High-temperature series expansion for spin glasses. I. Derivation of the series

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It is shown that the free energy and the Edwards-Anderson susceptibility for Ising spin glasses have a star-graph expansion. With the use of this, a method for generating the high-temperature series for these quantities is described. The scope and the applicability of this method to other properties involving quenched disorder is discussed.

I. INTRODUCTION

In a recent paper[1] the critical behavior of short-ranged Ising spin glasses was studied through the use of high-temperature series expansions. The series were obtained to high orders, comparable to the best available for the pure Ising model. The results of the analysis were in very good agreement with the numerical simulations[2] and provided confirmation of a finite-temperature phase transition in three-dimensional Ising spin glasses. In this paper we wish to discuss in detail the method for generating these series. In a companion paper[3] we shall present a detailed analysis of the series, including estimates of the exponents η and ν which were not given in Ref. 1.

The key ingredient of our calculation, which allows us to carry out the series expansions to high orders, is the existence of star-graph expansions for the free energy and various susceptibilities in zero magnetic field. This method is then implemented on the computer to generate the series for various quantities of interest for Ising spin glasses.

The organization of this paper is as follows. In Sec. II we introduce the method of star-graph expansion. In Secs. III and IV we show the existence of star-graph expansions for the free energy and the Edwards-Anderson susceptibility. In Sec. V we explain how the series is generated on the computer. Sec. VI contains a few brief concluding remarks. There are five appendices supplementing the discussion in the text.

II. SERIES EXPANSION: THE METHOD

A. Basic definitions

The Ising spin glass is described by the Hamiltonian

$$-\beta\mathcal{H}=\sum_{(i,j)} J_{ij}S_iS_j \ . \tag{2.1}$$

Here the sum runs over each nearest-neighbor pair once. S_i is the Ising spin at site i which takes values ± 1. J_{ij} are independent random variables and occur with the probability

$$P(J_{ij})=\tfrac{1}{2}[\delta(J_{ij}-J)+\delta(J_{ij}+J)] \ . \tag{2.2}$$

The free energy (F), the Edwards-Anderson susceptibility (χ_{EA}), and the auxilliary susceptibility (χ') are given by the expressions

$$-\beta F=\ln \operatorname{tr} e^{-\beta\mathcal{H}} \ , \tag{2.3}$$

$$N\chi_{\mathrm{EA}}=\sum_i\sum_j [\langle S_iS_j\rangle^2] \ , \tag{2.4a}$$

$$N\chi'=\sum_i\sum_j ([\langle S_iS_j\rangle^2])^2 \ . \tag{2.4b}$$

Here N is the total number of spins, the angular brackets refer to thermal averaging, and the square brackets refer to averaging with respect to the distribution of J_{ij}. We wish to derive a series expansion for the free energy and the susceptibilities in powers of w [$=\tanh^2(J/kT)$]. The susceptibility χ gives the exponent γ. The auxilliary susceptibility χ' can be used to obtain the exponents η and ν as discussed in the companion paper.[3]

B. Star-graph expansion

The method used for obtaining the series is the star-graph expansion which is extensively discussed in the literature.[4] For the sake of completeness we give here an overview of the underlying concepts and illustrate them in the context of the present problem.

The star-graph expansion is part of a more general method known as the excluded-volume expansion. The basic idea in these methods is the following. We are interested in calculating some extensive thermodynamic quantity A (which could be the free energy or the susceptibility) which is defined through the Hamiltonian in Eq. (2.1). In an excluded volume expansion we reduce the expression for A to the form

$$A=\sum_s W(s) \ . \tag{2.5}$$

The sum is over all subgraphs of the lattice and $W(s)$ is called the weight of the subgraph s. (For a precise definition of W see below.) The usefulness of this expression in obtaining the series expansion for the quantity A is based on the following observations.

(i) The weight of a graph with r edges starts as w^r, i.e., all lower powers of w have zero coefficient.

(ii) The weight of a graph only depends on its topology. It is important to realize that even though the present problem is inhomogeneous, after averaging with respect to the distribution of J_{ij}, homogenity is restored and all bonds become equivalent.

Hence, to obtain the series expansion for A, correct to order r, one needs to consider *all topologically distinct subgraphs* (g) of the lattice ($\mathcal{L}$) with r or less edges. One obtains their weights and the number of ways in which the graph g can be embedded in the lattice $\mathcal{L}$ [also denoted as $(g:\mathcal{L})$] and Eq. (2.5) becomes

$$A(\mathcal{L})=\sum_{g}(g:\mathcal{L})W(g) \ . \tag{2.6}$$

The following additional considerations greatly increase the effectiveness of the method in carrying out the expansion to high orders. A graph is said to have a point of articulation (k) if cutting all the edges incident at k and removing the piece connected to k splits the graph into disconnected parts. A graph with no such point of articulation is called a star graph. In special cases one can show that all disconnected and articulated graphs have zero weight. In this case Eq. (2.6) becomes

$$A(\mathcal{L})=N\sum_{g_s}\lambda(g_s)W(g_s) \ , \tag{2.7}$$

where λ is the lattice constant of the star graph g_s, and is given by the number of ways, per lattice site, that the star graph g_s can be embedded in the lattice. N is the total number of sites.

In the next two sections we shall discuss in detail the calculation of weights for the free energy and the susceptibility. Here we outline how a description in terms of graphs occurs naturally in such problems. In a graphical language each site is represented by a vertex (i) and each interaction J_{ij} is represented by an edge connecting the vertices at i and j. An arbitrary subset of the lattice defined by N_G spins and L_G interactions can be represented by a graph (G) with N_G vertices and L_G edges. Expanding A in terms of v_{ij} ($=\tanh J_{ij}$) would express it as a sum over infinitely many terms. Each term in the expansion can be associated with a graph obtained by drawing all the edges corresponding to the v_{ij} present in the term. In the excluded volume expansions the edges are drawn only once even though a given v_{ij} can occur many times. The sum of all the terms which get associated with a graph is called the weight of the graph. After the averaging with respect to the distribution of J_{ij} they become power series in w [$=\tanh^2(J/kT)$]. The following is clear from this discussion.

(i) Each nonvanishing term in the weight of a graph must get a nonzero power of w from each of the interaction lines present in the graph. Hence if there are r edges in the graph its weight would start as w^r.

(ii) For calculating the weight of a particular graph all the interactions J_{ij} corresponding to the edges not present in the graph can be set to zero.

The calculation then proceeds as follows. We associate a Hamiltonian with a graph (G) given by

$$-\beta\mathcal{H}_G=\sum_{(i,j)}J_{ij}S_iS_j \ . \tag{2.8}$$

Here all the spins and the interactions in the summation belong to G. We now define the quantity $A(G)$ by replacing the full Hamiltonian by that in Eq. (2.8). Furthermore, this quantity can be expressed as

$$A(G)=\sum_{s\subseteq G}W(s) \ . \tag{2.9}$$

Here the sum is over all subgraphs of G including G. [In fact, one can regard Eq. (2.9) as a definition of the weights with the condition that the weight of a subgraph s depends only on s and not on G. One can invert Eq. (2.9) to obtain the weights.] In the case of a star-graph expansion the sum in Eq. (2.9) is restricted to only star subgraphs. To obtain these weights one generates all distinct star graphs, in such a way, that at the time when one is considering a graph (G) weights of all its subgraphs have already been determined. Then one computes the quantity $A(G)$ and subtracts off the weights of all subgraphs to get $W(G)$:

$$W(G)=A(G)-\sum_{s}{}'W(s) \ . \tag{2.10}$$

The prime in the summation excludes the subgraph G. In the next two sections we discuss in detail the calculation of weights for the free energy and the susceptibility series.

III. STAR-GRAPH EXPANSION FOR THE FREE ENERGY

We start with the definition

$$-\beta F=\ln \mathrm{Tr}\exp\sum_{(i,j)}J_{ij}S_iS_j \ . \tag{3.1}$$

We use the identity

$$e^{J_{ij}S_iS_j}=(1+v_{ij}S_iS_j)\cosh J_{ij} \ , \tag{3.2}$$

where $v_{ij}=\tanh J_{ij}$. Using this, Eq. (3.1) becomes

$$-\beta F=L\ln\cosh J+N\ln 2+\ln\frac{1}{2^N}\mathrm{Tr}\prod_{(i,j)}(1+v_{ij}s_is_j) \ . \tag{3.3}$$

Here L is the total number of nearest-neighbor bonds. The factor of 2^N is introduced for normalization. The first two terms are smooth and we shall obtain a series expansion for the last term

$$F_1=\ln\frac{1}{2^N}\mathrm{Tr}\prod_{(i,j)}(1+v_{ij}s_is_j) \ . \tag{3.4}$$

As discussed earlier we define $F_1(G)$ for graph G by restricting to the sites and bonds in the graph G:

$$F_1(G)=\ln\frac{1}{2^{N_G}}\mathrm{Tr}\prod_{(i,j)}^{L_G}(1+v_{ij}s_is_j) \ . \tag{3.5}$$

It is evident that expanding the logarithm after calculating the trace in Eq. (3.4) would give rise to a power series in v_{ij}. The averaging over the bonds can be done by using the relations

$$\begin{aligned}\int dJ_{ij}P(J_{ij})v_{ij}^n&=0 \quad \text{for } n \text{ odd}\\ &=w^{n/2} \quad \text{for } n \text{ even} \ .\end{aligned} \tag{3.6}$$

This gives a power series in w. Since all the interactions become equivalent after the averaging, the weight of a graph is completely determined by specifying its topology.

It is also useful to keep in mind the definition of weights as a multivariable expansion. If we treat each w arising from the different bonds as a separate variable w_{ij} then we can perform a multivariable expansion for F_1 in Eq. (3.4). The weight of a subgraph is a sum of all those terms in this expansion for which the powers of all w_{ij} corresponding to the bonds in the subgraph is nonzero and the powers of all the other w_{ij} is zero.

Let us now consider any disconnected or articulated graph G which can be decomposed as $G=G_1\cup G_2$, where G_1 and G_2 have no edges in common. We show in Appendix A that

$$F_1(G)=F_1(G_1)+F_1(G_2) \ . \tag{3.7}$$

From the definition of the weights in terms of the multivariable expansion it is now evident that there are no terms on the right-hand side of Eq. (3.7) which have nonzero powers of all the bonds in the graph G. Hence, the weight of G is zero. Therefore, we have shown, that the free energy has a star graph expansion.[5]

IV. STAR-GRAPH EXPANSION FOR THE INVERSE EDWARDS-ANDERSON SUSCEPTIBILITY

The existence of a star-graph expansion for the inverse Edwards-Anderson susceptibility ($\chi_{\rm EA}^{-1}$) follows from a more subtle argument. If one uses the straightforward definition for the weights in terms of a multivariable expansion, the weights of articulated graphs appearing in the expansion of $\chi_{\rm EA}$ will not vanish. However, there exists a star-graph expansion for the *inverse* of the Edwards-Anderson susceptibility. This is because, in this case, for any homogeneous graph, where all sites are equivalent, the weights of all articulated subgraphs add up to zero. The proof of the existence of the star-graph expansion for the inverse of the Edwards-Anderson susceptibility, and its calculation are similar to those of pure Ising model.[6] We shall first construct a quantity that has a star-graph expansion, and then show that, for a homogeneous graph, this quantity is the true inverse of the Edwards-Anderson susceptibility.

To this purpose, for any graph G, let us define a matrix M with elements

$$M_{ij}=[\langle s_i s_j\rangle^2] \ . \tag{4.1}$$

Here i,j run over the sites in the graph and all the interactions not in the graph are set to zero. We associate an amplitude ψ with the graph (N_G is the number of vertices)

$$\psi_G=\sum_i\sum_j (M^{-1})_{ij}-N_G \ . \tag{4.2}$$

We shall show that the quantity ψ has a star-graph expansion. That is if we express ψ_G as

$$\psi_G=\sum_{g\subseteq G} W_\psi(g) \ , \tag{4.3}$$

where the sum runs over all subgraphs of G including G. Then the weights of all articulated and disconnected graphs vanish.

The weights can again be defined in terms of the multivariable expansion as before. In Appendix B we prove that for any articulated graph $G=G_1\cup G_2$ where G_1 and G_2 have no edges in common

$$\psi_G=\psi_{G_1}+\psi_{G_2} \tag{4.4}$$

which is again sufficient to ensure that the weight of G is zero. It should be remarked that our proof in Appendix B is entirely parallel to the corresponding proof for the star-graph expansion of the inverse susceptibility for the pure Ising model given by Rapaport.[6]

We now show that for an infinite lattice, which is a homogeneous graph with N equivalent sites, the inverse of the Edwards-Anderson susceptibility is given by

$$N\chi_{\rm EA}^{-1}=\sum_i\sum_j (M^{-1})_{ij} \ . \tag{4.5}$$

To see this we substitute Eq. (4.1) into Eq. (2.4a) to get

$$N\chi_{\rm EA}=\sum_i\sum_j M_{ij} \ . \tag{4.6}$$

For the infinite lattice

$$\sum_j [\langle s_i s_j\rangle^2]=\chi_{\rm EA} \ . \tag{4.7}$$

After averaging over the distribution of J_{ij} all sites become equivalent and the left-hand side does not depend on i.

It is convenient to use a matrix notation. Equations (4.6) and (4.7) can be written as

$$N\chi_{\rm EA}=\mathbb{1}^T M\mathbb{1} \ , \tag{4.8}$$

$$M\mathbb{1}=\chi_{\rm EA}\mathbb{1} \ . \tag{4.9}$$

Where $\mathbb{1}$ is an N-component vector given by

$$\mathbb{1}=\begin{pmatrix}1\\1\\1\\ \vdots\\1\end{pmatrix} \ . \tag{4.10}$$

Multiplying Eq. (4.9) by the inverse of the matrix M from the left we get

$$M^{-1}M\mathbb{1}=\chi_{\rm EA}M^{-1}\mathbb{1} \tag{4.11}$$

which then leads to

$$\mathbb{1}^T M^{-1}M\mathbb{1}=\chi_{\rm EA}\mathbb{1}^T M^{-1}\mathbb{1} \ . \tag{4.12}$$

Writing this explicitly gives

$$N=\chi_{\rm EA}\sum_{i,j}(M^{-1})_{ij}$$

or (4.13)

$$N\chi_{\rm EA}^{-1}=\sum_{i,j}(M^{-1})_{ij}$$

which is the required relation. Combining this with Eq. (4.2) and Eq. (4.3) gives

$$N\chi_{\rm EA}^{-1}=N\sum_{g_s}\lambda(g_s)W_\psi(g_s)+N \tag{4.14}$$

or

$$\chi_{\mathrm{EA}}^{-1}=1+\sum_{g_s}\lambda(g_s)W_{\psi}(g_s) . \tag{4.15}$$

Hence we have shown that there is a star-graph expansion for the inverse of the Edwards-Anderson susceptibility. It is also clear from our proof that for any susceptibility of the type

$$\chi'=\sum_i\sum_j [\langle s_i s_j\rangle^m]^n \tag{4.16}$$

with arbitrary distribution of J_{ij}, and any $m,n,(\chi')^{-1}$ will have a star-graph expansion.

V. GENERATING THE SERIES

The following steps were necessary to generate the series on the computer.

All possible topologies in order of increasing cyclomatic number (c) starting with $c=2$, were enumerated.

All possible star graphs that can be embedded in the lattice were generated as realizations of the topologies.

The lattice constant for each star graph was calculated.

The star subgraphs for all star graphs with nonzero lattice constant were found.

The weights of all the star graphs to a given order in w were evaluated.

Once the lattice constants and the weights are known the power series is immediately obtained from Eq. (2.7). We shall discuss each of these steps in some detail.

A. Generating topologies

The cyclomatic number (c) of a graph with V vertices and E edges is defined as $c=E-V+1$. The valency of a vertex in a graph is defined as the number of edges meeting at that vertex.

The process of generating all star graphs is carried out in two stages. First the bare topologies, in which vertices of valency 2 are suppressed, are obtained. These topologies consist of nodes, which are vertices of valency greater than 2, and lines connecting the nodes which are called bridges. In this subsection we shall discuss the generation of topologies. In Sec. V B we shall discuss how star graphs are obtained from such topologies.

All topologies of cyclomatic number c can be obtained from topologies of cyclomatic number $c-1$ by one or more of the following three basic operations.[7] A topology with n nodes and b bridges can be generated by the following.

(i) Adding a bridge to connect any two nodes in a topology with n nodes and $b-1$ bridges.

(ii) Adding a bridge to connect one of the nodes to the midpoint of one of the bridges in a topology with $n-1$ nodes and $b-2$ bridges.

(iii) Adding a bridge to connect the midpoints of any two bridges in a topology with $n-2$ nodes and $b-3$ bridges. The two bridges can also be the same.

Our strategy for generating all possible topologies is as follows. The unique topology for $c=2$ is first initialized. We then proceed in order of increasing cyclomatic number starting with $c=3$. For a fixed c we proceed in order of increasing number of nodes. To generate topologies with n nodes and b bridges we repeat the operations in (i), (ii), and (iii) with all the topologies of cyclomatic number $c-1$. In this process a large number of duplicates are generated which have to be removed (to be discussed below). The topologies, which are not duplicates, are stored for generating topologies with cyclomatic number $c+1$.

1. Condition for stopping

In order to calculate the series to a given power of w (say N_{order}) we need to find all star graphs with N_{order} or less edges. It is possible to estimate the highest cyclomatic number one needs to consider, in order not to miss any relevant star graph. We can also define a recursive criteria which will guarantee that there are no more star graphs left. This criteria is as follows. In order for a star graph with cyclomatic number c to exist, which can be embedded in the lattice, there must exist a star subgraph with cyclomatic number $c-1$, which can be embedded in the lattice. Furthermore, this star subgraph must have number of edges less than N_{order}. Hence, for the purpose of generating topologies of cyclomatic number c, we need to consider only those topologies of cyclomatic number $c-1$, which have star realizations, that can be embedded in the lattice, with number of edges less than N_{order}. This process would automatically stop when all star graphs with number of edges less than N_{order} are exhausted. In practice, by careful inspection, it is possible to terminate this generation of topologies a little earlier and not to miss any star graphs.

The topologies are stored in the computer by their adjacency matrix which is defined as follows:

$$A_{ij}=\text{number of bridges connecting nodes } i \text{ and } j \tag{5.1}$$

2. Lexicographic ordering of the adjacency matrix

For a topology with n nodes there are $n!$ ways of numbering the nodes which would give rise to $n!$ adjacency matrices. We define here an ordering which uniquely picks out one of these as the adjacency matrix for the topology. We first define an ordering on $N\times N$ matrices. The matrix C is defined to be greater than matrix $B(C>B)$ if for some k,l

$$C_{ij}=B_{ij} \quad \text{for } 1\le i\le k-1 \text{ and all } j ,$$
$$C_{kj}=B_{kj} \quad \text{for } 1\le j\le l-1 , \tag{5.2}$$

and

$$C_{kl}>B_{kl} .$$

Furthermore, two matrices are equal if and only if all their elements are equal. One can verify that this is a well-defined binary relation and it orders all $N\times N$ matrices.

We now define the adjacency matrix of the topology to be that matrix which among the $n!$ choices is the greatest according to the above ordering scheme. This scheme would be called the canonical ordering for a set of matrices.

3. Removal of duplicate topologies

With the above definition of the adjacency matrix, two topologies are duplicates of each other, if and only if their adjacency matrices are equal. Hence, for every new topology that is generated, we check its adjacency matrix against all previously stored ones with the same number of nodes and bridges. If it is equal to any one of the previous ones it is discarded. Otherwise it is stored as a new topology. Once a new topology has been generated we obtain various star graphs from that topology.

B. Generating star graphs as realizations of a topology

Generating star graphs from the topologies amounts to putting back vertices of valency 2 in the various bridges. This can be thought of as an assignment of lengths to the bridges. For a bridge of length l there are $l-1$ valency-2 vertices between the nodes it connects. Once the lengths have been assigned to all the bridges one obtains a star graph with N_{tot} number of vertices and N_{edge} number of edges. Where, for a topology with n nodes and b bridges

$$N_{\text{tot}}=n+\sum_{\text{bridges}}(l_{\text{bridge}}-1)\ , \tag{5.3a}$$

$$N_{\text{edge}}=\sum_{\text{bridges}} l_{\text{bridge}}\ . \tag{5.4b}$$

After the lengths are assigned these star graphs are used for finding the lattice constants and the weights. Hence the process of length assignment must take care of the following.

(i) Bridges should be ordered in a well-defined sequence so that the length assignments are in one to one correspondence with them. The data structure should be such that the lattice constant and the weight programs can be easily developed.

(ii) Each star graph should be considered exactly once and all duplicates must be removed.

(iii) In order to reduce the computer time it is useful to eliminate as many star graphs with zero lattice constant as possible before sending them to the lattice-constant program. We discard all star graphs which have closed loops of odd length, since such a graph cannot be embedded in a cubic lattice.

1. Assignment of length to the bridges

The assignment of lengths to all the bridges, connecting the same pair of nodes, was done as a single set; let $r=A_{ij}$. This represents a set of r bridges connecting the nodes i and j. Lengths were assigned to these bridges from a predetermined set of r-tuple of integers. This predetermination was done on the basis of the following considerations.

(i) Let us denote the r-tuple of integers by the set $\{l_i\}$. These integers were ordered as $l_1 \le l_2 \le l_3 \le \cdots \le l_r$. Since the bridges are indistinguishable, these sets of integers are sufficient to cover all possible length assignments.

(ii) The integers in a set are either all odd or they are all even. Since we are interested in embedding the star graph in a cubic lattice, there cannot be a graph with two bridges connecting the same pair of nodes, one of which is even and the other one odd.

(iii) Only one of the integers in the set was allowed to be of length one. It is obvious that there can be at most only one bridge of length one between any pair of nodes.

(iv) For a given order of the series, the maximum length of a single bridge can be easily deduced. All the integers were required to be less than or equal to this maximum integer (say $l_{\max}$).

The ordering of the bridges was done as follows. One visits the upper triangle of the adjacency matrix in a row-by-row sequence. Bridges are ordered as they are encountered. Suppose that i_{br} number of bridges have been encountered just before one reaches the (ij)th element of the matrix; let $A_{ij}=r$. Then bridges labeled $i_{\text{br}}+1$ to $i_{\text{br}}+r$ would be understood as connecting nodes i and j. The length assignment to these r bridges was done from the predetermined set of "r-tuples." It was done by a back-tracking algorithm that discarded partially completed length assignments, when there was no possibility of getting a completed star graph out of it, with number of edges less than or equal to N_{order}.

2. Loops of odd length

Subgraphs of the star graph were determined where each vertex had valency 2. If the total number of edges in such a subgraph is odd that star graph cannot be embedded in a cubic lattice and is hence discarded.

3. Removing duplicates of star graphs

The process of generating star realizations from a topology again gives rise to duplicates. In order to remove these duplicates we define a star adjacency matrix (SAM) A_s as follows. Let there be r bridges connecting nodes i and j. Also, let the maximum allowed length of the bridges ($l_{\max}$) be less than L (say $L=100$). We have already seen that the lengths of the bridges are in order

$$l_1 \le l_2 \le l_3 \cdots \le l_r\ . \tag{5.4a}$$

We define the (ij)th element of the star adjacency matrix as

$$(A_s)_{ij}=l_1L^{r-1}+l_2L^{r-2}+\cdots+l_r\ . \tag{5.4b}$$

Furthermore, this SAM can be brought to canonical order by the permutation of the nodes. Now, two star graphs would be identical if their star adjacency matrices are equal. Hence, once again, one can compare any star graph with the ones previously obtained and remove the duplicates.

Moreover, this star adjacency matrix would be useful in finding the symmetry factor of the star graph and hence we explain it through an example. Consider the topology in Fig. 1(a). Its adjacency matrix is

$$A=\begin{bmatrix} 0 & 2 & 1 & 0 \\ 2 & 0 & 0 & 1 \\ 1 & 0 & 0 & 2 \\ 0 & 1 & 2 & 0 \end{bmatrix}\ . \tag{5.5}$$

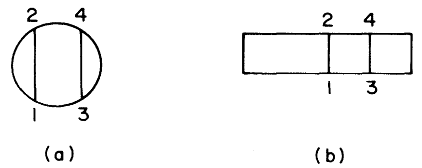

FIG. 1. (a) A typical topology; (b) a star graph obtained as a realization of the topology in Fig. 1(a).

The assignment of lengths (1,5,1,1,1,3) to the bridges gives the star in Fig. 1(b). This has the star adjacency matrix

$$A_s=\begin{bmatrix} 0 & 105 & 1 & 0 \\ 105 & 0 & 0 & 1 \\ 1 & 0 & 0 & 103 \\ 0 & 1 & 103 & 0 \end{bmatrix} . \tag{5.6}$$

We now show that there is a one to one correspondence between a star graph and its star adjacency matrix. By construction, for a given star graph, there is a unique star adjacency matrix. We will show the mapping from a star adjacency matrix to a star graph. This mapping requires one to obtain the adjacency matrix and the lengths of all the bridges from a given star adjacency matrix. Let $(A_s)_{ij}=p$. If $p=0$, then $r=0$; Otherwise, from Eq. (5.4b) the integer part of the logarithm of p, with respect to the base L, is $r-1$. Hence r is easily obtained. To get the length of the r bridges we can use

$$l_{r-i+1}=\left[\frac{p-\sum_{j=1}^{i-1} l_{r-j+1}L^{j-1}}{L^{i-1}}\right] \mathrm{mod} L \quad \text{for } i=1,\ldots,r . \tag{5.7}$$

Notice that the requirement that L be greater than the maximum allowed length of the bridges ($l_{\max}$) is important for the inverse mapping to be defined. This mapping proves the one to one correspondence between a star graph and its star adjacency matrix.

We define the symmetry factor of a star graph as the number of permutation of the nodes which leaves the star adjacency matrix unchanged. One can verify that the symmetry factor for the graph in Fig. 1(b) is 2. We shall make use of the symmetry factor in calculating the lattice constants.

C. *Lattice constant of a star graph*

The calculation of lattice constants is the most time consuming part in generating high-temperature series expansions. Hence, there is a need for an efficient algorithm. We used a modified version of an algorithm due to Martin.[8] We shall discuss here its salient features, especially those which are not discussed in the article by Martin.

We first generated all self-avoiding walks (SAW's) on the lattice up to a given length (say L). Details of how SAW's are generated is discussed in Appendix D. One can regard these SAW's as rigid segments that can be embedded in the lattice. The lattice constant of a given star graph was found through the following steps.

(i) All bridges with length greater than L were broken into parts which were smaller than or equal to L by promoting some of the ordinary vertices with valency 2 to nodes.

(ii) The bridges were ordered in a well-defined sequence through the adjacency matrix as discussed earlier. If a bridge is broken up, it is useful to accommodate the various parts as consecutive bridges.

(iii) The nodes were assigned distinct positions on the lattice. We will discuss this later.

(iv) Bridges were then fitted as rigid segments between the nodes. It is clear, from the way we have defined the problem, that there should not be any crossing of these segments. This was checked explicitly at each step, in our algorithm.

(v) By carefully exhausting all possible lattice positions for the nodes and all possible SAW's for the bridges we found all possible embeddings of a star in the lattice.

(vi) Certain symmetries of the graph lead to multiple counting. This is taken into account by dividing by the symmetry factor. We shall discuss this further, later.

1. *Assignment of node positions*

Let us denote the points on the d-dimensional cubic lattice by a set of d integers which we call $\mathbf{I}\equiv(i_1,\ldots,i_d)$. The first node is placed at the origin $\mathbf{0}$. Lattice positions are assigned to the other nodes in the order as they first arise as the endpoints of the bridges. The bridges are considered in the sequence defined earlier. It is important that when one is considering the lth bridge one of its endpoints has already been assigned a position. This point we shall call the starting point of the bridge and the other point the endpoint. The previously discussed ordering of the bridges is consistent with this requirement. In case this requirement is not met, one has to redo the lattice constant of that star graph, as this is essential to our algorithm. The endpoint is now assigned to all possible positions on the lattice with the criteria which we now define. Let us denote the starting point of the bridge by $\mathbf{I}$ and the endpoint of the bridge by $\mathbf{J}$. We define the distance between the two points as

$$d_{JI}=|j_1-i_1|+|j_2-i_2|+\cdots+|j_d-i_d| . \tag{5.8}$$

We require that the point $\mathbf{J}$ must be such that (a) $d_{JI}\le$ length of the bridge; (b) d_{JI} is even or odd depending on whether the length of the bridge is even or odd. We use a binary variable $I_{\text{taken}}(\mathbf{I})$ which takes values 1 or 0 depending on whether its argument $\mathbf{I}$ has already been assigned to a vertex or not. This allows us to check that no lattice site is assigned more than once to the nodes or vertices.

Exhaustive assignment of lattice sites to the nodes and then rigid segments (SAW's) to the bridges was done through a backtracking algorithm. In this process one orders the lattice sites and the SAW's and assigns them to various nodes and bridges one by one. At intermediate stages one determines when a partially completed assignment cannot lead to an embedding of the star in the lattice. One then traces a step backward by removing the last assignment, starting, once again, the exhaustive search. For a flow diagram of the basic algorithm the reader is referred to the article by Martin.[7]

2. Symmetry factors and multiple counting

Lattice constant is defined as the number of embeddings of a star in the lattice, per lattice site. The following symmetry transformations of the star give rise to multiple counting.

(i) If there is a permutation of nodes which leaves the star graph unchanged, then an assignment of node positions only differing by this permutation, would lead to multiple counting in our program. To illustrate this point we give two examples. Consider the diagram in Fig. 2(a), to be embedded in a two-dimensional square lattice. There are four different labelings of the vertices which leave the star adjacency matrix unchanged. These are shown in Fig. 2(b). In the process of counting all four of the assignments shown in Fig. 2(c) would be counted. However, all these diagrams are simple translations of each other and should be counted only once. Hence we need to divide the count by the symmetry factor.

Another example is shown in Fig. 3(a). In this case there is a symmetry transformation which does not involve node 1. The two labelings which leave the star adjacency matrix unchanged are shown in Fig. 3(b). In Fig. 3(c) we show the assignments of node positions which lead to multiple counting. Once again the correct lattice constant is obtained by dividing the count by the symmetry factor.

(ii) There is one other symmetry that can lead to multiple counts. This happens when two or more bridges of equal length connect the same pair of nodes. An example is given in Fig. 3(d). For a given assignment of node positions (as shown) the SAW segment corresponding to bridges A and B can be interchanged. If there are r bridges of equal length this would give rise to a multiple count of $r!$. However, this is avoided by using a suitable ordering of the bridges.

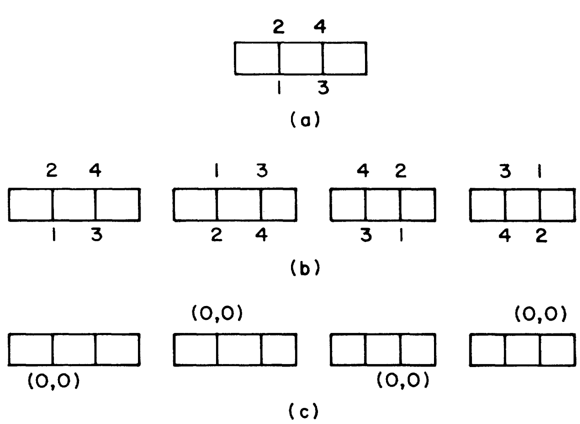

FIG. 2. (a) A star graph; (b) four different labelings of the vertices which leave the star adjacency matrix unchanged; (c) four different counts for the lattice constant which should have been counted only once.

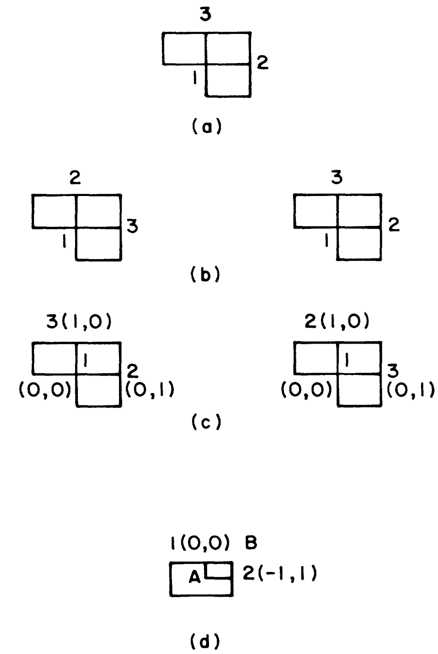

FIG. 3. (a) Another star graph; (b) two different labelings which leave the star adjacency matrix unchanged; (c) two different counts for the lattice constant which should have been counted only once; (d) an example of a star graph where two bridges of equal length, called A and B, connect the same pair of nodes, labeled 1 and 2.

D. Determination of star subgraphs

For a given star graph with n nodes and b bridges, the possible star subgraphs would be made out of b_1 of its bridges ($b_1 < b$). The procedure we use for determining star subgraphs is as follows.

(i) We generate all subgraphs which consist of b_1 bridges ($1 < b_1 < b$) of the star graph. A method for doing this is discussed in Appendix E.

(ii) For a given subgraph we determine the valency of all the vertices in the subgraph. We only need to consider the nodes of the star graph as all the other vertices have valency 2.

(iii) We discard subgraphs which have a node with valency 1, as these cannot be star graphs.

(iv) Subgraphs for which all the vertices are of valency 2, we determine whether or not they are connected. If such a subgraph is connected then it is a star subgraph. (It is a polygon whose length can be readily determined.) If it is not connected then it is not a star graph and is discarded.

(v) For all other subgraphs, let the number of vertices with valency greater than 2 be m. This means that the other $(n-m)$ nodes of the star graph have either valency 0 (in which case they can be ignored) or they have valency 2. In the latter case they are not nodes of the subgraph and one must redefine the bridges such that each of these nodes of valency 2 become ordinary vertices. We illustrate this by an example. Consider the star graph with

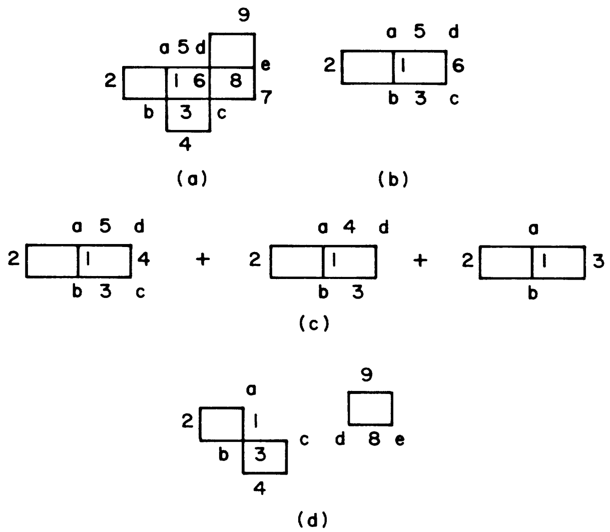

FIG. 4. (a) A given star graph where the letters label the nodes and the numbers label the bridges; (b) a subgraph of the star graph in Fig. 4; (c) relabeling the nodes of this subgraph; (d) a subgraph which has a closed loop and hence is not a star subgraph.

five nodes and nine bridges shown in Fig. 4(a). Its subgraph in Fig. 4(b) has only two nodes and three bridges. Nodes c and d have valency 2. They are ordinary vertices in the subgraph. In order to compare this subgraph with a list of star graphs a redefinition of nodes is needed. First, the node c is removed by redefining a single bridge connecting b to d. Then node d is removed redefining a bridge connecting b to a. This operation is shown in Fig. 4(c). In the process of redefining bridges one can also encounter close loops. An example is given in Fig. 4(d). Such subgraphs are however not star graphs and hence are discarded.

Once the nodes of the subgraphs have been identified, and the bridges redefined, one can construct the star adjacency matrix for the subgraph. This can now be compared with a list of all stars with that many nodes and bridges. If it agrees with one of them, one has identified the subgraph as a particular star graph. If it does not agree with any of them then it cannot be a star graph and hence is discarded.

E. Calculation of the weights for χ_{EA}

We are given a star graph in terms of its adjacency matrix, lengths of the various bridges, and a list of all its star subgraphs. We wish to compute the weight of the star graph. As discussed earlier, we first need to construct the matrix

$$M_{ij} = [\langle s_i s_j \rangle^2]_J$$

This is done through the following steps.

(i) We first calculate the trace

$$A' = \mathrm{Tr} \prod_{(l,m)} (1 + v_{lm} s_l s_m) = \sum_{s_g} \prod_{(i,j)}^{s_g} v_{ij} \ . \tag{5.9}$$

The sum s_g runs over all subgraphs in which all vertices are of even valency. It includes the empty subgraph.

(ii) We then store the expression

$$A = \left[\frac{1}{A'} \right]^2 = \sum_{s_g} \left[\prod_{(i,j)}^{s_g} v_{ij} \right] P_{s_g}(w) \ . \tag{5.10}$$

Here $P_{s_g}(w)$ is a power series in w obtained by setting $v_{ij}^2 = w$ for all v_{ij}.

(iii) For every pair of vertices (i,j) we then obtain the expression for

$$B = \left[\mathrm{Tr} s_i s_j \prod_{(l,m)} (1 + v_{lm} s_l s_m) \right]^2 \tag{5.11}$$

in the form

$$B = \sum_{s_g} \left[\prod_{(i,j)}^{s_g} v_{ij} \right] Q_{s_g}(w) \ . \tag{5.12}$$

The calculation of the trace gives rise to graphs in which all nodes have even valency, except the nodes i and j, which have odd valency. However, squaring the trace leaves all the nodes with even valency. Hence, once again the sum s_g runs over the same set of graphs in which all vertices are of even valency, and Q_{s_g} is a power series in w.

(iv) We now obtain the quantity

$$M_{ij} = [AB]_J = \sum_{s_g} w^{n_g} P_{s_g}(w) Q_{s_g}(w) \ , \tag{5.13}$$

where n_g is the number of edges in the subgraph s_g.

(v) Once M_{ij} is known as a power series in w we need to find its inverse. This is done as follows. The matrix M equals

$$M = I + X \ , \tag{5.14}$$

where I is the identity matrix and X goes to zero for $w = 0$. That is $X = O(w)$.

Hence

$$M^{-1} = (I - X)(I + X^2)(I + X^4)(I + X^8)(I + X^{16}) + O(w^{32}) \ . \tag{5.15}$$

Hence, by a sequence of matrix multiplications M^{-1} can be obtained.

(vi) Taking the trace over M^{-1}, we obtained the amplitude ψ_G. The weight of the subgraphs can now be subtracted to obtain the weight of the star graph.

In Appendix C we give an example for the calculation of such a weight.

VI. CONCLUSION

In this paper we have shown that the method of star graph expansion can be used to generate long series for Ising spin glasses. Since most of the proofs are quite general, it is easy to see that the method is applicable to a wide class of problems involving quenched disorder. In particular, the following problems can be studied.

(1) Ising spin glasses with arbitrary symmetric distributions.

(2) Ising spin glasses with asymmetric distributions.

(3) Vector spin glasses, such as X-Y or Heisenberg spin glasses.

(4) Disordered antiferromagnets.

A major limitation of this method is that it is not applicable to problems involving a finite magnetic field.

In a companion paper,[3] we shall analyze different series for Ising spin glasses. In the future, we hope, this method would be used to study other problems mentioned above.

ACKNOWLEDGMENTS

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APPENDIX A

We wish to show that for a graph G with N_G vertices and L_G edges articulated at a point t (see Fig. 5)

$$F_1(G)=F_1(G_1)+F_1(G_2)\ , \tag{A1}$$

where $G=G_1\cup G_2$. G_1 has N_1 vertices and L_1 edges and G_2 has N_2 vertices and L_2 edges. And $F_1(G)$ is defined as

$$F_1(G)=\ln\frac{1}{2^{N_G}}\mathrm{Tr}\prod_{(i,j)}^{L_G}(1+v_{ij}s_is_j)\ . \tag{A2}$$

We consider the trace

$$\frac{1}{2^{N_G}}\sum_{s_i=\pm 1}\prod_{(i,j)}^{L_G}(1+v_{ij}s_is_j)\ . \tag{A3}$$

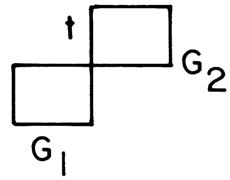

FIG. 5. A graph (G) articulated at t which can be decomposed as $G=G_1\cup G_2$, where G_1 and G_2 have no bonds in common.

We fix the spin at t to be summed over at the end, giving

$$\frac{1}{2^{N_G}}\sum_{s_t}\left[\sum_{s_i}{}'\prod_{(i,j)}^{L_G}(1+v_{ij}s_is_j)\right]\equiv\frac{1}{2^{N_G}}\sum_{s_t}\sum_{s_i}{}'A(s_t,s_i)\ . \tag{A4}$$

The primed summation excludes the spin at site t. We shall now show that $\sum_{s_i}' A(s_t,s_i)$ is independent of s_t. To see this we observe that A is invariant under the transformation $s_i\to -s_i$ and $s_t\to -s_t$. Hence

$$\sum_{s_i=\pm 1}A(s_t,s_i)=\sum_{s_i=\pm 1}A(-s_t,-s_i)\ . \tag{A5}$$

However, we can redefine the summation variable $\sigma_i=-s_i$ giving

$$\sum_{s_i=\pm 1}A(s_t,s_i)=\sum_{\sigma_i=\pm 1}A(-s_t,\sigma_i)\ . \tag{A6}$$

Hence the sum is independent of whether s_t equals $+1$ or -1. Furthermore, since the subgraphs G_1 and G_2 have no bonds in common the expression in Eq. (A4) becomes

$$\frac{1}{2^{N_G}}\sum_{s_t}\left[\sum_{s_i\subset G_1}{}'\prod_{(i,j)}^{L_1}(1+v_{ij}s_is_j)\right]\left[\sum_{s_i\subset G_2}{}'\prod_{(i,j)}^{L_2}(1+v_{ij}s_is_j)\right]\ . \tag{A7}$$

By the same argument as before each of the terms within brackets is independent of s_t. Hence the summation over s_t can be separately done for both of them giving

$$\frac{1}{2^{N_G+1}}\left[\sum_{s_i\subset G_1}\prod_{(i,j)}^{L_1}(1+v_{ij}s_is_j)\right]\left[\sum_{s_i\subset G_2}\prod_{(i,j)}^{L_2}(1+v_{ij}s_is_j)\right]=\left[\frac{1}{2^{N_1}}\sum_{s_i\subset G_1}\prod_{(i,j)}^{L_1}(1+v_{ij}s_is_j)\right]\left[\frac{1}{2^{N_2}}\sum_{s_i\subset G_2}\prod_{(i,j)}^{L_2}(1+v_{ij}s_is_j)\right]\ . \tag{A8}$$

Using this, the logarithm in (A2) becomes

$$F_1(G)=F_1(G_1)+F_1(G_2) \tag{A9}$$

which is the desired relation.

APPENDIX B

In this appendix we wish to show that for the articulated graph in Fig. 5

$$\psi_G=\psi_{G_1}+\psi_{G_2}\ . \tag{B1}$$

We begin by showing the relations

$$\begin{aligned}\langle s_is_j\rangle&=\langle s_is_j\rangle_1\quad\text{for } i,j\in G_1\\&=\langle s_is_j\rangle_2\quad\text{for } i,j\in G_2\\&=\langle s_is_t\rangle_1\langle s_ts_j\rangle_2\quad\text{for } i\in G_1,\ j\in G_2\ ,\end{aligned} \tag{B2}$$

where the subscript 1 (or 2) refers to the corresponding expression obtained for graphs G_1 (or G_2). We prove the last relation for $i\in G_1, j\in G_2$. The other proofs are identical:

$$\langle s_is_j\rangle=\frac{\sum_{s_k=\pm 1}s_is_j\exp\sum_{(l,m)}J_{lm}S_lS_m}{\sum_{s_k=\pm 1}\exp\sum_{(l,m)}J_{lm}S_lS_m}$$

$$=\frac{\frac{1}{2^{N_G}}\sum_{s_k=\pm 1}s_is_j\prod_{(l,m)}(1+v_{lm}s_ls_m)}{\frac{1}{2^{N_G}}\sum_{s_k=\pm 1}\prod_{(l,m)}(1+v_{lm}s_ls_m)}\ , \tag{B3}$$

where we have used Eq. (3.2) and canceled the cosine hyperbolic term in the numerator and the denominator. We have already shown in Appendix A that the denominator factorizes into a product. To see that the numerator also factorizes we fix s_t to be summed at the end giving

$$\frac{1}{2^{N_G}}\sum_{s_t=\pm 1}\left[\sum_{s_k=\pm 1}{}' s_i s_j \prod_{(l,m)}(1+v_{lm}s_l s_m)\right] . \tag{B4}$$

By the argument of Z_2 symmetry, as before, the quantity within brackets is again independent of s_t. This leads to the expression

$$\frac{1}{2^{N_G}}\sum_{s_t=\pm 1}\left[\sum_{s_k\subset G_1}{}' s_i \prod_{(l,m)}^{L_1}(1+v_{lm}s_l s_m)\right]\left[\sum_{s_k\subset G_2}{}' s_j \prod_{(l,m)}^{L_2}(1+v_{lm}s_l s_m)\right] \tag{B5}$$

Now the quantities within brackets are not z_2 invariant. However we can multiply the expression by s_t^2 $(=1)$ giving

$$\frac{1}{2^{N_G}}\sum_{s_t=\pm 1}\left[\sum_{s_k\subset G_1}{}' s_i s_t \prod_{(l,m)}^{L_1}(1+v_{lm}s_l s_m)\right]\left[\sum_{s_k\subset G_2}{}' s_j s_t \prod_{(l,m)}^{L_2}(1+v_{lm}s_l s_m)\right] \tag{B6}$$

and this factorizes into

$$\left[\frac{1}{2^{N_1}}\sum_{s_k\subset G_1} s_i s_t \prod_{(l,m)}^{L_1}(1+v_{lm}s_l s_m)\right]\left[\frac{1}{2^{N_2}}\sum_{s_k\subset G_2} s_j s_t \prod_{(l,m)}^{L_2}(1+v_{lm}s_l s_m)\right] . \tag{B7}$$

From this, the relation (B2) follows. Again, since G_1 and G_2 have no bonds in common it follows that

$$\begin{aligned}[\langle s_i s_j\rangle^2]&=[\langle s_i s_j\rangle^2]_1 \quad \text{for } i,j,\in G_1\\ &=[\langle s_i s_j\rangle^2]_2 \quad \text{for } i,j\in G_2\\ &=[\langle s_i s_t\rangle^2]_1[\langle s_t s_j\rangle^2]_2 \quad \text{for } i\in G_1,\ j\in G_2 .\end{aligned} \tag{B8}$$

With these relations the proof of star-graph expansion for the Edwards-Anderson susceptibility becomes identical to the proof for the susceptibility of the pure Ising model.[5]

We wish to compute

$$\psi_G=\mathbb{1}^T M^{-1}\mathbb{1}-N_G . \tag{B9}$$

Consider the vector f which is the solution to the equation

$$\sum_j M_{ij}f_j=\mathbb{1} . \tag{B10}$$

Then

$$\psi_G=\sum_n f_n-N_G . \tag{B11}$$

By Crammer's rule

$$f_n=\det M^{(n)}/\det M , \tag{B12}$$

where $M^{(n)}$ is obtained from M by replacing the nth column by the vector $\mathbb{1}$.

For the graph articulated at point t (see Fig. 8) we choose vertex labels such that vertices labeled 1 to t lie in G_1 and t to N lie in G_2. We have shown that the matrix M has the form

$$M=\begin{vmatrix} M_1' & M_{1t} & M_{1t}M_{2t}^T\\ M_{1t}^T & 1 & M_{2t}^T\\ M_{2t}M_{1t}^T & M_{2t} & M_2'\end{vmatrix} . \tag{B13}$$

Here M_d' $(d=1,2)$ is the matrix of graph 1 or 2, where the index (i,j) do not take the value t. Indices 1,2 refer to G_1,G_2.

By subtracting suitable multiples of the tth row from the others the determinant of this matrix becomes

$$\det M=\begin{vmatrix} M_1'-M_{1t}M_{1t}^T & 0 & 0\\ M_{1t}^T & 1 & M_{2t}^T\\ 0 & 0 & M_2'-M_{2t}M_{2t}^T\end{vmatrix} . \tag{B14}$$

From which it follows that

$$\det M=\det M_1\det M_2 , \tag{B15}$$

where $\det M_1$ and $\det M_2$ are the full determinants for the corresponding matrices of graphs G_1 and G_2. Similarly we can show that for any n with $n<t$

$$\det M^{(n)}=\det M_1^{(n)}\det M_2 \tag{B16}$$

and for $n>t$

$$\det M^{(n)}=\det M_1\det M_2^{(n)} ,$$

hence implying that

$$f_n=\frac{\det M_d^{(n)}}{\det M_d} \quad (d=1 \text{ or } 2) . \tag{B17}$$

Also for $n=t$

$$f_t=\frac{\det M_1^{(t)}}{\det M_1}+\frac{\det M_2^{(t)}}{\det M_2}-1 . \tag{B18}$$

To see this observe that

$$\det M^t = \begin{vmatrix} M_1' & 1 & M_{1t}M_{2t}^T \\ M_{1t}^T & 1 & M_{2t}^T \\ M_{2t}M_{1t}^T & 1 & M_2' \end{vmatrix}$$

$$= \begin{vmatrix} M_1' & 1 & M_{1t}M_{2t}^T \\ M_{1t}^T & 1 & M_{2t}^T \\ M_{2t}M_{1t}^T & M_{2t} & M_2' \end{vmatrix} + \begin{vmatrix} M_1' & M_{1t} & M_{1t}M_{2t}^T \\ M_{1t}^T & 1 & M_{2t}^T \\ M_{2t}M_{1t}^T & 1 & M_2' \end{vmatrix} - \begin{vmatrix} M_1' & M_{1t} & M_{1t}M_{2t}^T \\ M_{1t}^T & 1 & M_{2t}^T \\ M_{2t}M_{1t}^T & M_{2t} & M_2' \end{vmatrix} . \quad \text{(B19)}$$

From which it follows that

$$\det M^{(t)} = \det M_1^{(t)} \det M_2 + \det M_1 \det M_2^{(t)} - \det M_1 \det M_2 \quad \text{(B20)}$$

which leads to the Eq. (B18). Combining all these we get

$$\psi_G = \psi_{G_1} + \psi_{G_2} \quad \text{(B21)}$$

which is the desired result.

APPENDIX C

In this appendix we explicitly compute the susceptibility series to order w^7 for the three-dimensional (3D) cubic lattice. Star graphs with 7 or less edges which can be embedded in the 3D cubic lattice are shown in Table I. In Table II we write down their weights to order w^7.

We illustrate the calculation of the weight of the graph 4 with vertices labeled in Fig. 6. We need to construct a 6×6 matrix M with elements

$$M_{ij} = [\langle s_i s_j \rangle^2] \quad \text{(C1)}$$

We shall show the calculation of M_{13}. The others can be done in a similar fashion. The expression for M_{13} is given by

$$M_{13} = \left[\left[\frac{\frac{1}{2^6}\mathrm{Tr}\prod_{(i,j)} (1+v_{ij}s_is_j)s_1s_3}{\frac{1}{2^6}\mathrm{Tr}\prod_{(i,j)}(1+v_{ij}s_is_j)} \right]^2 \right] . \quad \text{(C2)}$$

We write this as

$$M_{13} = [(N/D)^2] , \quad \text{(C3)}$$

where

$$N = \frac{1}{2^6}\mathrm{Tr}(1+v_{13}s_1s_3)(1+v_{15}s_1s_5)(1+v_{56}s_5s_6)(1+v_{62}s_6s_2)(1+v_{24}s_2s_4)(1+v_{43}s_4s_3)(1+v_{12}s_1s_2)s_1s_3 . \quad \text{(C4)}$$

We can expand the product. All the terms in which all spins are paired would have the trace equal to 2^6, all other terms vanish, giving

$$N = v_{13} + v_{15}v_{56}v_{62}v_{24}v_{43} + v_{12}v_{24}v_{43} + v_{13}v_{15}v_{56}v_{62}v_{21} . \quad \text{(C5)}$$

This has a simple diagrammatic representation show in Fig. 7(a). Similarly the denominator is a sum over all subgraphs in which all sites have even valency as shown in Fig. 7(b). In computing M_{13}, everytime a line is paired, we can replace it by w. Let us denote the two squares shown in Fig. 7(b) by L and R. Then we get the result for $(1/D)^2$ as shown in Fig. 7(c). Again squaring the numerator gives the result shown in Fig. 7(d).

Multiplying the two and averaging over the bonds, only those terms are left where all bonds are paired. This leads to

TABLE I. Stars with seven or less edges which can be embedded in the 3D cubic lattice.

Star number	Star	Lattice Constant
1		3
2		3
3		22
4		18

TABLE II. Weights of the star graphs to order w^7.

Star number	Weight
1	$-2w+2w^2-2w^3+2w^4-2w^5+2w^6-2w^7$
2	$56w^4-240w^5+496w^6-752w^7$
3	$132w^6-552w^7$
4	$-672w^7$

$$M_{13}=(1+6w^4+3w^6)(w+w^3+2w^5)-w^4(2-6w^3)2(1+w^3)-w^4(2-6w^3)2(w+w^2)-w^6(2-6w)2(1+w)+\cdots$$
$$=w+w^3-4w^4+4w^5-8w^6+25w^7 \quad (C6)$$

Once the Matrix elements are calculated the inverse is obtained as follows.

Let

$$M=I+X\ , \quad (C7)$$

where I is the identity matrix and X is $O(w)$. Then

$$M^{-1}=(I-X)(I+X^2)(I+X^4)+O(w^8)\ , \quad (C8)$$

One then obtains

$$\psi_G\equiv\sum_{i,j}M_{ij}^{-1}-6$$
$$=-14w+14w^2-14w^3+126w^4-494w^5$$
$$+1138w^6-2742w^7\ . \quad (C9)$$

In Table III we show how the star-graph subtraction works. Hence the series to order w^7 is obtained by multiplying the lattice constants and the weights in Tables I and II:

$$\chi_{\mathrm{EA}}^{-1}=1-6w+6w^2-6w^3+174w^4-726w^5$$
$$+4398w^6-26502w^7 \quad (C10)$$

which can be inverted to give

$$\chi_{\mathrm{EA}}=1+6w+30w^2+150w^3+582w^4$$
$$+2454w^5+6870w^6+25782w^7 \quad (C11)$$

which is our series to order w^7.

APPENDIX D

In this appendix we shall discuss self-avoiding walks. A self-avoiding walk of length l is a non-self-intersecting path of l steps on the lattice. It is defined by the position of the steps 1 to l with respect to the starting point. The idea behind using them in finding lattice embeddings of star graphs is to avoid repeating a very large number of times the same operation of finding all possible nonintersecting paths between two given lattice sites. In the course of finding the lattice constants, one frequently needs to insert a bridge of length l between two nodes which have been assigned positions **I** and **J**, respectively. In this situation one only needs to try all the self-avoiding walks of length l whose destination points are distant $\mathbf{J}-\mathbf{I}$ with respect to the starting point. We now discuss how these self-avoiding walks are generated.

We first need to define a canonical ordering for the neighbors of each point. Since there are only a few neighbors of any site this is easily done. For the square lattice the neighbors of the site (i,j) are ordered as $(i+1,j)$, $(i,j+1)$, $(i-1,j)$, and $(i,j-1)$. The starting point for all self-avoiding walks is taken as the origin of the lattice. The first self-avoiding walk is obtained by walking through the first allowed neighbor of each point as it is encountered. With the ordering defined above this implies that the first walk of length l on the square lattice is given by

$$(0,0)\to(1,0)\to(2,0)\to\cdots\to(l,0)\ .$$

The next move is to backtrack one step at a time, and while at point k to pivot about this point, exhausting all self-avoiding walks with the segment from the origin to k intact. The pivoting is done with the help of canonical ordering. If the previous step from k was to k' the next will be to k'', where k'' occurs next to k' in the canonical ordering for the neighbors of k. If the point k'' is already occupied one goes to the point next to k''. This is done until all neighbors of k are exhausted. As this pivot point moves past the origin one exhausts all possible self-avoiding walks of length l. In Fig. 8 we show all self-avoiding walks of length 2 for the square lattice in the order as they occur in our enumeration. By using this method we generated all self-avoiding walks of length up to 4 for the 4D hypercubic lattice, 5 for the cubic lattice, and 7 for the square lattice.

APPENDIX E

We will now discuss generating all subgraphs with complete bridges. In the star-subgraph determination program, and in the weight calculation program, one frequently needs to find all possible subgraphs of one type or

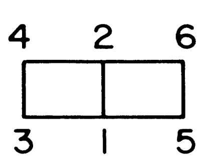

FIG. 6. The graph whose weight is calculated in Appendix C.

TABLE III. Subtraction of the weights of star subgraphs to obtain the weight of the graph 4.

Power of w	$\psi(G)$	Weight of graph $\underline{m}\times$ number of subgraphs of type $\underline{m}$: $\underline{1}\times 7$	$\underline{2}\times 2$	$\underline{3}\times 1$	$W(G)$
1	-14	-14	0	0	0
2	14	14	0	0	0
3	-14	-14	0	0	0
4	126	14	112	0	0
5	-494	-14	-480	0	0
6	1138	14	992	132	0
7	-2742	-14	-1504	-552	-672

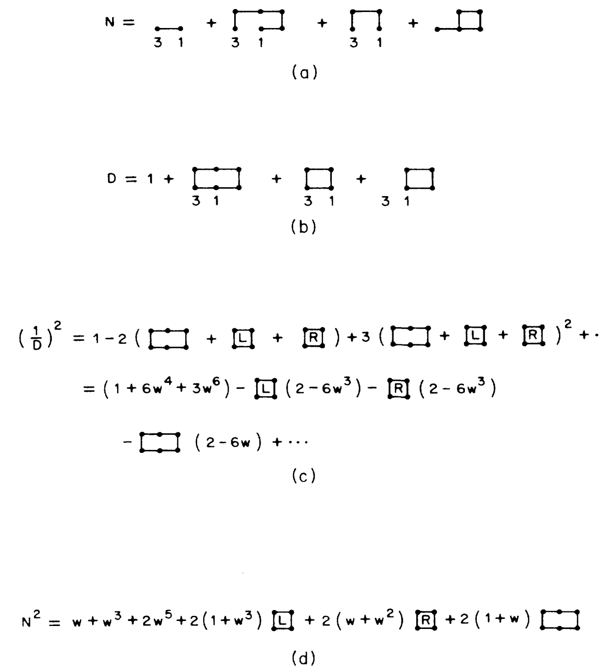

FIG. 7. (a) Graphical representation of N; (b) graphical representation of D; (c) graphical representation of $(1/D)^2$; (d) graphical representation of N^2.

the other. In this appendix we discuss a generic method for finding all possible subgraphs which are made out of complete bridges of the original graph.

We define a variable $N_{\text{taken}}(I)=0$ or 1 for $I=1$ to l (l is the number of bridges). Then a subgraph in which k of these l bridges are present and $l-k$ absent can be defined by letting the binary variable take values

$$N_{\text{taken}}(I)=1 \quad \text{for } I \in k$$

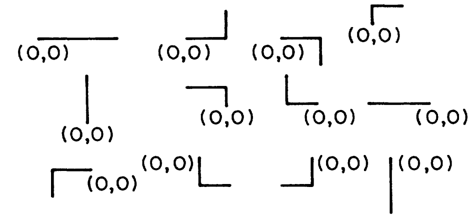

FIG. 8. Self-avoiding walks of length 2 in the order as they occur in our enumeration. The origin is taken as the starting point for all walks.

and

$$N_{\text{taken}}(I)=0 \quad \text{for } I \in l-k \; .$$

Excluding the empty set which has no bridges [$N_{\text{taken}}(I)=0$ for all I] there are 2^l-1 such subgraphs. These can be put in one to one correspondence with integers from 1 to 2^l-1. The mapping is given by

$$\mathcal{I}\{N_{taken}(I)\} = \sum_I 2^{(I-1)} N_{\text{taken}}(I) \; . \tag{E1}$$

The inverse mapping can be defined recursively as follows. Let

$$M(I) = \frac{\mathcal{I} - \sum_{I'=1}^{I-1} 2^{(I'-1)} N_{\text{taken}}(I')}{2^{(I-1)}} \; , \tag{E2}$$

then

$$N_{\text{taken}}(I) = M(I) \bmod 2 \; . \tag{E3}$$

Hence all subgraphs with complete bridges of the original graph can be generated by letting an integer variable take values from 1 to 2^l-1 and using the mapping in Eqs. (E2) and (E3).

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