High-temperature series expansion for spin glasses. II. Analysis of the series

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We discuss the critical behavior of an Ising spin glass with a $\pm J$ distribution. The results are based on an extensive analysis of the high-temperature series carried out for two-, three-, and four-dimensional systems. The estimates for the transition temperature and the exponents γ,η, and ν are presented.

I. INTRODUCTION

For many years the study of disordered magnetic systems has been a very active area of research.[1] By now, it is well understood, that the presence of disorder and frustration in a system gives rise to a new type of thermodynamic state, called a spin-glass[2] state. In recent years, one of the most debated questions has been the existence of an equilibrium phase transition, at a finite temperature, in three-dimensional Ising spin glasses. Recent numerical simulations[3] and high-temperature series expansions[4] have produced overwhelming evidence in favor of a finite temperature transition. The agreement between the two methods in the estimates of the critical parameters was found to be very good. Since the two methods face problems of very different types, one is led to believe that these results give a very effective description of the underlying physics.

The purpose of this paper is to explain the details of the high-temperature series *analysis* in these systems. In an earlier paper[5] we have discussed in detail how the series were generated. A short account of the analysis of the Edwards-Anderson (EA) susceptibility[2] for the $\pm J$ distribution Ising spin glasses in two, three, and four dimensions was already presented in Ref. 4. Since then, we have derived another high-temperature series for a different susceptibility. On the basis of a simple scaling assumption, this has allowed us to determine the exponents η and ν. These exponents are also in good agreement with the numerical simulations.[3]

The organization of this paper is as follows. In Sec. II we shall give an introduction to the use of series analysis in the study of disordered systems. In Sec. III we shall define the quantities of interest and explain how the various exponents can be obtained. In Sec. IV we shall present the detailed analysis of the various series. Finally, in Sec. V we shall present our conclusions.

II. SERIES ANALYSIS IN DISORDERED SYSTEMS

In the study of critical phenomena, high-temperature series expansions have played a very important role. For several lattice models, such as the pure Ising model, this method has led to the most accurate and reliable estimates of the critical parameters.[6] The method involves extrapolating from a finite number of coefficients in the high-temperature series expansion of some thermodynamic function to its asymptotic coefficients. The asymptotic form of the series contains the information on the singularities of the function. The existence of a phase transition at a critical temperature T_c implies that the function should be nonanalytic at that point. Hence, by locating the position of the singularity on the real axis, and studying the nature of the singularity, the critical parameters are determined.

These extrapolation techniques are not rigorous in the strict sense of the word. To a certain extent, one's confidence in them to give the right critical behavior is based on experience. The methods, however, are well tested on test functions where the singularities are known. They also give excellent results for statistical models where the exact solutions are available, such as in the two-dimensional (2D) Ising model. Nevertheless, one can question the use of these extrapolation techniques in disordered systems, where the complete singularity structure of the function may be very complicated. For example, these could be Griffiths-type singularities.[7] Any extrapolation of the series to obtain the critical behavior represents an analytic continuation which ignores any such subtle singularities in the high temperature phase. We still hope that the series analysis is able to locate the dominant singularity which marks the onset of a new type of broken symmetry in the system. Other effects simply contribute to the noise in the estimates. To make sure that one is not seeing a spurious effect, one can check one's analysis on the basis of several criteria.

(i) *Convergence of the analysis.* Most analysis techniques allow for several determinations of the critical parameters. From these one can study how good the convergence of the analysis is. Furthermore, one can study whether adding more and more terms to the series leads to an improved estimate or not. In this way the method of analysis itself provides a check.

(ii) *Agreement between different methods of analysis.* The same series can be analyzed by different methods. One can then check if they give the same answer.

(iii) *Analysis of more than one series.* If, for a given model, one can obtain more than one series, which are expected to have a singularity at the same point, then that can also be a useful check.

(iv) *Verification of universality.* The notion of universality of critical exponents is a very important concept in the study of phase transitions. In this regard, one can study a given model on various lattices and see if the results obtained are consistent with universality.

(v) *Verification of scaling relations.* On the basis of renormalization group theory, one expects certain scaling relations to hold between the exponents. If one can obtain independent estimates for these exponents, one can check if these scaling relations hold.

(vi) *Agreement with other studies.* One of the most useful consistency checks on an approximate study of a complicated problem is to see that different ways of approaching the problem give rise to the same answer.

(vii) *Agreement with experiments.* The ultimate test of any theoretical study is its agreement with experiments. However, as a prerequisite for such a comparison it is important to make sure that the model actually represents the physical system.

In the next few sections we shall analyze the critical behavior of Ising spin glasses and see how the results fare compared to the criteria discussed above.

TABLE I. Coefficients a_n for d-dimensional "cubic" lattices.

n	$d=2$	$d=3$	$d=4$
1	4	6	8
2	12	30	56
3	36	150	392
4	52	582	2408
5	116	2454	152 72
6	−108	6870	853 52
7	228	257 82	508 808
8	−2380	343 74	262 589 6
9	4084	202 486	151 119 76
10	−146 60	−323 730	720 676 72
11	800 52	342 826 2	421 464 680
12	−185 268	−821 774 6	185 160 319 2
13	877 428	110 253 462	118 105 832 08
14	−305 585 2	−241 502 106	463 466 253 20
15	944 515 6	249 563 893 4	347 729 503 368
16	−422 307 48	−122 174 979 30	
17	141 760 852	480 174 252 06	
18	−545 628 100		
19	214 027 682 0		

III. BASIC DEFINITIONS

The Ising spin-glass is defined by the Hamoltonian

$$-\beta\mathcal{H}=\sum_{(i,j)} J_{ij}S_iS_j \ , \tag{3.1}$$

where the sum runs over each nearest neighbor pair once. S_i is the Ising spin at site i which takes values ± 1, and J_{ij} is the nearest-neighbor interaction which is a quenched random variable distributed with the probability

$$P(J_{ij})=\tfrac{1}{2}[\delta(J_{ij}-J)+\delta(J_{ij}+J)] \ . \tag{3.2}$$

We shall be interested in the Edwards-Anderson susceptibility ($\chi_{\rm EA}$) and an auxilliary susceptibility (χ') given by the expressions

$$N\chi_{\rm EA}=\sum_i\sum_j [\langle S_iS_j\rangle^2] \tag{3.3}$$

and

$$N\chi'=\sum_i\sum_j [\langle S_iS_j\rangle^2]^2 \ . \tag{3.4}$$

Here the angular brackets refer to thermal averaging and the square brackets refer to an average with respect to the distribution of J_{ij}'s. The series are obtained in powers of w [$=\tanh^2(J/kT)$]. The χ' series for three dimensions was found to be

$$\chi'=1+6w^2+102w^4-192w^5+1998w^6-7584w^7+42\,822w^8-221\,856w^9+114\,787\,8w^{10}-598\,060\,8w^{11}$$
$$+323\,189\,10w^{12}-167\,464\,128w^{13}+906\,131\,742w^{14}-484\,995\,830\,4w^{15}+259\,528\,897\,98w^{16}-141\,648\,771\,168w^{17} \ . \tag{3.5}$$

The series for χ has already been reported, but for the sake of completeness we give it in Table I, where

$$\chi=1+\sum_n a_n w^n \ . \tag{3.6}$$

We also give the free-energy series for three dimensions which we have, so far, found difficult to analyze:

$$F_1/N=-1.5w^4-11w^6+36w^7-83.25w^8+656w^9-2250w^{10}+9852w^{11}-54767w^{12}$$
$$+213\,276w^{13}-112\,917\,3w^{14}+553\,872\,8w^{15}-253\,367\,86.125w^{16}+137\,498\,784w^{17} \ .$$

Let us characterize the divergence of the two susceptibilities by the exponents γ and γ', respectively. That is

$$\lim_{T\to T_c}\chi_{\rm EA}\sim(T-T_c)^{-\gamma} \ ,$$
$$\lim_{T\to T_c}\chi'\sim(T-T_c)^{-\gamma'} \ . \tag{3.7}$$

We shall assume a scaling relation of the form

$$[\langle S_iS_j\rangle^2]\sim\frac{1}{r^{d-2+\eta}}G(r/\xi) \ , \tag{3.8}$$

where r is the distance between the points i and j. ξ is the correlation length. As usual the correlation-length exponent ν is given by

$$\lim_{T\to T_c} \xi \sim (T-T_c)^{-\nu} . \tag{3.9}$$

Then from Eqs. (3.3)–(3.8) we get

$$\gamma=(2-\eta)\nu \tag{3.10}$$

and

$$\gamma'=(4-d-2\eta)\nu . \tag{3.11}$$

We shall analyze the series for $\chi_{\rm EA}$ and χ' and use the relations (3.10) and (3.11) to obtain the exponents η and ν.

IV. SERIES ANALYSIS

In this section we shall discuss the analysis of the susceptibilities and obtain the critical temperature and the critical exponents. We are interested in locating power-law divergences of the functions, such as the one given in Eq. (3.9). The nature of these singularities is hidden in the asymptotic form of the series. Also, the asymptotic behavior is ultimately dominated by the closest singularity to the origin. In cases where the physical singularity is the closest to the origin, the task for estimating the critical parameters becomes easier. One can obtain the critical temperature and the critical exponents by studying the ratio of successive terms in the series. However, in the present problem, as we shall see, the singularity which controls the radius of convergence of the series lies in the complex plane. Hence, the critical parameters become harder to estimate.

If one had knowledge about the exact analytic behavior of the function, one could make a change of variables such that in terms of the new variable the physical singularity became the closest to the origin. Then the series could be analyzed much more easily. In practice, one may adopt the following procedure. By using a method such as the d-log Padé method (described below) one first determines the approximate locations of the dominant singularities close to the origin. One then makes a change of variables to move the unphysical singularities away, hoping that it would bring the physical singularity closest to the origin. One then analyzes the transformed series in terms of the new variable. In the next paragraph we would like to argue that such a procedure is filled with uncertainties and that in the present problem it can lead to incorrect answers.

For our susceptibility series the first few terms contain little information on spin-glass ordering. In fact one cannot hope to see any spin-glass behavior until one gets contributions from diagrams involving closed loops. This is because frustration is an essential feature of spin glass and only occurs in closed loops. Also, the coefficients of the first three terms of the series for the EA susceptibility are identical to those for the susceptibility series for the pure Ising model. Hence, any analysis which depends very sensitively on the first few terms in determining the critical behavior is likely to give incorrect answers. It has been shown by Nickel[8] that the use of an Euler transform of the form $z=w/(1+bw)$ to go from a series in w to a series in z amounts to weighting all the higher-order coefficients with the lower-order ones. If the value of b is chosen to be large, then the higher-order terms in the original series are almost entirely suppressed. Hence, an apparent convergence on the basis of such an analysis may be misleading. We have avoided such a change of variables in our analysis.

The method that we found to be most suitable to this problem is that of first-order inhomogeneous differential approximants.[9,10] In this method the function, whose power series is known, is expressed as the solution to a first-order inhomogeneous differential equation. Let us consider the function whose first $N+1$ terms in the power series are known as

$$f(w)=\sum_{n=0}^{N} a_n w^n . \tag{4.1}$$

We demand that this function satisfy the differential equation

$$P_1(w)\frac{df}{dw}+P_2(w)f=P_3(w)+O(w^{M+L+J+2}) , \tag{4.2}$$

where

$$P_1(w)=1+\sum_{i=1}^{M} p_{1,i}w^i , \tag{4.3a}$$

$$P_2(w)=\sum_{i=0}^{L} p_{2,i}w^i , \tag{4.3b}$$

$$P_3(w)=\sum_{i=0}^{J} p_{3,i}w^i . \tag{4.3c}$$

The coefficients $P_{\alpha,i}$ ($\alpha=1,2,3$) are obtained by equating the coefficients of w^j with $0\le j\le M+L+J+1$ in Eq. (4.2). Since only the first $N+1$ terms of the series are known and Eq. (4.2) contains a derivative (df/dw) we can choose L,M,J such that $L+M+J+2\le N$.

One can also show that at points w_c, where $P_1(w_c)=0$, the solution to the differential equation (4.2) has a singularity of the form $(w_c-w)^{-\gamma}$ with γ by

$$\gamma=P_2(w_c)/P_1'(w_c) . \tag{4.4}$$

Here the prime refers to a derivative. We now find the polynomials P_1, P_2, and P_3 by solving the linear equations obtained by equating the coefficients of w^i in Eq. (4.2). Then we get the zeros of the polynomial P_1. The zero of P_1 on the real axis is identified as the critical point. The exponent γ is evaluated using Eq. (4.4).

It is useful to compare this method with the more frequently used d-log Padé method. In this method one represents f as the solution to the differential equation

$$P_1(w)\frac{df}{dw}+P_2(w)f=0 . \tag{4.5}$$

Here P_1 and P_2 are polynomials of order M and L, respectively, with $P_1(0)=1$. They are constructed by equating the coefficients of w^i with $0\le i<M+L$ in Eq. (4.5). Here, again, at points w_c where $P_1(w_c)=0$, the solution to the differential equation has a singularity of the type $(w_c-w)^{-\gamma}$ with $\gamma=P_1(w_c)/P_1'(w_c)$.

In the method of inhomogeneous approximants, the term P_3 only affects the coefficients of w^i with $i<j$. Hence, P_1 and P_2 are entirely determined by demanding that

$$P_1(w)\frac{df}{dw}+P_2(w)f=0 \tag{4.6}$$

be satisfied for coefficients of w^i with $J+1\leq i\leq M+L+J+1$. Comparing Eqs. (4.5) and (4.6) we see the similarity of the two methods. By allowing for an inhomogeneous term in the differential equation, one allows for a smoothly varying background term. This makes the critical behavior less sensitive to the first few terms of the series. The stability of inhomogeneous differential approximants against changes in the first few terms of the series has been observed by Fisher and Au-Yang.[9]

For a series of given length a large number of approximants with different values of L, M, and J can be constructed. The order of the series used by an approximant is equal to $L+M+J+2$. This number will be called the order of the approximant. All the approximants are not equally well behaved. Hunter and Baker[10] showed that the approximants of the type (M, $L=M-2$, $J=M-2$) are invariant under Euler transformation. They also studied the convergence of these approximants for several test functions. They then suggested the use of approximants with values of L, M, and J close to the Euler invariant ones. We have studied all the approximants and our results are based on those which are the best behaved. In addition, the following type of approximants are considered defective and hence are discarded.

(i) Approximants where a zero of P_2 comes very close to the physical singularity, giving rise to an anomalously small value of the exponent γ.

(ii) Approximants where a zero of P_1 occurs close to the origin on the positive real axis hence hindering the integration of the differential equation in Eq. (4.2) from the origin to the critical region.

(iii) Approximants where a zero of P_1 and a zero of P_2 occur within a distance of 0.001 anywhere in the complex plane.

After we make our preliminary estimates of w_c and γ from the unbiased approximants we also study biased differential approximants. In this case one of the zeros of P_1 is constrained to be at a predetermined value of w_c. The exponent γ is then obtained using Eq. (4.4). We find that the same approximants which were well behaved before biasing are also well behaved after biasing. However, some of these approximants develop a zero of P_1 on the positive real axis closer than w_c. These are very poorly behaved and are not considered. There are also some approximants where a zero of P_1 and a zero of P_2 come very close to each other. These approximants are also not utilized in estimating γ.

A. Analysis of 3D series

For the 3D series we construct all approximants with $4\leq M\leq 7$, $2\leq L\leq 5$, and $2\leq J\leq 6$. We discard the defective and ill-behaved approximants. In Table II we give a listing of the well behaved approximants. On the basis of fifteenth- and higher-order approximants we get

TABLE II. Unbiased estimates for w_c and γ for the 3D $\chi_{\rm EA}$ series. The number of terms of the series used in the approximant (L,J,M) is $L+J+M+2$.

L	J	M	w_c	γ
2	4	6	0.410	2.03
2	5	5	0.410	2.04
2	6	4	0.438	2.49
3	3	6	0.436	2.42
3	4	5	0.409	2.02
4	4	4	0.521	4.94
3	4	6	0.426	2.31
4	3	6	0.502	3.25
3	4	7	0.456	2.79
4	3	7	0.473	2.85
4	4	6	0.465	2.84
5	2	7	0.450	2.51
5	3	6	0.465	2.76
5	4	5	0.464	3.47
3	6	6	0.447	2.44
4	4	7	0.482	2.81
4	5	6	0.590	4.07
5	3	7	0.486	2.94
5	4	6	0.465	2.20

$$w_c=0.48\pm0.04\ (T_c=1.2\pm0.1)\ ,\qquad \gamma=2.9\pm0.5\ . \tag{4.7}$$

Here the uncertainty refers to the standard deviation. It should be remarked that this uncertainty gives a measure of the spread in the values of T_c and γ obtained in our analysis. However, since there are no underlying sta-

TABLE III. Biased estimates of γ for the 3D $\chi_{\rm EA}$ series. T_c is biased at 1.175 ($w_c=0.478$). An asterisk indicates a defective approximant as explained in the text. A dagger indicates that a zero of P_1 and a zero of P_2 come very close to each other in the complex plane.

L	J	M	γ
2	4	6	3.11
2	5	5	*
2	6	4	2.91
3	3	6	*
3	4	5	2.92
4	4	4	3.85
3	4	6	*
4	3	6	2.95
3	4	7	3.19
4	3	7	2.92
4	4	6	3.00
5	2	7	3.01
5	3	6	2.94
5	4	5	3.84†
3	6	6	2.60
4	4	7	2.83
4	5	6	3.02
5	3	7	2.90
5	4	6	*

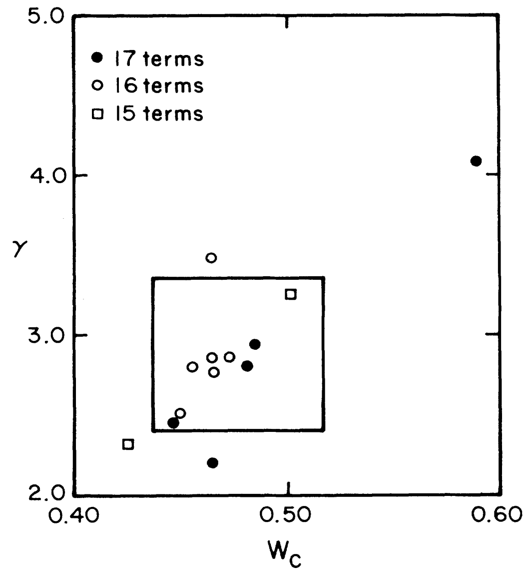

FIG. 1. The unbiased estimates of w_c and γ obtained from the various approximants for 3D χ_{EA} series. The inset indicates the results quoted in the text.

tistical distributions, it is not a measure of the convergence of the approximants. It has been suggested by Baker and Greves-Morris,[11] that the systematic errors in T_c and γ are in the ratio $\delta T_c/T_c$: $\delta\gamma = 1{:}N$, where N is the number of terms of the series used. By this criteria, the uncertainty in γ could be 4 times larger than those quoted.

Our result can be compared with the Monte Carlo calculations[3(c)]

$$T_c = 1.175 \pm 0.025\ ,\quad \gamma = 2.9 \pm 0.3\ . \tag{4.8}$$

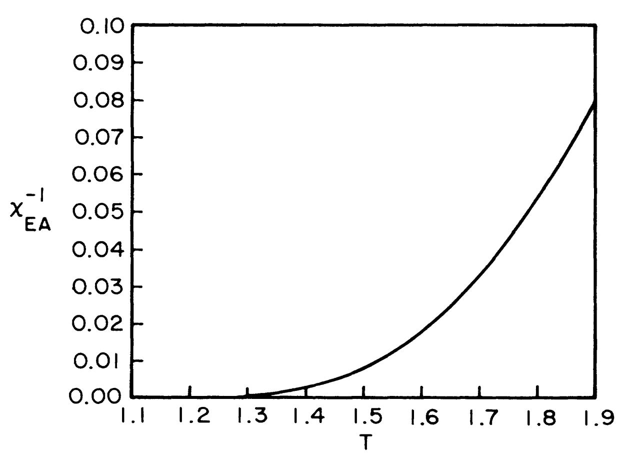

FIG. 2. A representative plot for χ_{EA}^{-1} in three dimensions, obtained by integrating ($L=5$, $M=6$, $J=3$) differential equation in Eq. (4.2) with the boundary condition $f(0)=1$. On the scale of the plot ten different approximants essentially coincide with the curve. The maximum deviation occurs near $T=1.5$.

TABLE IV. Biased estimates of γ' in three dimensions. T_c is biased at 1.175 ($w_c = 0.4/8$).

L	J	M	γ'
3	5	6	1.98
3	6	6	1.98
3	5	7	2.00
4	4	6	1.59
4	4	7	2.00
4	5	6	2.00
5	3	6	2.27
5	3	7	2.06
5	4	6	2.04
6	3	6	1.72

In Table III we present the estimates of γ for T_c biased at 1.175. From these, our estimate, based on fifteenth- and higher-order approximants is

$$\gamma = 2.94 \pm 0.15\ . \tag{4.9}$$

In Fig. 1 we present the scatter of the various approximants for w_c and γ. All the estimates seem to lie on a curve, which suggests that the estimation of w_c is well correlated with the estimation of γ. This is also reflected in the fact that the estimates for γ from biased approximants have a much smaller uncertainty. In Fig. 2 we give a representative plot of the inverse susceptibility as a function of temperature, obtained by integrating the differential equation (4.2).

B. Analysis of the χ' series

In Table IV we present the well-behaved estimates for the exponent γ', which characterizes the divergence of the

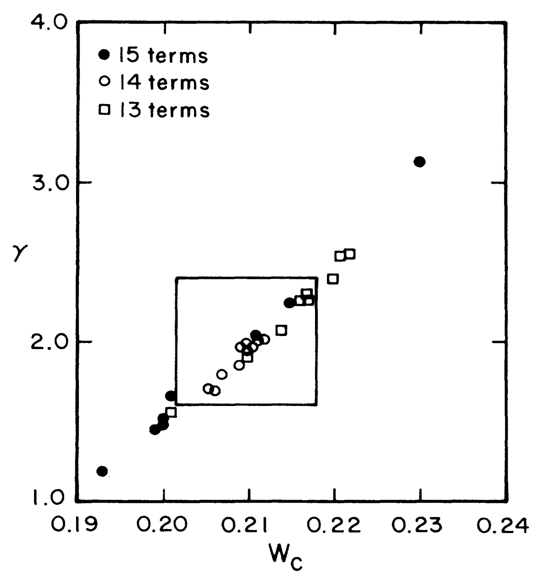

FIG. 3. The unbiased estimates of w_c and γ obtained from the various approximants for the 4D χ_{EA} series. The inset indicates the results quoted in the text.

TABLE V. Unbiased estimates of w_c and γ for the 4D χ_{EA} series. The number of terms of the series used in the approximants (L,J,M) is $L+J+M+2$.

L	J	M	w_c	γ	L	J	M	w_c	γ
2	1	7	0.224	2.62	2	3	7	0.211	2.02
2	2	6	0.205	1.84	2	4	6	0.212	2.02
2	4	4	0.239	3.55	3	1	8	0.206	1.74
3	1	6	0.216	2.26	3	3	6	0.210	1.94
5	1	4	0.238	3.39	4	1	7	0.210	1.95
2	1	8	0.201	1.55	4	2	6	0.210	1.95
2	2	7	0.217	2.27	4	3	5	0.209	1.85
2	3	6	0.214	2.14	5	1	6	0.210	1.95
2	4	5	0.222	2.55	2	2	9	0.200	1.47
3	1	7	0.216	2.26	2	5	6	0.199	1.45
3	2	6	0.217	2.30	2	7	4	0.230	3.14
4	1	6	0.210	1.94	3	2	8	0.215	2.24
4	3	4	0.221	2.53	4	1	8	0.193	1.18
5	1	5	0.220	2.38	4	3	6	0.211	2.03
2	1	9	0.207	1.80	6	2	5	0.201	1.66
2	2	8	0.206	1.74	6	3	4	0.200	1.50

χ' series, with T_c biased at 1.175. We estimate

$$\gamma' = 1.96 \pm 0.19 \; . \tag{4.10}$$

From these we can now obtain the estimates for ν and η as

$$\nu = 1.3 \pm 0.2 \; , \tag{4.11}$$

$$\eta = -0.25 \pm 0.17 \; . \tag{4.12}$$

This can be compared with Monte Carlo result[3(c)]

$$\nu = 1.3 \pm 0.1 \; , \tag{4.13}$$

$$\eta = -0.22 \pm 0.05 \; . \tag{4.14}$$

C. Analysis of the 4D series

For the 4D series we construct all approximants with $4 \le M < 9$, $2 \le L \le 6$, and $1 \le J \le 5$. In Table V we give a listing of all the well-behaved approximants. On the basis of thirteenth and higher-order approximants we estimate[12]

$$w_c = 0.21 \pm 0.01 \; (T_c = 2.02 \pm 0.06) \; , \quad \gamma = 2.0 \pm 0.4 \; . \tag{4.15}$$

Our result is in sharp contrast to the earlier work of Fisch and Harris,[13] who on the basis of a ten term series concluded that $\gamma \to \infty$ in four dimensions. Indeed, we find larger values of γ ($\gamma > 10$) if we consider only ten terms. However, these approximants are mostly defective. The uncertainty in estimating γ from a short series is again a result of incorrect estimation of w_c. In Fig. 3 we present the scatter of w_c and γ. The points again lie on a curve. In Table VI we present the results for the biased estimates for $w_c = 0.208$. It is clear that from twelfth- to fifteenth-order there is good agreement between the estimates. Our

TABLE VI. Biased estimates of γ for the 4D χ_{EA} series. T_c is biased at 2.03 ($w_c = 0.208$). An asterisk indicates a defective approximant as explained in the text.

L	J	M	γ	L	J	M	γ
2	1	7	2.10	2	3	7	1.88
2	2	6	1.92	2	4	6	1.89
2	4	4	*	3	1	8	1.86
3	1	6	1.82	3	3	6	1.84
5	1	4	1.96	4	1	7	1.85
2	1	8	1.84	4	2	6	1.85
2	2	7	1.95	4	3	5	1.79
2	3	6	1.94	5	1	6	1.86
2	4	5	2.03	2	2	9	1.89
3	1	7	1.93	2	5	6	1.88
3	2	6	1.95	2	7	4	*
4	1	6	1.88	3	2	8	1.89
4	3	4	1.91	4	1	8	1.86
5	1	5	1.96	4	3	6	1.74
2	1	9	1.88	6	2	5	*
2	2	8	1.86	6	3	4	*

estimate based on fourteenth- and fifteenth-order approximants is

$$\gamma = 1.855 \pm 0.041 \ . \tag{4.16}$$

D. Analysis of the 2D series

A similar analysis in two dimensions does not show any convergent singularity in the region of interest. We interpret this as an absence of finite temperature transition or an essential singularity at zero temperature. The variable w, then, is not the appropriate expansion variable, as it has an essential singularity at $T=0$. Hence, we express the series in terms of a new variable z $(=1/T^2)$. We estimate γ for the zero-temperature transition using a method due to Baker *et al.*[14] We expect that

$$\lim_{z\to\infty} \chi(z) \to z^{\gamma/2} \tag{4.17}$$

or

$$\lim_{z\to\infty} \frac{d\ln\chi}{d\ln z} = \frac{\gamma}{2} \ . \tag{4.18}$$

We construct a power series in z for the expression $d\ln\chi/d\ln z$. Since we wish to estimate the value of this expression in the limit $z\to\infty$, we must use an analytic continuation of the series that goes to a constant for large z. This is done by a diagonal $[M/M]$ Padé approximant. However, those Padé estimates which develop singularity along the positive real axis, hence hindering the analytic continuation to large z, are ignored. Our estimates from various diagonal Padé approximants are shown in Table VII. This method only uses series of even orders. To get estimates from series of odd orders we consider the function $A(z)$ $(=d\chi/dz)$. Then

$$\lim_{z\to\infty} \frac{d\ln A}{d\ln z} = \frac{\gamma}{2} - 1 \ . \tag{4.19}$$

Our estimates of γ from diagonal $[M/M]$ Padé approximant of $d\ln A/d\ln z$ are also shown in Table VII. Our best guess is

$$\gamma = 5.3 \pm 0.3 \ . \tag{4.20}$$

The uncertainties in our analysis are much larger than the corresponding uncertainties for a series of the same length for the pure Ising model. A better analysis of the problem may require a deeper understanding of the singularities in the complex plane, Griffiths singularities, corrections to scaling, etc.

TABLE VII. Estimate of γ for the zero-temperature transition in two dimensions. Asterisks imply that the Padé approximant has a pole on the real axis. A dagger implies that there is a pole-zero pair on top of each other.

Estimate for the quantity	Order M of the $[M/M]$ Padé approximant	Estimates γ
$\lim_{z\to\infty} \frac{d\ln\chi}{d\ln z}$	6	1.43
	7	14.44
	8	4.14
	9	$4.97^{\dagger}$
$\lim_{z\to\infty} \frac{d\ln A}{d\ln z}$	6	6.02
	7	5.63
	8	*
	9	5.54

V. CONCLUSION

In Sec. II of this paper we laid down several criteria to check the effectiveness of the series analysis. We find that our analysis fulfills many of the criteria well. Although the convergence of the analysis is not good, it still constitutes an evidence in favor of a finite temperature transition in 3D Ising spin glass. We have analyzed two different series, which together give estimates of the exponents ν and η. The agreement with the numerical simulations is good. If we combine the numerical simulations and the series analysis, the scaling relations are shown to hold. In the future, we hope to check the universality of critical exponents by studying the EA susceptibility on other lattices. A preliminary analysis of a 13-term bcc series gives similar results.

To conclude, we would like to summarize our work. We have obtained the critical behavior of Ising spin-glasses in two, three, and four dimensions. We hope our work would stimulate further interest in the use of series analysis in studying the critical behavior of disordered systems.

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Note added. Recently, in an independent work, a novel Monte Carlo simulation by Swendsen and Wang[15] has led to $\chi_{SG} \sim T^{-5.3}$ in two dimensions. This is in agreement with our estimate of γ which is also 5.3.

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