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Renormalization-group treatment of the long-ranged one-dimensional Ising model with random fields

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We construct a renormalization-group theory for a one-dimensional Ising model with long-ranged interactions and random fields with short-ranged correlations. We find recursion relations near the zero-temperature, finite randomness fixed point, valid for σ near $\frac{1}{2}$, where the interaction $J/(r) = J/r^{1+\sigma}$. We also show that the probability distribution for the random fields becomes strongly non-Gaussian under the scaling procedure.

One-dimensional systems in which the interactions between the spins are long ranged are often used as a testing ground for theories of higher dimensional short-ranged models. In recent years there has been much theoretical interest in the random-field Ising model (RFIM). In particular, the value of the lower critical dimension d^* of the short-ranged problem has been the subject of much controversy,[1,2] settled recently by Imbrie[3] who established that the three-dimensional system has a phase transition. The behavior of the system close to $d = d^*$ remains somewhat controversial, but it is clear[3] that one should consider the role of spins whose orientation is pinned by the local random field. This motivates us to consider the one-dimensional Ising model with long-ranged interactions, for which the mathematical description of domains is straightforward. We construct a renormalization-group (RG) approach for the zero-temperature random-field problem.

The long-ranged problem without random fields was treated[4] using the method introduced by Anderson and Yuval[5] to develop recursion relations valid for small fugacity. In this method, the configurations of the system are described in terms of the positions of $2n$ domain walls $r_1, \ldots, r_{2n}$, with $|r_i - r_j| > \tau$. One then considers increasing the short-distance cutoff to $\tau + d\tau$, and by making small changes in the parameters of the system, one can include the effects of the close pairs of domain walls, giving the desired recursion relations. If the Ising spins interact with a strength $J(r) = J/r^{1+\sigma}$, where r is the separation of the spins, it is found that the value $\sigma = 1$ separates a regime with a finite critical temperature from one with a critical temperature of zero. The renormalization-group calculation in Ref. 4 is valid near $\sigma = 1$, which is analogous to the lower critical dimension d^* of the short-ranged model.

For the model with random fields, the critical value of σ is $\frac{1}{2}$. This can be seen by extending the familiar Imry-Ma domain argument[2] to the long-ranged case. Consider flipping a block of spins of linear dimension L in an otherwise ordered state. The energy gained due to the fluctuations in the random fields, which have a distribution characterized by a width of h_R, will scale as $h_R L^{1/2}$, while the energy lost due to the interactions will scale as $JL^{1-\sigma}$. Thus for $\sigma > \frac{1}{2}$, the ground state is unstable against the formation of domain walls, and there is no long-ranged order, while for $\sigma < \frac{1}{2}$ long-ranged order is expected to occur at $T = 0$, and to persist up to a nonzero critical temperature.

In our model we follow Bray and Moore and assume that the critical behavior is governed by a fixed point at zero temperature and finite randomness.[6-9] The above Imry-Ma scaling ideas can be interpreted as scaling relations for J and h_R as functions of $l = \ln L$:

$$\frac{\partial J}{\partial l} = (1-\sigma)J \;, \tag{1}$$

$$\frac{\partial h_R}{\partial l} = h_R/2 \;, \tag{2}$$

from which we have

$$\frac{\partial \omega}{\partial l} = -\left(\tfrac{1}{2} - \sigma\right)\omega \;, \tag{3}$$

with $\omega = h_R/J$. The quantity w, thus, scales up or down[10] depending on whether one is above or below the critical value of $\frac{1}{2}$.

For the short-ranged problem near two dimensions, Bray and Moore[6] find corrections to Eq. (3) due to the roughening of the domain walls which are cubic in w. Since we expect that no domain roughening occurs in our problem, we need to calculate the exponentially small terms which arise physically from the effects of domains within domains, and were ignored in that work.

We use the same Hamiltonian as in Ref. 4 with the addition of a random field:

$$H = -\sum_{i \neq j} \frac{J n_i n_j}{\sigma(1-\sigma)} \left(\left| \frac{r_i - r_j}{\tau} \right|^{1-\sigma} - 1 \right) + 2n\mu - h \sum_{i=0}^{2n} n_i \left(\frac{r_{i+1} - r_i}{\tau} \right) - \sum_{i=0}^{2n} n_i \int_{r_i}^{r_{i+1}} \frac{dx}{\tau} h(x) \;, \tag{4}$$

where $r_1, \ldots, r_{2n}$ are the positions of the domain walls and satisfy $|r_i - r_j| > \tau$. $n_i = (-1)^i$, $r_0 = 0$, and $r_{2n+1} = L$, and h is a uniform external field. We assume that the chemical potential μ is large to ensure a low density of domain walls. The random fields $h(x)$ are defined as the sum of the fields at the lattice sites inside a block of size τ, which are, thus, already *smeared out* over length scales of size τ. While our general procedure holds for any distribution of random fields with short-ranged correlations, we will work initially with a Gaussian distribution satisfying

$$\overline{h(x)h(x')}=\delta\left(\frac{x-x'}{\tau}\right)h_R^2 \ . \tag{5}$$

We now consider an effective Hamiltonian defined with a cutoff $\tau+d\tau$, which allows for the possibility of a close pair of walls whose separation lies between τ and $\tau+d\tau$.

$$H_{\text{eff}}(r_1,\ldots,r_{2n},\tau+d\tau)=H(r_i,\ldots,r_{2n})+\left[\Delta E+2n_i\int_r^{r'}\frac{dx}{\tau}h(x)\right]\theta\left[-\Delta E-2n_i\int_r^{r'}\frac{dx}{\tau}h(x)\right] \tag{6}$$

in which $r_1,\ldots,r_{2n}$ have a minimum spacing $\tau+d\tau$, and ΔE is the change in energy of the system, neglecting the random-field contribution, due to the addition of the close pair:

$$\Delta E\equiv 2\mu+2hn_i-\sum_{j=1}^{2n}\frac{2Jn_in_j}{\sigma(1-\sigma)}\left[\left|\frac{r_j-r'}{\tau}\right|^{1-\sigma}-\left|\frac{r_j-r}{\tau}\right|^{1-\sigma}\right] . \tag{7}$$

Since we are working at zero temperature, the small pair can only appear if it lowers the total energy; this accounts for the step function in Eq. (6). We are, thus, expressing the fact that the ground state of the system must be stable against the flip of any set of spins. Here we restrict our attention to some block of neighboring spins of size in the range τ to $\tau+d\tau$, which will flip if there is a favorable net random field in the block of sufficient size. Thus, the small pair will occur only in regions of extreme random fields of opposite sign to the spins in the background domain. The parameters J, μ, h, and h_R include the effects of all the possible flips of blocks of spins of size less than τ. Since μ/h_R is large, the probability of a small domain is small, and to first order in J and h, we may write

$$\theta(-\Delta E-2n_ih_r)=\theta(-2\mu-2n_ih_r)+(2\mu-\Delta E)\delta(-2\mu-2n_ih_r) \ , \tag{8}$$

where $h_r=\int_r^{r'}(dx/\tau)h(x)$.

To calculate the recursion relations for J, μ, and h, we average the effective Hamiltonian (6) over the random field, and absorb the term proportional to $dl=d\tau/\tau$ into the first term whose average is the Hamiltonian of the pure model. Note that the averaging process includes integration over the position and length of the small domain. This gives us a Hamiltonian of the same form with renormalized parameters J, μ, and h, apart from a constant shift in the ground-state energy which we shall drop.

To obtain the recursion relations for h_R, we return to the unaveraged expression (6) and antisymmetrize it with respect to the sign of the spins in the background domain that contains the proposed small block. We now subtract off the mean which isolates the terms which contribute to the random field on the new length scale, and the variance of this field lets us calculate the recursion relation for h_R. This gives

$$\int_a^b h'(x)\frac{dx}{\tau+d\tau}=\int_a^b h(x)\frac{dx}{\tau}-dl\int_a^b\frac{dr}{\tau}[(h_r+\mu-\Delta/2)\theta(-h_r-\mu)+(h_r-\mu-\Delta/2)\theta(h_r-\mu)+\Delta\overline{\theta(h_r-\mu)}] \ , \tag{9}$$

where $\Delta\equiv 2\mu-\Delta E$ and the term proportional to dl accounts for the possible presence of the small block of reversed spins.

We work to linear order in the field h, and first order in the effective *fugacity*[2] defined by

$$y^2=\frac{h_R}{\sqrt{2\pi}\mu}\exp(-\mu^2/2h_R^2) \ .$$

Since μ/h_R is assumed to be large, this expression is simply the probability that a random field will fall in the tail of the Gaussian probability distribution $P(h)=(\sqrt{2\pi}h_R)^{-1}\exp(-h^2/2h_R^2)$, and is our small parameter. We find recursion relations

$$\frac{\partial J}{\partial l}=J(1-\sigma-4y^2),\quad \frac{\partial h_R}{\partial l}=h_R(\tfrac{1}{2}-2y^2) \ ,$$
$$\frac{\partial h}{\partial l}=h(1-2y^2),\quad \frac{\partial\mu}{\partial l}=\frac{J}{\sigma} \ . \tag{10}$$

Let us define $\omega=h_R/J$, and $v=h_R/\mu$, then

$$\frac{\partial\omega}{\partial l}=\omega(\sigma-\tfrac{1}{2}+2y^2) \ ,$$
$$\frac{\partial v}{\partial l}=v\left[\frac{1}{2}-\frac{v}{\sigma\omega}-2y^2\right] \ , \tag{11}$$

which has a fixed point at $2y^{*2}=\epsilon=\frac{1}{2}-\sigma$, which fixes v^*, and $\omega^*=4v^*$. Linearizing around the fixed point we find exponents $\theta=\frac{1}{2}-\epsilon=\sigma, \nu=v^{*2}/\epsilon\sim(2\epsilon\ln\epsilon)^{-1}$, and $\lambda_h=\frac{1}{2}+\sigma$. Here θ is the exponent that enters the modified hyperscaling equation (Refs. 6, 9, and 11) $(d-\theta)\nu=2-\alpha$, and characterizes the scaling of all energies at the fixed point, e.g., $J=Je^{l\theta}$. These three exponents are the only independent ones in our theory.

We now use the scaling relations that are derived in Ref. 6 to find the other exponents of interest. The exponents η and $\bar{\eta}$ are defined in terms of the connected and disconnected correlation function as follows:[12]

$$\overline{\langle S_0 S_r\rangle - \langle S_0\rangle\langle S_r\rangle} \propto \frac{T}{r^{d-2+\eta}} \ ,$$
$$\overline{\langle S_0\rangle\langle S_r\rangle} \propto \frac{1}{r^{d-4+\bar\eta}} \ . \tag{12}$$

We use the relations $\bar\eta = 4+d-2\lambda_h$ which gives $\bar\eta = 4-2\sigma$, and $\eta=\theta+\bar\eta-2$ which gives $\eta=2-\sigma=\bar\eta/2$. We also find that $\beta=(-2\ln\epsilon/2)^{-1}$, $\gamma=\sigma\nu$, $\delta=(2\epsilon)^{-1}$, and $\alpha=2-2\beta-\gamma$.

The result $\eta=2-\sigma$ agrees with the value found by Bray,[7] and is expected to be valid in all systems with long-ranged interactions. The violation of hyperscaling exponent θ, which takes the value σ in our calculation, agrees with the results of Bray at $\sigma=\frac{1}{2}$ and $\frac{1}{3}$. When σ is near $\frac{1}{2}$, we find that the exponent θ is modified by the presence of the small walls, to become $\theta=\frac{1}{2}-\epsilon$, where the first term is the naive value. This is to be contrasted with the results of Bray who finds further corrections in a $1/n$ expansion of the form $\theta=\frac{1}{2}-\epsilon+(1-2\sigma)/n$, for σ near $\frac{1}{2}$ where n is the number of spin components. This value for the exponent θ is in agreement with that found in Ref. 13; however, $1/n$ expansions cannot be relied upon in the Ising system. Note also that the exponent θ obviously satisfies the relation $\theta \le d/2$, which is intuitively obvious in a domain-within-domain picture, and was obtained by Villain,[8] and Berker and McKay.[11] Bray notes that θ "will in general take on nontrivial (i.e., different from σ) values at intermediate dimensions" that is for $\frac{1}{3}<\sigma<\frac{1}{2}$, however, we find that this is not the case in the Ising model to order ϵ. Schwartz[14] has derived an exact inequality $\bar\eta \le 2\eta$, which, as in Ref. 6, we find is satisfied to order ϵ as an equality. This leads immediately to $\theta=2-\eta$, and a hyperscaling relation of $2-\alpha=(d-2-\eta)\nu$.[1,14,15] The exponent $\bar\eta$ also satisfies the inequality $\bar\eta \ge 4-d$.[12]

So far we have assumed that the random field distribution at the fixed point is Gaussian. We now go on to discuss the non-Gaussian distribution that is generated by the renormalization group. This consequence of the RG flow close to d^* or σ^* does not appear to have been discussed previously.

We calculate the higher even moments of the random field on the new length scale from Eq. (9), and from this we can find the renormalization-group equations for all the cumulants. We work to linear order in the higher cumulants, but allow any power of the variance. This lets us explicitly calculate the probability as a function of the field for a distribution that is sufficiently close to a Gaussian. We then use this approximate probability distribution to calculate the averages needed in the procedure that we have described. We find the following recursion relation for the cumulants[16]

$$\frac{\partial \hat C_{2n}}{\partial l} = -\hat C_{2n}(n-1) - 4ny^2x^{2n-2}(1+\hat C_4 x^4/4!+\hat C_6 x^6/6!\cdots) \ , \tag{13}$$

where we have kept only the highest power of $x \equiv \mu/h_R$. The quantities $\hat C$ are the cumulants scaled by the appropriate power of the variance:

$$\hat C_n = \frac{C_n}{h_R^n} = \frac{C_n}{C_2^{n/2}} \ .$$

Note that the central limit theorem tells us that all the cumulants above the second are irrelevant when $y^2=0$, which in terms of the scaling of $\hat C$ means that

$$\frac{\partial \hat C_{2n}}{\partial l} = \hat C_{2n}(1-n) \ ,$$

this accounts for the first term in the above.

To illustrate the difficulty of dealing with all the cumulants, we can look for a fixed-point distribution where we restrict our attentions to the subset of distributions in which only some finite set of cumulants are nonzero. Equation (13) can then be recast in the form of a matrix equation, which can then be solved to yield[17]

$$\hat C_n^* \simeq \frac{4n}{n-2} y^2 x^{n-2}$$

with $n>2$, and x large. When we substitute the value of y^2 (or x) at the fixed point of Eq. (11),[16] we see that for fixed ϵ, the high-order cumulants become very large, and our approximation of linearizing in the cumulants becomes inappropriate. This clearly shows that the distribution is becoming very non-Gaussian. Note that we cannot solve the problem with an infinite set of cumulants, as the asymptotics used in the above would then be valid only at $x=\infty$, where we know that the Gaussian distribution is unchanged.

We have also studied a functional renormalization group for the full distribution in an attempt to see how the distribution develops under the renormalization group. For any distribution for which all the cumulants are finite, we find

$$\frac{\partial \hat P(z)}{\partial l} = \sum_{n=1}^{\infty} \frac{\hat C_{2n}}{(2n)!} \frac{\partial^{2n}\hat P(z)}{\partial z^{2n}} + \phi \frac{\partial}{\partial z}[z\hat P(z)] - \hat P(z)\theta\left(\frac{\mu^2}{h_R^2} - z^2\right) + [\delta(\mu/h_R+z)+\delta(-\mu/h_R+z)]\int_{\mu/h_R}^{\infty} \hat P(z')dz' \ , \tag{14}$$

where we have defined

$$\hat P(z) \equiv h_R P(h), \ z \equiv \frac{h}{h_R}, \ \phi \equiv \frac{\partial \ln h_R}{\partial l} \ .$$

The first two terms again come from the naive scaling of the cumulants, and the last two terms account for the small change in the probability due to the presence of the small block. If we restrict our attention to probability distributions close to Gaussian, this equation gives the same flows for the cumulants as previously. Equation (14) is a nonlinear diffusion equation containing drift and rescaling as well as higher derivatives. An apparently innocuous simplification, as far as the higher cumulants are concerned, is to drop the terms involving $\hat{C}_{2n}$ with $n > 1$ in the first term removing the higher derivatives and the nonlinearity, except where it enters implicitly through ϕ. This corresponds to replacing $n-1$ by n in the first term in Eq. (13) since the -1 comes from the term with $2n$ derivatives. It would be interesting to study the renormalization-group flow described by Eq. (14) numerically.

We can readily see from this equation that an initial Gaussian distribution is unstable under the renormalization-group procedure. This may be a signal that, in fact, the transition is first order. If the flow was such as to cause the weight in the tails of the distribution to grow sufficiently under the RG, taking one out of the region of validity of our small fugacity expansion, this would indicate that the system has become disordered before the correlation length has diverged, and would be a first-order transition driven by fluctuations.

In conclusion, we have derived recursion relations for the long-ranged random field Ising model in one dimension in an expansion around $\sigma = \frac{1}{2}$ under the assumption that the distribution of random fields is Gaussian. The exponents derived in their theory satisfy all the strict inequalities for the exponents of random field systems. This theory, however, incorrectly neglects the non-Gaussian effects generated by the renormalization procedure which may produce corrections to the exponents. We claim that the non-Gaussian effects are not negligible, and may even indicate the presence of a first-order transition in the problem. We have given a functional formulation of the field distribution renormalization problem, and this should be useful in the further treatment of both long- and short-ranged interaction problems.

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