartheta$.

The response function is also investigated. The density response function for coupled layers does not depend strongly on the z momentum and is quite similar to that for a single plane. Over a substantial region of phase space, it can be very well reproduced by the single (plasmon) mode approximation.

When the interplane coupling is strong enough, there will be an interaction between the intraplane charge fluctuations on different layers and hence a tendency toward the formation of a Wigner solid with true long-range positional order. We explore this transition by treating the interplane coupling in the mean-field approximation. It is found that this tendency is much stronger in the fractional case than in the integral case, chiefly because the density response function is much larger in the latter situation.

When the interplane-hopping matrix element gets high there can be charge transfer and fluctuations between layers. For the fully filled band, hopping is not important because there are no unoccupied states for the electrons to hop into. For the fractional case, hopping can create a significant difference and seems to be an important effect for some recently studied experimental systems. We conclude with a brief discussion of impurity bound states in coupled layers.

II. THE FULLY FILLED CASE

We first focus on the case for which the Landau level is nearly fully filled. If we ignore the coupling between different Landau levels, the ground-state $|0\rangle$ is particularly simple and corresponds to a determinant of all possible states occupied. We shall investigate how the structure factor is changed; how the particle-hole excitation energy is affected by the electron-electron (e-e) interaction in the coupled-layer problem and the instability toward the formation of a charge-density wave (CDW). Let us first look at the ground-state properties.

A. Structure factor

It is well known that the structure factor for a single layer is given by

$$S(q)=1-\exp(-q^2/2)+(2\pi)\delta(\mathbf{q}).$$

The evaluation of this is recapitulated in Appendix A. The factor $\exp(-q^2/2)$ comes from the exchange hole; the δ function comes from the direct contribution. The structure factor for coupled layers depends only on the transverse momentum and is essentially identical to that for a single layer (Appendix A). More precisely

$$S(q,p)=1-\exp(-q^2/2)+(2\pi)\delta(\mathbf{q})\sum_n \delta^d(p-2n\pi/d)N_z \tag{1}$$

where p is the momentum in the z direction, N_z is the number of planes in the z direction. The superscript d indicates a discrete δ function. Because all states are occupied, there is no way for the different layers to couple to each other. The exchange-hole contribution is identical to the single-plane case whereas the direct contribution reflects the additional scattering from the different planes.

B. Excitation spectrum

We shall pick units such that the effective mass of the electron, Planck's constant, the cyclotron frequency, and the charge are all unity. In general particle-hole excitations are of the form $c^\dagger_{k,m,1}c_{k',m',0}\,|\,0\rangle$ for general z momentum k, k' and angular momentum m, m'. Here $c_{k,m,n}$ is the electron destruction operator for a state with z momentum k, angular momentum m, and Landau level n. For excitations of the form $c^\dagger_{l,m,1}c_{l,m',0}\,|\,0\rangle$ with the same z momentum, the excitation energy is independent of m in the absence of the Coulomb interaction. A proper linear combination of them should be taken so that degenerate perturbation calculation with respect to the e-e interaction can be performed. Because of the translational invariance of the Coulomb e-e interaction, the xy momentum $\mathbf{q}$ is still a good quantum number. Define

$$\langle m'1\,|\exp(i\mathbf{q}\cdot\mathbf{C})v^\dagger\,|\,m0\rangle=V_{m,m'}.$$

The excited state with xy momentum $\mathbf{q}$ can be written as

$$|\,q,k,a\,\rangle=A\sum_{m,m'}V(m,m')c^\dagger_{m',k+a,1}c_{m,a,0}\,|\,0\rangle.$$

Here A is a normalization factor. $\mathbf{C}$ is the center-of-gyration operator.[4] It is introduced to separate the different Landau levels from each other. For excitations in the z direction of the form $c^\dagger_{l,m,1}c_{k+l,m,0}\,|\,0\rangle$ with the z momentum changed from l to $l+k$, the associated change in the kinetic energy of the electrons is a function of l. The energy of this state is the sum of the kinetic energy of motion along the z direction,

$$t\{\cos[(k+l)d]-\cos(kd)\}+\Delta,$$

(t is the hopping integral) and the Coulomb energy E_C.

The set of basis states $|\,q,k,a\,\rangle$ with different q's are orthogonal to each other. Since $\exp(i\mathbf{q}\cdot\mathbf{C})$ is a translation operator, the maximum value of $q_{x(y)}$ is the transverse dimension L of the sample. The spacing of $q_{x(y)}$ is $2\pi/L$. Hence the total number of such states for a given k,a is equal to $A/(2\pi)^2$. This is the same as the total number of excitations of the form $c^\dagger_{m',1}c_{m,0}$. Hence the states that we have been discussing completely exhaust all the spectral weight of particle-hole excitations. When restricted to the 2D case, the magnetoexciton (plasmon)[5,6] defined by

$$|\,q\,\rangle=(1/2N)^{1/2}\sum_j\exp(-i\mathbf{q}\cdot\mathbf{C}_j)\mathbf{v}^\dagger_j\,|\,0\rangle \tag{2}$$

thus exhaust all the spectral weight of the single-particle–single-hole excitations. There is no particle-hole continuum in the excitation spectrum.

In general, many such excitations can be present at the same time. Because of the existence of the gap $\hbar\omega_c$, the probability amplitude of such processes is reduced. Such processes shall be ignored. From these basis states, it is possible to construct three-dimensional collective and noncollective excitations. To do this, we calculate the Hamiltonian matrix element $\langle q,x,k\,|\,H\,|\,q,x+a,k\,\rangle$ in Appendix B. We find that

$$\begin{aligned}\langle q,x,k\,|\,H\,|\,q,x+a,k\,\rangle=&-\int dp\frac{1-\exp(-2\,|\,\mathbf{p}d\,|\,)}{1+\exp(-2\,|\,\mathbf{p}d\,|\,)-2\cos(ad)\exp(-\,|\,\mathbf{p}d\,|\,)}(1-|\,\mathbf{p}\,|^2/2)\\&\times\exp(-|\,\mathbf{p}\,|^2/2)J_0(pq)/N_z\\&+\frac{1-\exp(-2\,|\,\mathbf{q}d\,|\,)}{1+\exp(-2\,|\,\mathbf{q}d\,|\,)-2\cos(kd)\exp(-\,|\,\mathbf{q}d\,|\,)}\pi\,|\,\mathbf{q}\,|\\&\times\exp(-|\,\mathbf{q}\,|^2/2)/N_z+\delta(a)\left[\int dp(1-|\,\mathbf{p}\,|^2/2)\exp(-|\,\mathbf{p}\,|^2/2)\right].\end{aligned} \tag{3}$$

The Hamiltonian matrix is then diagonalized numerically for a system with 31 planes for different values of k and q with parameters appropriate for the GaAs heterojunctions studied by Stormer *et al.* ($d/l=2.64$, $l=85.6$ Å, z-band width $w=2t=2.5$ meV). We find that for a given k,q the spectrum consists of a continuum of states. For intermediate values of q, one state is split off the top. This spectrum is shown in Figs. 1–4 for

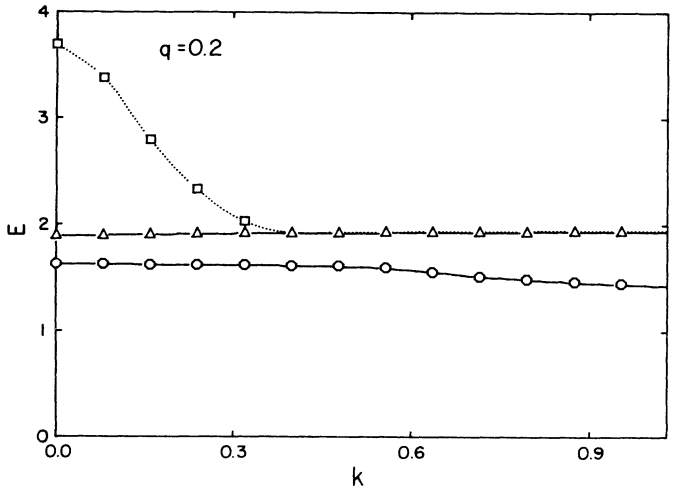

FIG. 1. Excitation energy E $(e^2/\epsilon l)$ for coupled planes with fully filled Landau levels as a function of the wave vector k $(2\pi/d)$ for $q=0.2$ (l^{-1}). The lower two values form the bounds of the particle-hole continuum. The uppermost band corresponds to the "plasmon" excitation. The origin of the energy is at $\hbar\omega_c+1.31$.

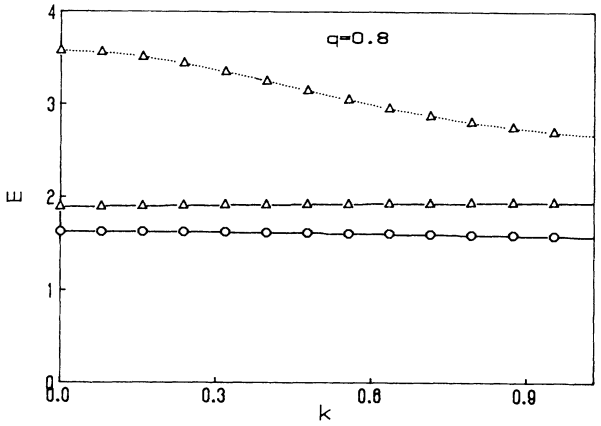

FIG. 3. Excitation energy E $(e^2/\epsilon l)$ for coupled planes with fully filled Landau levels as a function of the wave vector k $(2\pi/d)$ for $q=0.8$ (l^{-1}). The lower two values form the bounds of the particle-hole continuum. The uppermost band corresponds to the "plasmon" excitation. The origin of the energy is at $\hbar\omega_c+1.31$.

$q=0.2$, 0.4, 0.8, and 1.8, where we have plotted the energy of the isolated collective excitation as well as the boundaries of the continuum of states as a function of k. For small q, the collective excitation at the top is separated from the continuum only over a range of k. As q is increased, this range is increased. This collective excitation band no longer intersects the particle-hole continuum. As q is further increased, the collective excitation band comes down and merges with the particle-hole continuum eventually. Even though it is not obvious from the figures that we have presented so far, an examination of the imaginary part of the density response function (Sec. II C) indicated that the plasmon band is split off from the bottom of the continuum at small values of q.

To examine finite-size effects, we have investigated a run with 61 planes. The eigenvalues are found to change by less than 1%.

For $t=0$ or $k=0$, the kinetic energy of the particle-hole pair is not a function of a. It is then possible to construct an exact eigenstate of the Hamiltonian as

$$|q,k\rangle=\sum_a |q,k,a\rangle/\sqrt{N_z}$$
$$=A\sum_{m,m'} V(m,m')\rho_{m,m'}|0\rangle/\sqrt{N_z}, \quad (4)$$

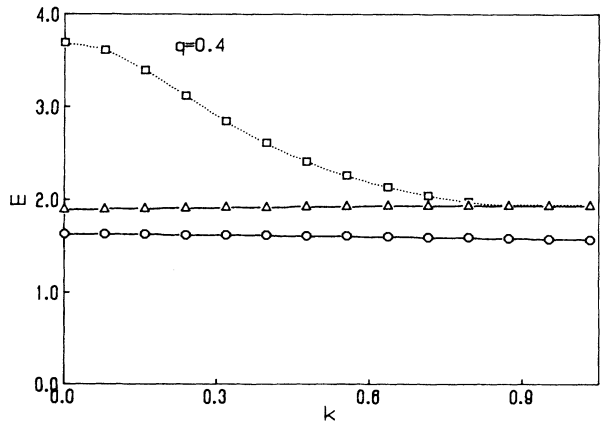

FIG. 2. Excitation energy E $(e^2/\epsilon l)$ for coupled planes with fully filled Landau levels as a function of the wave vector k $(2\pi/d)$ for $q=0.4$ (l^{-1}). The lower two values form the bounds of the particle-hole continuum. The uppermost band corresponds to the "plasmon" excitation. The origin of the energy is at $\hbar\omega_c+1.31$.

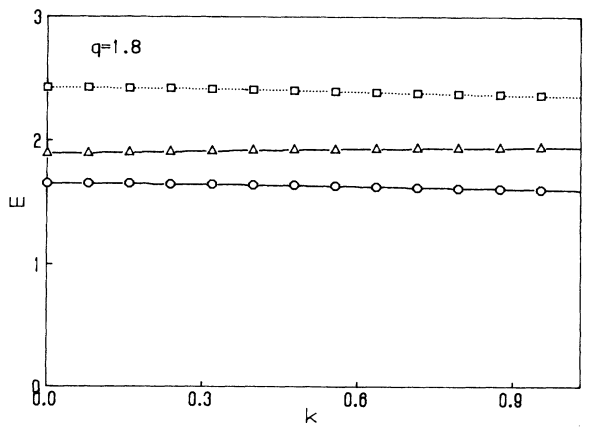

FIG. 4. Excitation energy E $(e^2/\epsilon l)$ for coupled planes with fully filled Landau levels as a function of the wave vector k $(2\pi/d)$ for $q=1.8$ (l^{-1}). The lower two values form the bounds of the particle-hole continuum. The uppermost band corresponds to the "plasmon" excitation. The origin of the energy is at $\hbar\omega_c+1.31$.

where

$$\rho_{m,m'}=\sum_a c^{\dagger}_{m',a+k,1}c_{m,a,0}.$$

This operator is very similar to the "plasmon" operator $\rho=\sum_a c^{\dagger}_{a+k}c_a$ studied in the one-dimensional (1D) Tomonaga model. An examination of our numerical results at $k=0$ indicates that amplitude of the collective excitation at the top as a function of a is indeed the same for all the basis states with different a. The state $|q,k\rangle$ can be written in a first-quantized form as

$$|k,q\rangle=(1/2N)^{1/2}\sum_j \exp(ikz_j-i\mathbf{q}\cdot\mathbf{C}_j)\mathbf{v}^{\dagger}_j|0\rangle \quad (4a)$$

and is very similar to the 2D magnetoexciton (plasmon) defined in Eq. (2). The energy of the 2D excitation is equal to[7]

$$E_{2q}=0.5\left[q\exp(-q^2/2)+\int_0^{\infty}dp[1-J_0(pq)](2-p^2)\exp(-p^2/2)\right]$$
$$=0.5\left[\frac{\pi}{2}\right]^{1/2}\left[1-e^{-q^2/4}[(1+q^2/2)I_0(q^2/4)-q^2I_1(q^2/4)/2]+\left[\frac{2}{\pi}\right]^{1/2}qe^{-q^2/2}\right]. \quad (5)$$

The energy of the 3D excitation in Eq. (4) can be similarly calculated and is found to be

$$E(k,q)=0.5\left[2\sum_n q^2\exp(-q^2/2)/d\,[q^2+(k+2\pi n/d)^2]+\int dp\exp(-p^2/2)[1-J_0(pq)](2-p^2)\right]. \quad (6)$$

This is shown in Fig. 5 as a function of k and q for an interplane spacing $d=4$. Whereas in two dimensions the excitation energy of $|q\rangle$ approaches zero as q approaches zero, in the present case, in the small-q limit $E_c\rightarrow|\mathbf{q}|^2/(|\mathbf{q}|^2+k^2)=\cos^2\vartheta$ where ϑ is the angle of propagation of the plasmon with respect to the xy plane. For $k<|\mathbf{q}|$, a gap exists in the excitation spectrum in the long-wavelength limit. Because the particle and hole can exist on different layers, the particle-hole pair will be farther apart on the average and their Coulomb attraction is reduced. Mathematically, this term comes from a bubble diagram (Appendix B) and is equal to

$$\frac{(2\pi)^3}{d}\sum_n v(q,k+2n\pi/d)|\mathbf{q}|^2\exp(-|\mathbf{q}|^2/2)$$

in three dimensions and

$$(2\pi)^2v_2(q)|\mathbf{q}|^2\exp(-|\mathbf{q}|^2/2)$$

in two dimensions. Here $v_2(q)=(2\pi q)^{-1}$ is the Coulomb interaction of a point charge in two dimensions. Because of the different degree of divergence of the two interactions, the difference of the result between two and three dimensions is obtained. A $\cos^2\vartheta'$ dependence is seen in the excitation spectrum of the plasmon in quasi-1D systems.[8] However, the angle ϑ' in that case is measured with respect to the z direction and not respect to the xy plane. We believe that the existence of a gap along a certain direction of propagation in both cases is due to the long-range nature of the Coulomb interaction. The details of the two situations are different enough that the angular dependences are different in the two cases.

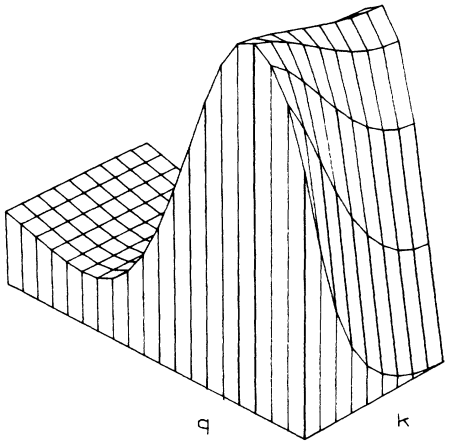

FIG. 5. The excitation energy E $(e^2/\epsilon l)$ of the collective excitation for the fully filled Landau level as a function of the wave vector q $(0.2/l)$ and k $(0.1\pi/d)$ for coupled planes. The origin is at $q=1$, $k=1$.

Kohn[9] has argued that the excitation is not shifted from $\hbar\omega_c$ for a short-range interaction. Our results can be generalized to a short-range interaction. In particular, it is not difficult to verify that E_c approaches zero in the long-wavelength limit in that case.

The two-dimensional (2D) excitation energy E_{2q} exhibits a rotonlike dip at around $q=2.2l^{-1}$, reflecting a short-range cage structure of the exchange-correlation hole. A corresponding dip is also seen in the present calculation where the collective excitation merges back with the continuum of excitations once again as q is increased beyond $2.2l^{-1}$.

In these calculations, we have assumed a large enough interplane hopping so that there is phase coherence along the z direction. Quinn[10] and co-workers have recently investigated the excitations of coupled layers but with the electron and hole confined to the same layer. As, we shall argue later on in this paper, for some experimental systems the interplane hopping matrix elements are much higher than room temperature. It is thus of experimental relevance to consider the case such that the wave function possesses phase coherence along the z direction.

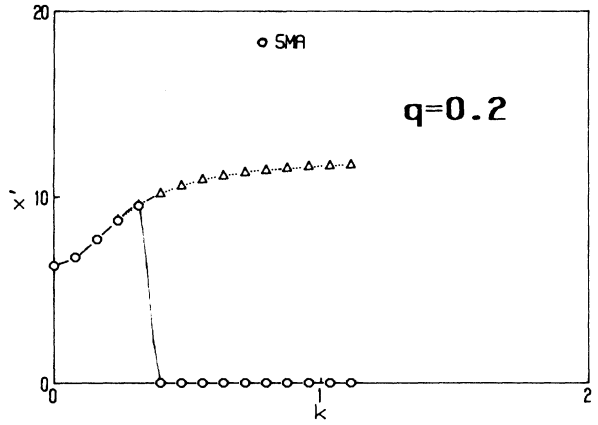

FIG. 6. The density response function multiplied by a numerical factor $62\chi(q,k,\omega=0)/q^2\exp(-q^2/2)$ [in units of $(e^2/\epsilon l)^{-1}$] as a function of k $(2\pi/d)$ for $q=0.2$ with the single-mode approximation (lower curve) and with all states included.

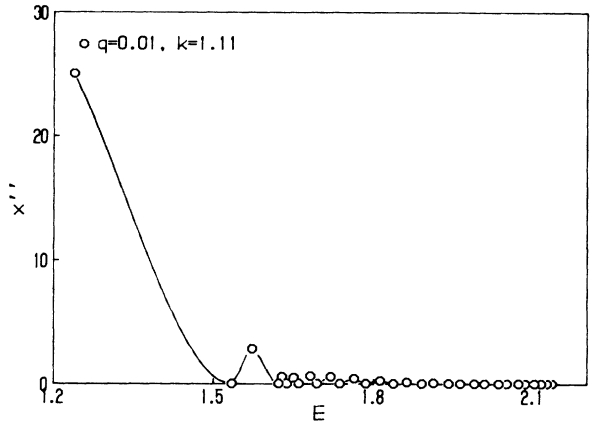

FIG. 8. A typical imaginary part of the density response function $62\chi''(q,k,\omega=0)/q^2\exp(-q^2/2)$ at small q $(0.01l^{-1})$, with $k=1.11(2\pi/d)$. The lines are drawn through the points to guide the eye.

C. Response function

From the eigenfunctions it is possible to calculate the response functions. In this section we shall concentrate on the dielectric function which is basically the density response function at zero frequency defined by

$$\chi(q,k,\omega=0)=\sum_i |\langle q,k,i|\bar{\rho}(q,k)|0\rangle|^2/E_i \tag{7}$$

and

$$\chi''(q,k,\omega)=\sum_i |\langle q,k,i|\bar{\rho}(q,k)|0\rangle|^2\delta(\omega-E_i).$$

From our discussion of the plasmon state it is obvious that for the real part χ the sum over i is dominated by this state in the long-wavelength limit. Hence we expect the single-mode approximation to work well in that case. This contribution can be evaluated analytically and has been carried out in Appendix B. We obtain

$$\chi_c(q,k,\omega)=q^2\exp(-q^2/2)/2\hbar(-\omega+\omega_c+E_c). \tag{8}$$

To test the validity of this single-mode approximation, we have calculated the response functions with only the topmost state and then with all the states from our numerical calculation ($\hbar\omega_c=15.6$ meV). The results for $62\chi/q^2\exp(-q^2/2)$ as a function of k for $q=0.4$ and 0.8 are shown in Figs. 6 and 7. Whenever this topmost state does not merge with the continuum, the single-mode approximation is quite accurate. When it merges with the continuum, the topmost state may not be the collective excitation any more.

The imaginary part of the density response function χ'' provides much information about the nature of the excited states. This can be seen from Figs. 8–13 where we have plotted χ'' as a function of E for different values

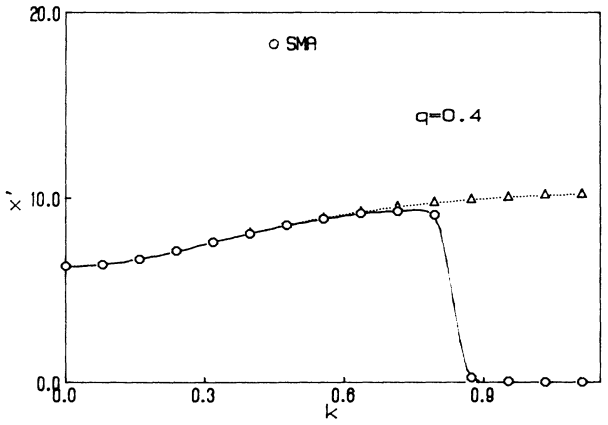

FIG. 7. The density response function multiplied by a numerical factor $62\chi(q,k,\omega=0)/q^2\exp(-q^2/2)$ [in units of $(e^2/\epsilon l)^{-1}$] as a function of k $(2\pi/d)$ for $q=0.4$ with the single-mode approximation (lower curve) and with all states included.

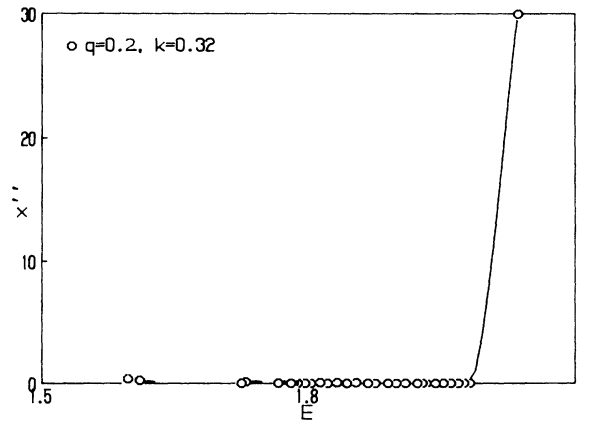

FIG. 9. $62\chi''(q,k,\omega=0)/q^2\exp(-q^2/2)$ at $q=0.2l^{-1}$, with $k=0.32\times 2\pi/d$. The lines are drawn through the points to guide the eye.

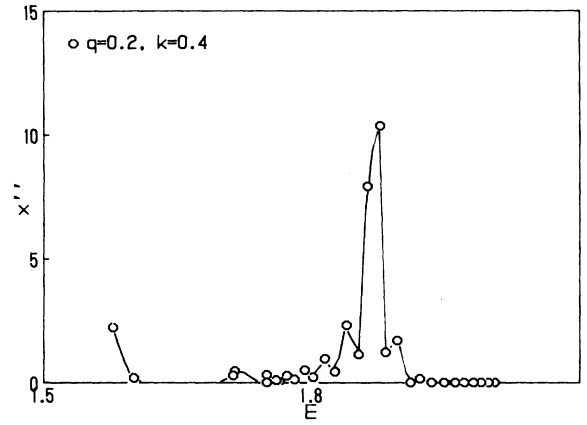

FIG. 10. $62\chi''(q,k,\omega=0)/q^2\exp(-q^2/2)$ at $q=0.2l^{-1}$, with $k=0.4\times 2\pi/d$. The lines are drawn through the points to guide the eye.

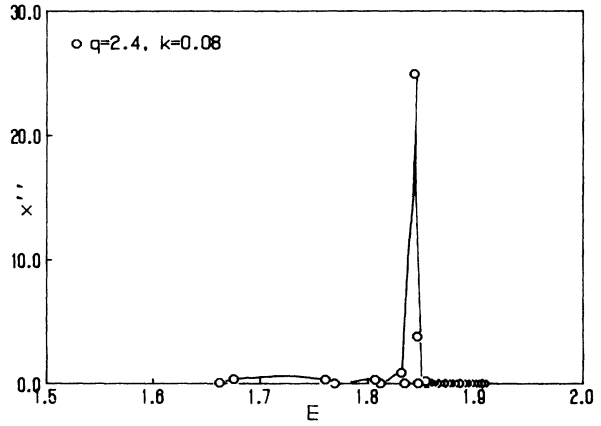

FIG. 12. $62\chi''(q,k,\omega=0)/q^2\exp(-q^2/2)$ at $q=2.4l^{-1}$, with $k=0.08\times 2\pi/d$. The lines are drawn through the points to guide the eye.

of q,k. The lines are drawn through the points to guide the eye. Figure 8 represents a typical situation at small q, with $k=1.11$. Similar results are obtained for other values of $k>q$ at this q. We see that for small values of q (0.01) there is a very sharp peak at the bottom of the band whenever $k>q$. Thus the plasmon is actually split off the bottom of the continuum under those situations. This is very different from the scenario in the quasi-1D systems where the particle-hole continuum is usually below the plasmon band and is not a function of ϑ. When the collective excitation merges with the continuum its identity is still quite well defined. This can be seen from Figs. 9, 10, and 11, where we looked at $q=0.2$ but with $k=0.32$, 0.4 and 1.1. For $k=0.32$, the plasmon is above the continuum and χ'' is very sharp. For $k=0.4$, it has merged with the continuum. χ'' is slightly smeared but is still quite well defined. For $k=1.1$, the plasmon has split off the bottom of the band. Another example of this can be seen from Figs. 12 and 13, where $q=2.4$ and $k=0.08$ and 0.4. For this q, the excitation has reached the rotonlike minimum and has merged with the continuum for all values of k. The peaks of χ'' are clearly discernible. The width of the peak increases as k is increased. We think the sharpness of the peak is due to the small value of the hopping integral that we have used. To test this, we have calculated χ'' but with t twice as big. The results for $q=0.2$, $k=0.4$ is shown in Fig. 14. The width of the peaks is much broader than those in Fig. 10, as we expected.

D. Instability condition

When the interplane coupling is strong enough, there will be an interaction between the intraplane charge fluctuations on different layers and hence a tendency toward the formation of a Wigner solid with true long-range positional order. The onset of the instability depends on the density autocorrelation function. Chui[11] has investigated the effect of interplane coupling in the mean-field approximation for models that consist of layers of coupled spins. He found that the magnetic transition temperature T_c of quasi-2D n-component spin models for

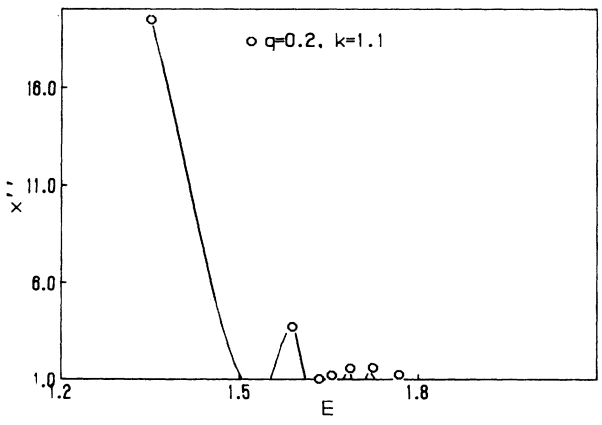

FIG. 11. $62\chi''(q,k,\omega=0)/q^2\exp(-q^2/2)$ at $q=0.2l^{-1}$, with $k=1.1\times 2\pi/d$. The lines are drawn through the points to guide the eye.

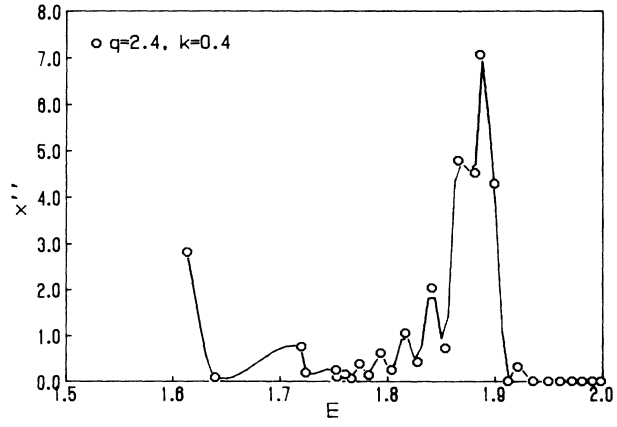

FIG. 13. $62\chi''(q,k,\omega=0)/q^2\exp(-q^2/2)$ at $q=2.4l^{-1}$, with $k=0.4\times 2\pi/d$. The lines are drawn through the points to guide the eye.

$n \geq 3$ to agree with the renormalization group calculation of Kosterlitz and Santos.[12] For $n=2$, the result agrees with Monte-Carlo simulation[13] to within 20% in the limit of weak interplane coupling. This approximation has been applied earlier to the coupled-chain problem by Scalapino *et al.*[14] Good agreement is found with the exact results of the anisotropic 2D Ising model. This approximation is not restricted to spin models alone. It can also be generalized to the case such that the interaction is not restricted to that acting between adjacent planes. When applied to the present case, the effective transition temperature for an instability with a periodicity characterized by a wave vector $\mathbf{K}$ in the direction parallel to the plane is controlled by the susceptibility $\chi(\mathbf{K},\omega=0)$ of a single plane; when

$$n\bar{V}(\mathbf{K})\chi(\mathbf{K},\omega=0)=1, \tag{9}$$

a phase transition must have occurred

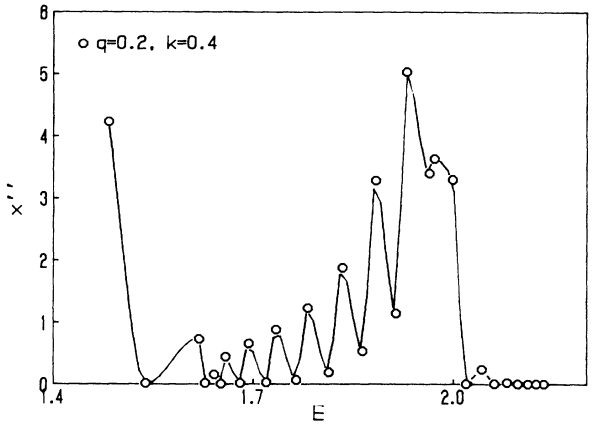

FIG. 14. $62\chi''(q,k,\omega=0)/q^2\exp(-q^2/2)$ at $q=0.2l^{-1}$, with $k=0.4\times 2\pi/d$ and t, the hopping integral equal to 0.25. The lines are drawn through the points to guide the eye.

$$\bar{V}=\sum_{z\neq 0}\int V(\mathbf{K},q)\exp[i(q-p)z]dq$$
$$=\{\tanh(d\mathbf{K}/2)[1+\tan^2(dp/2)][\tan^2(dp/2)+\tanh^2(d\mathbf{K}/2)]^{-1}-1\}/2\pi\mathbf{K}.$$

For the present case $\mathbf{K}$ is the smallest reciprocal lattice vector of the Wigner lattice. p is the period of the order parameter along the z direction. n is the number of reciprocal lattice vectors $\mathbf{K}$ of the same magnitude. For a triangular lattice at filling factor ν, there are six such vectors of magnitude $K^2=4\pi\nu/1.7321$. We have assumed that the order parameter associated with these six directions are of the same magnitude. Using the variational wave function discussed in the previous section, we find that the matrix element

$$\langle q\,|\exp(i\mathbf{q}\cdot\mathbf{r})\,|\,0\rangle=|\,\mathbf{q}\,|\exp(-|\,\mathbf{q}\,|^2/4)/2^{0.5}$$

(Appendix B). Collecting terms, we obtain

$$\chi(q,\omega)=|\,\mathbf{q}\,|^2\exp(-|\,\mathbf{q}\,|^2/2)/2\hbar(\omega-\omega_c) \tag{10}$$

where we have neglected E_q in comparison with $\hbar\omega_c$.

It is quite likely that the electronic states close to the band edge are localized. The above formulas should be unaffected if the localization length l is larger than the typical lattice constant of the Wigner solid that we are interested in.

The optimal instability condition in the present approximation corresponds to picking the period $p=0$. In that case $\bar{V}=[\coth(d\mathbf{K}/2)-1]/2\pi\mathbf{K}$. Substituting the response function into Eq. (9), we found that a phase transition will occur at zero temperature when the interplane spacing is given by

$$\coth(1.35d_c/l)-1=175\frac{\hbar\omega_c}{e^2/\epsilon l}. \tag{11a}$$

Approximating $\coth x$ by $1/x$, we obtain the condition

$$1.35d_c/l=\left[175\frac{\hbar\omega_c}{e^2/\epsilon l}+1\right]^{-1}. \tag{11b}$$

This corresponds to quite a small d_c, hence it is difficult to see this effect for the integrally filled case.

III. THE FRACTIONAL CASE

There are two possible types of instability, the CDW instability due to correlation of the location of charges on different planes and that due to interplane charge fluctuation. We first turn our attention to the CDW instability.

There are different guesses as to the nature of the ground state.[15] None of these guesses possess long-range translational order even though they all seem to manifest a finite shear modulus. To obtain a conservative estimate of the criterion of instability toward forming a CDW with long-range order, we take the wave function with the least amount of long-range order suggested by Laughlin. The response function χ depends on the excited state as well. Girvin *et al.*[16] have suggested a simple guess of the excited states. They find that the response function is equal to $\chi=\bar{S}(\mathbf{K})/E_{\mathbf{K}}$, where $\bar{S}$ is the projected structure factor. Using their result and substituting into Eq. (9), we obtain a critical spacing d given by

$$\coth(0.77d_c/l)-1=0.16/\bar{S}(\mathbf{K}), \tag{12}$$

where we estimated $E_{\mathbf{K}}$ to be $0.02e^2/l$. At $\frac{1}{3}$ filled, $\bar{S}(\mathbf{K})=0.4$. Hence $d_c/l=1.3$. The effect of interplane coupling is much stronger in the fractional case. This is mainly due to a much smaller gap and hence a much larger χ in the present case.

For the 3D Wigner solid, the charges are localized in the z direction as well. In the original argument of Wigner, the kinetic energy expended in localizing the electron is of the order $n^{-2/d}$ (d is the spatial dimension)

whereas the potential energy gained is of the order of $n^{-1/d}$. For n small enough, the potential energy prevails and the solid phase is stabilized. Because of the magnetic field, there is no extra energy cost in "localizing" the electrons in the xy plane. Extra energy cost is required to localize the electrons along the z axis, however. Depending on the parameters, we thus expect a crossover from a 3D Wigner solid to a 2D cylinder-type Wigner solid. On the other hand, if the temperature is low enough, there will be a Peierls instability associated with the one-dimensional motion and hence a concomitant CDW in the z direction as well.

If one starts off from the small-t limit, then the conventional 2D picture should still be a reasonable approximation. At odd denominator filling factors, the excitation spectrum possesses a gap Δ. From simple consideration of perturbation theory, we expect the effective dimensionless coupling constant for interplane charge fluctuation to be $(1-\nu)t/\Delta$. The factor of $1-\nu$ comes from the Fermi exclusion principle. If this coupling is large compared with one, then we expect the gap Δ to be destroyed.

In the sample studied by Stormer *et al.*,[2] the bandwidth w due to interplane hopping is of the order of $0.1\hbar\omega_c$, the interplane spacing d is 226 Å and the planar electron density is 4.5×10^{11} cm^{-2}. For this density, the cyclotron radius l at $\frac{1}{3}$ filled will be 60 Å. Hence $d_c/d=0.27$ and we expect the interplane charge correlation not to be very strong. On the other hand, the effective dimensionless coupling for interplane hopping is very big. Indeed, the largest theoretical estimate for the gap at $\frac{1}{3}$ filled is of the order of $0.1e^2/\epsilon l\cong0.03\hbar\omega_c$. This gap is believed to be further reduced due to the finite z extent of the wave function and impurities. In fact, the experimental estimate[17] of the gap at $\frac{1}{3}$ filled is an order of magnitude smaller than the theoretical value deduced from finite cluster calculations. Hence $w/\Delta>1$. No fractional quantized Hall effect has so far been observed in three-dimensional systems. This may be due to the destruction of the gap as a result of the interplane charge fluctuation.

IV. CONCLUSION

Stormer *et al.* recently found the integral quantized Hall effect in three-dimensional heterojunctions. They found that the gap deduced from the activation-type temperature dependence of ρ_{xx} to be much smaller than that expected from simple band-structure calculations and Shubnikov–de Haas measurements. As we have argued in this paper, the origin of this effect is most likely not due to the e-e interaction effect on particle-hole excitations. Nor is it likely a single impurity effect.

The two-dimensional bound-state energy scales as $B^{1/2}$. Indeed, if one ignores the inter-Landau-level spacing, the bound-state energy is just $\langle m \mid V \mid m \rangle$ (Ref. 18) where V is the impurity potential. For the Coulomb potential, the lowest ($m=0$) bound state thus scales as the inverse of the magnetic length and hence $B^{1/2}$. By contrast, in three-dimensional systems, the bound-state energy in the presence of a Coulomb potential scales as $\ln B$ in the high-field limit. Hence any midgap impurity "bound" states caused by a single impurity will be less bound for a quasi-2D system than it is in three. When the density of impurities is high, this picture may be changed in that a "bound" state will interact with impurities on different layers. The shift in energy will be increased.

The reduction in the gap may be due to a surface or interface band, or a reduction in the mobility gap (localization is much more difficult in three than in two dimensions).

The interplanar distance d for the graphite intercalation compounds is smaller than that of the heterostructures. On the other hand, the electron density is also higher. It is not clear if the tendency toward the formation of CDW is higher.

APPENDIX A: STRUCTURE FACTORS

In this appendix the structure factor of different trial wave functions will be evaluated. Let us first recapitulate the calculation of the pair correlation function for a single layer. The expectation value $\langle \exp[i\mathbf{q}\cdot(\mathbf{r}_1-\mathbf{r}_2)] \rangle$ can be written as

$$\sum_{m,m'} \langle m \mid \exp(i\mathbf{q}\cdot\mathbf{r}_1) \mid m \rangle \langle m' \mid \exp(-i\mathbf{q}\cdot\mathbf{r}_2) \mid m' \rangle - \langle m \mid \exp(i\mathbf{q}\cdot\mathbf{r}_1) \mid m' \rangle \langle m' \mid \exp(-i\mathbf{q}\cdot\mathbf{r}_1) \mid m \rangle . \tag{A1}$$

Since $\mid m \rangle = z^m \exp(-r^2/4)/(2\pi 2^m m!)^{1/2}$,

$$\sum_m \langle m \mid \exp(i\mathbf{q}\cdot\mathbf{r}) \mid m \rangle = \int d^2r \left[\sum_m r^{2m}/(2^m m!) \right] \exp[-(r^2/2 - i\mathbf{q}\cdot\mathbf{r})]/2\pi \tag{A2}$$

$$= \int d^2r \exp(-i\mathbf{q}\cdot\mathbf{r})/2\pi = (2\pi)\delta(\mathbf{q}). \tag{A3}$$

Similarly

$$\sum_m \mid m \rangle \langle m \mid = \exp(z'z^*/2 - r^2/4 - r'^2/4) \tag{A4}$$

and we get, for the exchange term

$$\langle m \mid \exp(i\mathbf{q}\cdot\mathbf{r}_1) \mid m' \rangle \langle m' \mid \exp(-i\mathbf{q}\cdot\mathbf{r}_1) \mid m \rangle = \int \exp[-(r-r')^2/2 - i\mathbf{q}\cdot(\mathbf{r}-\mathbf{r}')] \tag{A5}$$

$$= \exp(-q^2/2). \tag{A6}$$

Hence for $q\neq 0$, $S(q)=1-\exp(-q^2/2)+(2\pi)\delta(\mathbf{q})$.

We next turn our attention to the coupled plane problem. The expectation value $\langle \exp\{i[\mathbf{q}\cdot(\mathbf{r}_1-\mathbf{r}_2)+p(l-l')]\}\rangle$ can now be written as

$$\sum_{mk,m'k'}\langle mk\mid \exp[i(pl+\mathbf{q}\cdot\mathbf{r})]\mid mk\rangle\langle m'k'\mid \exp[-i(pl'+\mathbf{q}\cdot\mathbf{r}')]\mid m'k'\rangle \tag{A7}$$

$$-\langle mk\mid \exp[i(pl+\mathbf{q}\cdot\mathbf{r})]\mid m'k'\rangle\langle m'k'\mid \exp[-i(pl+\mathbf{q}\cdot\mathbf{r}')]\mid mk\rangle. \tag{A8}$$

Since

$$\mid mk\rangle=\sum_l z^m\exp[-(r^2/4+ikl)]/(2\pi 2^m m!N_z)^{1/2}, \tag{A9}$$

$$\sum_{m,k}\langle mk\mid \exp[i(pl+\mathbf{q}\cdot\mathbf{r})]\mid mk\rangle=(2\pi)\delta(\mathbf{q})\sum_n\delta^d(p-2n\pi/d)\alpha L, \tag{A10}$$

where the superscript d denotes a discrete δ function. α is the density of the electrons. Similarly

$$\sum_{m,k}\mid mk\rangle\langle mk\mid =\exp(z'z^*/2-r^2/4-r'^2/4)\delta(l-l') \tag{A11}$$

and we get, for the exchange term

$$\langle mk\mid \exp[i(pl+\mathbf{q}\cdot\mathbf{r})]\mid mk'\rangle\langle mk'\mid \exp[-i(pl'+\mathbf{q}\cdot\mathbf{r}')]\mid mk\rangle=\exp(-q^2/2) \tag{A12}$$

where p is the perpendicular component of q. Hence

$$S(q,p)=1-\exp(-q^2/2)+(2\pi)\delta(\mathbf{q})\sum_n\delta^d(p-2n\pi/d)\alpha L. \tag{A13}$$

Consider next the case for which all the m's but only νth of the k's (from $-k_F$ to k_F) are occupied. Then the direct contribution remains the same, i.e.

$$\sum_{m,k}\langle mk\mid \exp[i(pl+\mathbf{q}\cdot\mathbf{r})]\mid mk\rangle=(2\pi)\delta(\mathbf{q})\sum_n\delta(p-2n\pi/d)\alpha L. \tag{A14}$$

For the exchange contribution,

$$\sum_{m,k}\mid mk\rangle\langle mk\mid =\exp(z'z^*/2-r^2/4-r'^2/4)\sin[k_F(l-l')L]/\pi(l-l') \tag{A15}$$

and we get, for the exchange term

$$\langle mk\mid \exp[i(pl+\mathbf{q}\cdot\mathbf{r})]\mid mk'\rangle\langle mk'\mid \exp[-i(pl'+\mathbf{q}\cdot\mathbf{r}')]\mid mk\rangle=\exp(-q^2/2)f(p), \tag{A16}$$

$$f(p)=\alpha d\sum_z\exp(ipz)\sin^2(k_Fz)/(k_Fz)^2. \tag{A17}$$

Hence

$$S(q,p)=1-\exp(-q^2/2)f(p)+(2\pi)^2\delta(\mathbf{q})\sum_n\delta^d(p-2n\pi/d)\alpha L. \tag{A18}$$

APPENDIX B

We calculated the energy of the excitations in this appendix. We first derive a few commonly used identities involving the center-of-gyration operators. The position operator r is a sum of the center of gyration operator and the velocity operator

$$\mathbf{r}=\mathbf{C}+\mathbf{e}_z\times\mathbf{v},$$

where the operator $v_x+(-)iv_y$ lowers (raises) the occupation of the Landau level. We shall adopt the convention where vector operators in the plane such as $x+iy$ would be written as z, and p would represent p_x+ip_y, etc. When restricted to the lowest Landau level, the C and $C^\dagger$ operators, defined as $C_x\pm iC_y$, multiply the wave function by z and differentiate the polynomial part by z, respectively. The operators C and v satisfy the commutation relations: $[v,v^\dagger]=2$, $[C^\dagger,C]=2$, $[C_x,C_y]=-i$. We have

$$\exp(i\mathbf{q}\cdot\mathbf{r})=\exp(-q^2/4)\exp(i\mathbf{q}\cdot\mathbf{C})\exp(qv^\dagger/2)\exp(-q^*v/2) \tag{B1}$$

and

$$\exp(-pv^\dagger/2)v\exp(pv^\dagger/2)=v+p. \tag{B2}$$

$C_{x(y)}$ is the generator for moving the center of gyration in the $y(x)$ direction. The operator

$$T(s)=\exp(-i\mathbf{s}\times\mathbf{C}) \tag{B3}$$

moves the center of gyration by s. Hence, from the identity $\exp(A+B)=\exp(A)\exp(B)\exp(-[A,B]/2)$, one gets

$$\exp(-i\mathbf{q}\cdot\mathbf{C})f(\mathbf{C})\exp(i\mathbf{q}\cdot\mathbf{C})=f(C_x+q_y,C_y-q_x)=f(\mathbf{C}+\mathbf{q}_\perp) \tag{B4}$$

One also has

$$\langle\exp(i\mathbf{p}\cdot\mathbf{C})\rangle=\exp(p^2/4)\langle\exp(i\mathbf{p}\cdot\mathbf{r})\rangle.$$

Because of the commutation relations, $[C,z^\dagger]=-2$, but $[C,z]=0$; $[C^\dagger,z]=2$, but $[C^\dagger,z^\dagger]=0$, T is a translation operator associated with a displacement s as well.

The Hamiltonian can be expressed in terms of the density operator restricted to the lowest two Landau levels

$$\bar\rho(p,l)=\exp(-p^2/4)[\,|\,1\rangle\langle 1\,|\,(1-|\,\mathbf{p}\,|^2/2)+|\,0\rangle\langle 0\,|+|\,1\rangle\langle 0\,|\,p/\sqrt{2}-|\,0\rangle\langle 1\,|\,p^*/\sqrt{2}]\sum_j\exp(-i\mathbf{q}\cdot\mathbf{C}_j-ilz_j).$$

In second quantized notation, this can be written as

$$\bar\rho(p,l)=\exp(-p^2/4)\sum_{m,m'}V_{m,m'}(p)[c^\dagger_{m,k+l,1}c_{m',k,1}(1-|\,\mathbf{p}\,|^2/2)+c^\dagger_{m,k+l,0}c_{m',k,0}+c^\dagger_{m,k+l,1}c_{m',k,0}p/\sqrt{2}$$
$$-c^\dagger_{m,k+l,0}c_{m',k,1}p^*/\sqrt{2}].$$

Here $V_{m,m'}(p)=\langle m,0\,|\exp(-i\mathbf{q}\cdot\mathbf{C}_j)\,|\,m,0\rangle$. The Hamiltonian is given by

$$H=0.5\int d^2p\,dl\,v(p,l)[\bar\rho(-p,l)\bar\rho(p,l)]$$
$$-[\,|\,0,1\rangle\langle 0,1\,|\,(1-|\,\mathbf{p}\,|^2/2)-|\,0,0\rangle\langle 0,0\,|-|\,1,1\rangle\langle 1,1\,|\,(1-|\,\mathbf{p}\,|^2/2)^2]\exp(-p^2/2),$$

where $v(p,l)=(1/2\pi^2)\exp(i\mathbf{p}\cdot\mathbf{r}_\perp+ilz)/(p^2+l^2)$. The matrix element $\langle x,x+k\,|\,H\,|\,x+a,x+a+k\rangle$ of the excited state consists of a contribution from a ladder and a bubble contribution illustrated schematically in Fig. 15. The bubble diagram provides a contribution

$$E_b=\frac{(2\pi)^3}{L_z}\sum_n v(q,k+2n\pi/d)\,|\,\mathbf{q}\,|^2\exp(-|\,\mathbf{q}\,|^2/2)\left[\sum_{m,m'}V_{m,m'}V_{m',m}\right]^2\Big/2.$$

The term in large parentheses is equal to 1 from the completeness of the set $|\,m\rangle$ when restricted to the lowest Landau level. The sum over n can be performed by using the identity $\sum_n[\exp(ilnd)]/N_z=\sum_m\delta^d(l-2\pi m/d)$ where the superscript d indicates a discrete δ function. To go from the discrete to the continuous δ function, we multiply by $\Delta l=2\pi/L_z$. We obtain

$$E_b=\sum_n\int dl\frac{\exp(ilnd)}{N_z}[\,|\,\mathbf{q}\,|^2+(l+k)^2]^{-1}\,|\,\mathbf{q}\,|^2$$
$$\times\exp(-|\,\mathbf{q}\,|^2/2)$$
$$=\sum_n\exp(-ink d-|\,n\mathbf{q}d\,|)\pi\,|\,\mathbf{q}\,|\exp(-|\,\mathbf{q}\,|^2/2)/N_z$$
$$=\frac{1-\exp(-2|\,\mathbf{q}d\,|)}{1+\exp(-2|\,\mathbf{q}d\,|)-2\cos(kd)\exp(-|\,\mathbf{q}d\,|)}$$
$$\times\pi\,|\,\mathbf{q}\,|\exp(-|\,\mathbf{q}\,|^2/2)/N_z.$$

The ladder diagram provides a contribution

$$E_l=-\int d^2p\,v(p,2n\pi/d+a)$$
$$\times(1-|\,\mathbf{p}\,|^2/2)\exp(-|\,\mathbf{p}\,|^2/2)$$
$$\times\left[\sum_m V_{m1,m2}(q)V_{m2,m3}(p)V_{m3,m4}(q)V_{m1,m4}(p)\right].$$

The sum in the large parentheses is equal to

$$\sum_m\langle m\,|\exp(-i\mathbf{q}\cdot\mathbf{C}_j)\exp(-i\mathbf{p}\cdot\mathbf{C}_j)\exp(i\mathbf{q}\cdot\mathbf{C}_j)$$
$$\times\exp(i\mathbf{p}\cdot\mathbf{C}_j)\,|\,m\rangle=\exp(-i\mathbf{p}\times\mathbf{q})$$

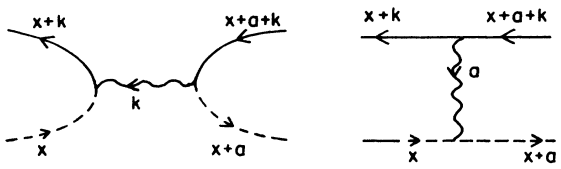

FIG. 15. Bubble and ladder diagrams illustrating the different contributions to the energy.

where we have used the commutation relations in (B4). We thus get

$$E_l=-\frac{2\pi}{L_z}\sum_n\int d^2p\, v(p,a+2n\pi/d)(1-|\mathbf{p}|^2/2)\exp(-|\mathbf{p}|^2/2)\exp(-i\mathbf{p}\times\mathbf{q}).$$

The sum over n can be simplified and we get

$$\begin{aligned}E_l&=-\sum_n\int d^2p\, dl\,\exp(ilnd)v(p,a+l)(1-|\mathbf{p}|^2/2)\exp(-|\mathbf{p}|^2/2)\exp(-i\mathbf{p}\times\mathbf{q})/N_z\\&=-\sum_n\int dp\exp(-|pnd|-idna)(1-|\mathbf{p}|^2/2)\exp(-|\mathbf{p}|^2/2)J_0(pq)/N_z\\&=-\int dp\frac{1-\exp(-2|\mathbf{p}d|)}{1+\exp(-2|\mathbf{p}d|)-2\cos(ad)\exp(-|\mathbf{p}d|)}(1-|\mathbf{p}|^2/2)\exp(-|\mathbf{p}|^2/2)J_0(pq)/N_z.\end{aligned}$$

Combining these terms and including the contribution from subtracting off the background and the projection terms finally, we get

$$\begin{aligned}E=&-\int dp\frac{1-\exp(-2|\mathbf{p}d|)}{1+\exp(-2|\mathbf{p}d|)-2\cos(ad)\exp(-|\mathbf{p}d|)}(1-|\mathbf{p}|^2/2)\exp(-|\mathbf{p}|^2/2)J_0(pq)/N_z\\&+\frac{1-\exp(-2|\mathbf{q}d|)}{1+\exp(-2|\mathbf{q}d|)-2\cos(kd)\exp(-|\mathbf{q}d|)}\pi|\mathbf{q}|\exp(-|\mathbf{q}|^2/2)/N_z\\&+\delta(a)\left[\int dp(1-|\mathbf{p}|^2/2)\exp(-|\mathbf{p}|^2/2)\right].\end{aligned}$$

The response function for the plasmon state can be similarly calculated. The matrix element

$$\begin{aligned}\left\langle q,k\left|\sum_i\exp(i\mathbf{q}\cdot\mathbf{r}_i)\right|0\right\rangle\Big/\sqrt{N}&=\left\langle 0\left|\sum_{i,j}\exp(-i\mathbf{q}\cdot\mathbf{C}_j-ikz_j)v_j\exp(-q^2/4+i\mathbf{q}\cdot\mathbf{C}_i+qv_i^\dagger/2)\right|0\right\rangle\Big/2^{1/2}N\\&=q\exp(-q^2/4)/2^{1/2}\end{aligned}$$

from Eq. (B2). The response function is thus given by

$$\chi(q,k,\omega)=q^2\exp(-q^2/2)/[2\hbar(\omega-\omega_c-E_c)].$$

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$$\int_0^\infty dp\, J_0(pq)\exp(-ap^2)=0.5\left[\frac{\pi}{a}\right]^{1/2}I_0(q^2/8a)\exp(-q^2/8a).$$

Differentiating with respect to a on both sides and using the fact that $I_0'=I_1$, we obtain

$$\begin{aligned}&\int_0^\infty dp\, p^2J_0(pq)\exp(-ap^2)\\&\quad=0.5\left[\frac{\pi}{a}\right]^{1/2}\exp(-q^2/8a)\\&\qquad\times[I_0(q^2/8a)(0.5/a-q^2/8a^2)+I_1(q^2/8a)q^2/8a^2].\end{aligned}$$

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