Current-carrying state in a polar-phase superconductor at $T=0$

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We discuss the current-carrying state in a polar-phase superconductor at zero temperature. This phase has a line of nodes on the Fermi surface, and its properties are quite anisotropic. We compute the relation between $\mathbf{J}$ and $\mathbf{v}_s$ both for a pure metal and for one with impurities. This allows us to find the critical current for a variety of situations. We also consider the density of states in the presence of superflow.

I. INTRODUCTION

Superconductivity in the heavy-fermion metals has recently been the subject of keen interest.[1,2] In particular, much evidence points to the possibility that in at least some of these superconductors the gap has a nontrivial $\mathbf{k}$ dependence. A recent paper argues that a gap with a line (or lines) of nodes can explain much of the experimental data.[3]

In this paper we investigate the current-carrying state in a polar phase superconductor at $T=0$. The polar phase has a simple $l=1$ order parameter with a line of nodes on the Fermi surface; thus it serves as a prototype for more complicated phases which also have lines of nodes. We consider the situation of a spatially uniform $\mathbf{v}_s$, and compute how the supercurrent and order parameter change as $\mathbf{v}_s$ increases. This allows us to find the critical velocity.

Because of the gap anisotropy, the direction of $\mathbf{v}_s$ matters. We treat the two extreme cases: we let $\mathbf{v}_s$ lie in the plane of the line of nodes, and perpendicular to this plane. We also allow for the presence of impurities; we treat the impurity scattering using the full t matrix, since recent work indicates the importance of avoiding the Born approximation.[3,4]

The most important questions we address are: (1) how anisotropic is the supercurrent as a function of $\mathbf{v}_s$, and (2) how does the presence of impurities affect the results. The polar phase is quite sensitive both to supercurrents and impurities. An arbitrarily small concentration of impurities causes this phase to be gapless,[5,6] as does an arbitrarily small $\mathbf{v}_s$ in the proper direction. This $T=0$ computation illustrates the effect of the zero-energy excitations on the supercurrent.

II. FORMALISM

We take the polar phase gap to be

$$\Delta_{\alpha\beta}(\mathbf{k},\mathbf{R})=i\mathbf{x}\cdot\boldsymbol{\sigma}_{\alpha\mu}\sigma^{y}_{\mu\beta}\Delta(\mathbf{k},\mathbf{R}) \ , \tag{1}$$

$$\Delta(\mathbf{k},\mathbf{R})=\Delta(\mathbf{k}\cdot\mathbf{z})e^{i2M/\hbar\mathbf{v}_s\cdot\mathbf{R}} \ . \tag{2}$$

We will consider the following two cases.

(1) $\mathbf{v}_s=\mathbf{v}_s\mathbf{x}$, in which the supercurrent is in a direction of vanishing gap.

(2) $\mathbf{v}_s=v_s\mathbf{z}$, in which the supercurrent is in a direction of maximum gap.

To do our calculations we follow the quasiclassical approach.[7] The key quantity to compute is the quasiclassical propagator $\hat{g}(\mathbf{k},\mathbf{R},\epsilon)$ which for our situation is a 2×2 matrix in particle-hole space. For a uniform supercurrent we can remove the $\mathbf{R}$ dependence of the problem by making a gauge transformation as follows:

$$\bar{g}(\mathbf{k},\epsilon)\equiv e^{-iM/\hbar\mathbf{v}_s\cdot\mathbf{R}\bar{\tau}_3}\hat{g}e^{iM/\hbar\mathbf{v}_s\cdot\mathbf{R}\bar{\tau}_3} \ . \tag{3}$$

$\bar{g}$ is determined by its equation of motion,

$$[(i\epsilon-q)\bar{\tau}_3-iD\bar{\tau}_2-\bar{s},\bar{g}]=0 \ , \tag{4}$$

as well as the normalization condition

$$\bar{g}\bar{g}=-\pi^2\bar{1} \ . \tag{5}$$

Here, $q=p_F\mathbf{k}\cdot\mathbf{v}_s$ and $D=\Delta\mathbf{k}\cdot\mathbf{z}$

The magnitude of the gap, Δ, and the impurity self-energy $\bar{s}$ must be determined self-consistently from $\bar{g}$. The weak-coupling gap equation is

$$\Delta\mathbf{k}\cdot\mathbf{z}=-i\lambda T\sum_{\epsilon}\int d\mathbf{p}\mathbf{k}\cdot\mathbf{p}g_2 \ , \tag{6}$$

where λ is the coupling constant and g_i is the $\bar{\tau}_i$ component of $\bar{g}$. The impurity self-energy is given by

$$\bar{s}=c\bar{t}(\epsilon) \ , \tag{7}$$

where c is the concentration of impurities and $\bar{t}$ is the impurity t matrix. The t matrix is computed via the following equation:

$$\bar{t}(\epsilon)=v+N_0v\int d\mathbf{k}/4\pi\bar{g}(\mathbf{k},\epsilon)\bar{t}(\epsilon) \ . \tag{8}$$

For simplicity, we have taken the impurity scattering to be purely s wave, characterized by strength v.

Thus we must solve (4)–(8) self-consistently for $\bar{g}$. The supercurrent is then computed from

$$\mathbf{J}=2N_0v_FT\sum_{\epsilon}\int d\mathbf{k}/4\pi\mathbf{k}g_3 \ , \tag{9}$$

where N_0 is the density of states at the Fermi surface in the normal phase. Note that for the two directions of $\mathbf{v}_s$ we consider, we always have $\mathbf{J}\parallel\mathbf{v}_s$. The Appendix contains details on solving these equations.

A point to stress is that one must check whether a polar type of gap in fact satisfies the gap equation (6) in the presence of superflow and impurities. It turns out that when $\mathbf{v}_s \| \mathbf{z}$ or when $\mathbf{v}_s \| \mathbf{x}$, the polar-phase gap does satisfy (6). However, if we pick $\mathbf{v}_s$ to lie in a less symmetrical direction, then the polar phase does not satisfy the gap equation. Of course, these symmetry considerations will be different when one considers this problem for a crystalline point group rather than the spherically symmetric normal state treated in this paper.

III. PURE LIMIT

When the impurity concentration is zero, the equations simplify considerably. The impurity self-energy vanishes, and we need only solve self-consistently for the gap amplitude. In the following we let Δ_0 be the gap amplitude when $v_s = 0$ and $c = 0$.

When $\mathbf{v}_s = v_s \mathbf{z}$ the problem is symmetrical enough to be solved analytically. The critical velocity is $v_c = \Delta_0 / p_F$ and the supercurrent is simply $\mathbf{J} = \rho \mathbf{v}_s$, where ρ is the electron density, right up to the critical velocity. The gap amplitude is $\Delta = \Delta_0$ right up to v_c. These results are illustrated in Fig. 1.

When $\mathbf{v}_s = v_s \mathbf{x}$ the situation is more interesting. With $\mathbf{v}_s$ in the direction of a node, the supercurrent can be diminished by excitations, even at $T = 0$, for arbitrarily small v_s. This has already been pointed out in a previous paper,[8] which discussed a term in J which is nonanalytic in v_s. Numerical calculations of J and Δ for the full range of v_s are shown in Fig. 1. The critical velocity is $v_c \approx 0.61 \Delta_0 / p_F$ while the critical current is $J_c \approx 0.28\, \rho \Delta_0 / p_F$.

The fact that zero-energy excitations can immediately start to reduce the supercurrent when $\mathbf{v}_s \| \mathbf{x}$ is reflected in the single-particle density of states. When v_s is small, we can derive the following expression for the density of states (for one spin population) at zero energy:

$$N(E=0) = N_0 p_F \, | v_s | / \Delta_0 \ . \tag{10}$$

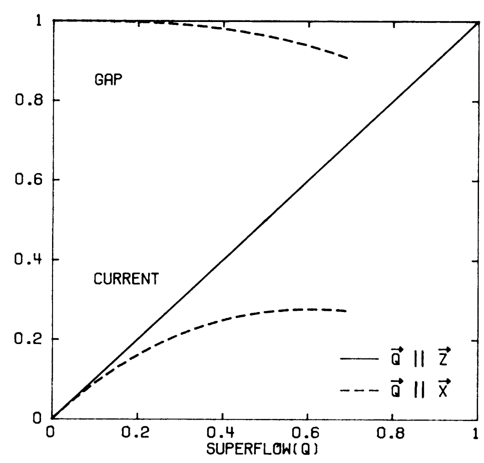

FIG. 1. Order parameter and supercurrent as functions of v_s for the polar phase without impurities. The solid lines are for $\mathbf{v}_s \| \mathbf{z}$, while the dashed lines are for $\mathbf{v}_s \| \mathbf{x}$. The dimensionless variable plotted on the horizontal axis is $Q = p_F v_s / \Delta_0$, where Δ_0 is the value of the order parameter unperturbed by impurities or superflow. On the vertical axis are plotted Δ/Δ_0 and $J p_F / \rho \Delta_0$. When $\mathbf{v}_s \| \mathbf{x}$, $J_c \approx 0.28 \rho \Delta_0 / p_F$ and $Q_c \approx 0.61$.

IV. IMPURITY RESULTS

When the concentration of impurities is nonzero, we must resort to a numerical solution of (4)–(9). Since both c, the concentration of impurities, and v, the strength of the potential, affect the outcome, we must fix two parameters to specify the situation. A convenient choice is the following:[9]

$$\sigma = (N_0 \pi v)^2 / [1 + (N_0 \pi v)^2] \ , \tag{11}$$

$$1/(2\tau\Delta_0) = c\sigma / (\pi N_0 \Delta_0) \ . \tag{12}$$

The first parameter, σ, is simply the cross section for scattering from one impurity, normalized by its value in the unitarity ($v \to \infty$) limit. The second parameter is the normal state elastic scattering rate, normalized by the unperturbed energy gap to form a dimensionless number. In the Born limit ($\sigma \ll 1$) only the single parameter cv^2 matters, whereas if the scattering potential is not weak, results will in general depend on the two parameters defined above in a more complicated way.

Figures 2–5 show some representative results for the supercurrent and order parameter as functions of v_s, for several choices of σ and τ. Figures 2 and 3 are in the extreme unitary limit of $\sigma = 1$; Fig. 2 has a low concentration of impurities to make $1/(2\tau\Delta_0) = 0.01$, while in Fig. 3 the concentration is raised to make $1/(2\tau\Delta_0) = 0.2$. Figures 4 and 5 are in the Born limit with $\sigma = 0.001$, and cover the same choices for τ as the previous two figures.

Several observations about these results are in order. There is little difference between Figs. 2 and 4. This shows that when $1/2\tau\Delta_0$ is small, it doesn't seem to matter whether σ is small or large; only the (small) value of $1/\tau$ seems to matter. Conversely, there is a marked

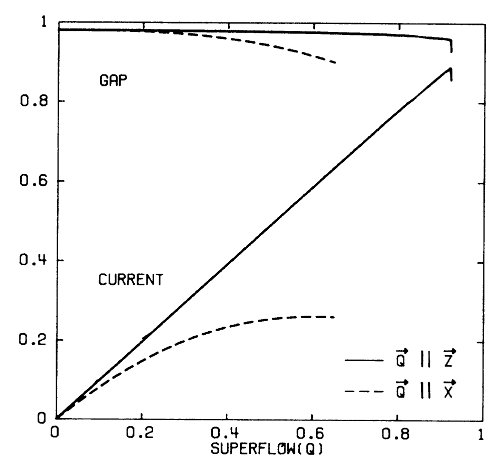

FIG. 2. Order parameter and supercurrent as functions of v_s for the polar phase with impurities. This figure is for the unitarity limit ($\sigma = 1$), and has $1/(2\tau\Delta_0) = 0.01$. The variables are scaled as in Fig. 1. As can be seen, when $\mathbf{v}_s \| \mathbf{z}$, $Q_c \approx 0.92$ and $J_c \approx 0.89 \rho \Delta_0 / p_F$, when $\mathbf{v}_s \| \mathbf{x}$, $Q_c \approx 0.60$ and $J_c \approx 0.26 \rho \Delta_0 / p_F$.

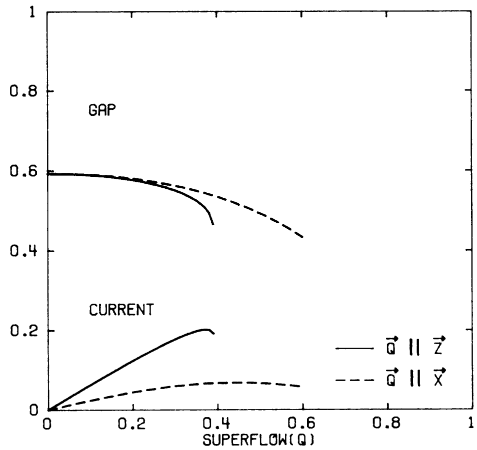

FIG. 3. Order parameter and supercurrent as functions of v_s for the polar phase with impurities. This figure is for the unitarity limit ($\sigma=1$), and has $1/(2\tau\Delta_0)=0.2$. The variables are scaled as in Fig. 1. When $\mathbf{v}_s\|\mathbf{z}$, $Q_c\approx0.38$ and $J_c\approx0.20\,\rho\Delta_0/p_F$; when $\mathbf{v}_s\|\mathbf{x}$, $Q_c\approx0.44$ and $J_c\approx0.07\rho\Delta_0/p_F$.

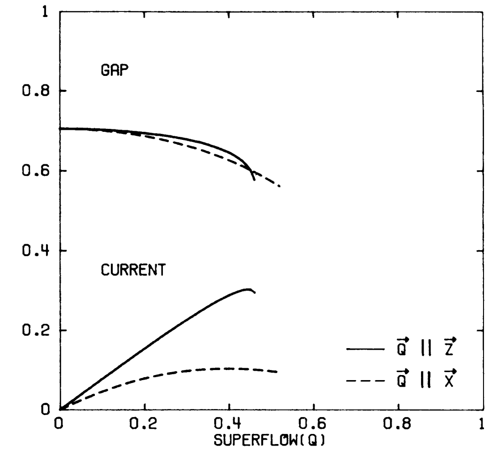

FIG. 5. Order parameter and supercurrent as functions of v_s for the polar phase with impurities. In this figure we have taken $\sigma=0.001$ and $1/(2\tau\Delta_0)=0.2$. The variables are scaled as in Fig. 1. When $\mathbf{v}_s\|\mathbf{z}$, $Q_c\approx0.45$ and $J_c\approx0.30\rho\Delta_0/p_F$, whereas when $\mathbf{v}_s\|\mathbf{x}$, $Q_c\approx0.39$ and $J_c\approx0.10\rho\Delta_0/p_F$.

difference between Figs. 3 and 5, which both have $1/2\tau\Delta_0=0.2$ So when τ is very short, it does matter whether its short value is due to a high concentration of impurities or to a strong scattering potential.

Fig. 3 is particularly interesting. Perhaps surprisingly, the order parameter has a larger value when $\mathbf{v}_s\|\mathbf{x}$ than when $\mathbf{v}_s\|\mathbf{z}$, at each value of v_s. In addition, the critical velocity (although not the critical current) is larger when $\mathbf{v}_s\|\mathbf{x}$. The curve of J versus v_s when $\mathbf{v}_s\|\mathbf{x}$ has an extremely flat, low trajectory.

One consistent feature of the calculations is that the critical current for $\mathbf{v}_s\|\mathbf{z}$ is always about three times bigger than the critical current for $\mathbf{v}_s\|\mathbf{x}$, regardless of the values of σ and τ.

Figures 6 and 7 show plots of the density of states (for one spin population) versus energy; both have the same value for τ, but in Fig. 6 $\sigma=1.0$, while in Fig. 7 we have $\sigma=0.001$. In both pictures we can see that when $\mathbf{v}_s\|\mathbf{x}$, the density of states at zero energy is much higher than when $\mathbf{v}_s\|\mathbf{z}$ or when $\mathbf{v}_s$ vanishes.

ACKNOWLEDGMENT

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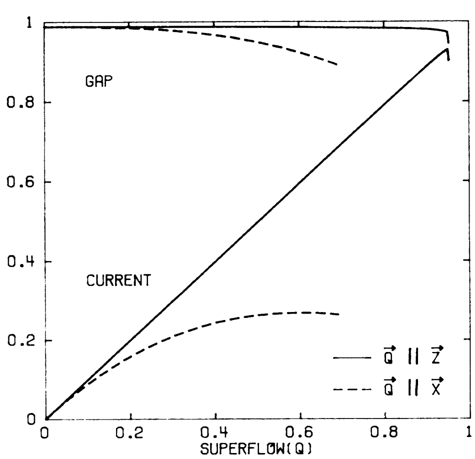

FIG. 4. Order parameter and supercurrent as functions of v_s for the polar phase with impurities. In this figure we have taken $\sigma=0.001$ and $1/(2\tau\Delta_0)=0.01$. The variables are scaled as in Fig. 1. When $\mathbf{v}_s\|\mathbf{z}$, $Q_c\approx0.95$ and $J_c\approx0.93\rho\Delta_0/p_F$, when $\mathbf{v}_s\|\mathbf{x}$, $Q_c\approx0.60$ and $J_c\approx0.27\rho\Delta_0/p_F$.

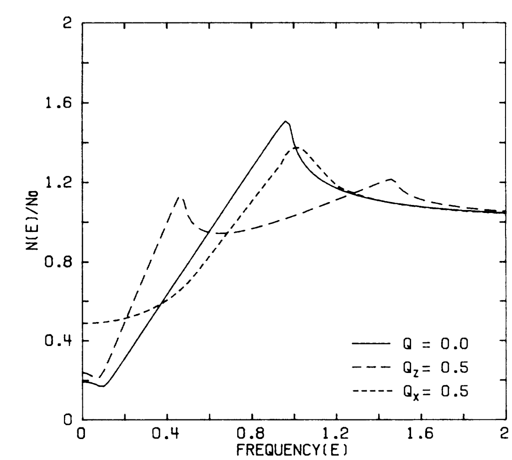

FIG. 6. Normalized density of states, $N(E)/N_0$ vs E/Δ_0. For this figure, $\sigma=1$ and $1/(2\tau\Delta_0)=0.01$. Results are shown for no superflow ($Q=0$), for $\mathbf{v}_s\|\mathbf{z}$ ($Q=0.5$) and for $\mathbf{v}_s\|\mathbf{x}$ ($Q=0.5$).

APPENDIX

Here we give a brief discussion of some of the details involved in solving Eq. (4)–(9). In general, the impurity self-energy $\bar{s}(\epsilon)$ can be written

$$\bar{s}=a_0\bar{1}+a_2\bar{\tau}_2+a_3\bar{\tau}_3 \ . \tag{A1}$$

Of course, the a_0 piece drops out of the commutator in (4), and so is irrelevant. The solution of (4) is then

$$\bar{g}=\frac{-\pi[(i\epsilon-q-a_3)\bar{\tau}_3-(iD+a_2)\bar{\tau}_2]}{[(\epsilon+iq+ia_3)^2+(D-ia_2)^2]^{1/2}} \ . \tag{A2}$$

It is easy to see that if $\bar{g}$ has only $\bar{\tau}_3$ and $\bar{\tau}_2$ components, then Eq. (8) will lead to no $\bar{\tau}_1$ terms in the $\bar{t}$ matrix. Hence, (A1) needs no $\bar{\tau}_1$ term. The gap Eq. (6) becomes

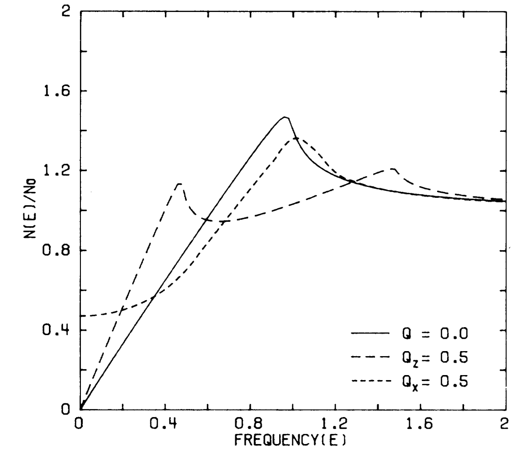

FIG. 7. Density of states vs. energy, each axis normalized as in Fig. 6. All parameters are the same as Fig. 6 except that $\sigma=0.001$. It should be noted that when $\mathbf{v}_s\|\mathbf{z}$ or when $\mathbf{v}_s=0$, $N(E=0)$ is not zero, just very small.

$$\Delta\mathbf{k}\cdot\mathbf{z}=\pi\lambda T\sum_{\epsilon}\int dp\left[\frac{(\mathbf{k}\cdot\mathbf{p})(\Delta\mathbf{p}\cdot\mathbf{z}-ia_2)}{[(\epsilon+ip_F\mathbf{p}\cdot\mathbf{v}_s+ia_3)^2+(\Delta\mathbf{p}\cdot\mathbf{z}-ia_2)^2]^{1/2}}\right] . \tag{A3}$$

When $\mathbf{v}_s\|\mathbf{x}$ one can check that a solution of (7) and (8) gives $a_2=0$; the gap equation then becomes

$$\Delta\cos\theta=\Delta\pi\lambda T\sum_{\epsilon}\int d\theta'\int d\phi'\left[\frac{\sin\theta'\cos\theta'[\cos\theta\cos\theta'+\sin\theta\sin\theta'\cos(\phi-\phi')]}{[(\epsilon+ip_Fv_s\sin\theta'\cos\phi'+ia_3)^2+\Delta^2\cos^2\theta']^{1/2}}\right] . \tag{A4}$$

The terms on the right-hand side proportional to $\sin\theta$ vanish by symmetry. This shows that a polar type of gap does indeed solve the gap equation when $\mathbf{v}_s\|\mathbf{x}$ and impurities are present.

When $\mathbf{v}_s\|\mathbf{z}$ the gap equation becomes

$$\Delta\cos\theta=\pi\lambda T\sum_{\epsilon}\int d\theta'\int d\phi'\left[\frac{\sin\theta'(\Delta\cos\theta'-ia_2)[\cos\theta\cos\theta'+\sin\theta\sin\theta'\cos(\phi-\phi')]}{[(\epsilon+ip_Fv_s\cos\theta'+ia_3)^2+(\Delta\cos\theta'-ia_2)^2]^{1/2}}\right] . \tag{A5}$$

The terms on the right-hand side proportional to $\sin\theta$ vanish because of the ϕ' integral, so we see that again the polar phase satisfies the gap equation.

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