p-spin-interaction spin-glass models: Connections with the structural glass problem

T. R. Kirkpatrick
Department of Physics and Astronomy and Institute for Physical Science and Technology, University of Maryland, College Park, Maryland 20742

D. Thirumalai
Department of Chemistry and Biochemistry and Institute for Physical Science and Technology, University of Maryland, College Park, Maryland 20742

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The static and the dynamical theories for the mean-field p-spin ($p>2$) interaction spin-glass model are studied. A broken-replica-symmetric equilibrium solution leads to a glass transition at a temperature T_g' where the Edwards-Anderson order parameter is discontinuous but where there is no latent heat and there is a discontinuous specific heat. The dynamical theory leads to a continuous slowing down and predicts a glass transition at $T_g > T_g'$. A reinterpretation of the equilibrium solution allows us to relate the dynamical transition to the equilibrium theory. The mathematical structure of the mean-field dynamical theory is closely related to certain recent dynamical theories of the structural glass transition.

I. INTRODUCTION

In this paper we discuss both the statics and the dynamics of the mean-field p-spin ($p>2$) interaction spin-glass (SG) model.[1] The basic motivation for this study is that it has recently been appreciated that in many SG models the Edwards-Anderson (EA) order parameters[2] $q_{\rm EA}$ is discontinuous[3] at the SG-transition temperature T_g. The usual $p=2$ or Sherrington-Kirkpatrick (SK) model,[4] where $q_{\rm EA}$ $(T=T_g)=0$, is an exceptional case. Models exhibiting discontinuous SG transitions include Potts glasses with more than four components,[5] quadrupolar glass models,[5,6] and p-spin interaction SG models with $p>2$.[3,7] Here we consider the p-spin models since both the static and dynamical properties of these models can be easily studied. The mean-field Potts glass model and quadrupolar model will be considered elsewhere.

Possible experimental systems where these models might be relevant include orientational glasses such as K(Br,CN) mixed crystals,[4,8] electric dipole glasses,[9] and mixed ortho-para hydrogen crystals.[10] It has also been recently suggested that there is a close connection between the dynamical theories of the structural glass transition and mean-field theories for the Potts glass and p-spin interaction SG models.[11] The motivation for this suggestion is the discontinuous nature of the transition in all three models. In this paper we show a particularly close connection between the structural glass theories and the p-spin ($p>2$) SG model. This connection is due to the fact that the nonlinearities in the dynamical equations for the p-spin interaction model are identical in structure to the nonlinearities in the dynamical equations for the structural glass theories.[12–16] We therefore hope that a study of these spin models will lead to a deeper understanding of the structural glass theories. Of particular interest to us is the relationship between the dynamical transition and the equilibrium transition in models where the order parameter is discontinuous at T_g. We note, however, a crucial difference between spin glasses and structural glasses: In the spin-glass problem the randomness is self-generated rather than put in by hand in spin-glass models.

The main conclusion of this paper is that the dynamical theory apparently predicts a higher glass-transition temperature than the equilibrium theory. By reinterpreting the equilibrium solution, we show, however, that there is a connection between the dynamical transition and equilibrium theory. The dynamical transition occurs where the Parisi free energy is maximized when one of the variational parameters is fixed to be at its physical endpoint. The usual equilibrium SG transition is then interpreted as a second transition which occurs at a lower temperature. This situation can also arise in other models like the Potts glass.[17]

The plan of this paper is as follows. In Sec. II we derive the equations of motion for the average (over the random interactions) correlation functions in the soft-spin version of the p-spin interaction SG model. In Sec. III we solve these equations for the $p=2+\epsilon$, with $\epsilon \ll 1$. We show that there is a dynamical glass transition at a temperature T_g. In Sec. IV we use replica methods to discuss the equilibrium SG transition in the $p=2+\epsilon$ model. A broken-replica-symmetric (BRS) solution leads to an equilibrium glass-transition temperature T_g' with $T_g > T_g'$. In this section we also discuss connections between the static and dynamic transitions. The paper is concluded in Sec. V with a discussion. In the Appendix the stability of the SG phase is considered.

II. THE DYNAMICAL MODEL

The Hamiltonian of the mean-field p-spin SG model is

$$H=-\sum_{i_1<i_2\cdots<i_p} J_{i_1\cdots i_p}\sigma_{i_1}\cdots\sigma_{i_p}-\sum_{i=1}^{N} h_i\sigma_i \ , \quad \text{(2.1a)}$$

where $\sigma_i=\pm 1$ $(i=1,\ldots,N)$ and h_i is an external magnetic field. The $\{J_{i_1\cdots i_p}\}$ are independent random in-

teractions whose distribution is

$$P(J_{i_1\cdots i_p})=\left[\frac{N^{p-1}}{\pi p! J^2}\right]^{1/2} \times\exp\left[\frac{-(J_{i_1\cdots i_p})^2 N^{p-1}}{J^2 p!}\right] . \tag{2.1b}$$

The N dependence in Eq. (2.1b) is chosen such that there is a well-defined thermodynamic limit.

Here we first consider a soft-spin version of the p-spin model that is defined by

$$\beta H=\sum_i\left[\frac{r_0}{2}\alpha_i^2+u\sigma_i^4\right]-\beta\sum J_{i_1\cdots i_p}\sigma_{i_1}\cdots\sigma_{i_p} -\beta\sum_{i=1}^{N} h_i\sigma_i , \tag{2.2}$$

where $\beta=T^{-1}$ and Boltzmann's constant is taken to be unity. The length of the soft spin σ_i is allowed to vary continuously from $-\infty$ to ∞. We can recover the constraint $\sigma_i^2=1$ by letting $u\to\infty$ and $r_0\to-\infty$ with $u/r_0\to$ const. This limit will be dubbed the hard limit.

The relaxational dynamics for $\sigma_i(t)$ is assumed to be given by the Langevin equation,

$$\Gamma_0^{-1}\partial_t\sigma_i(t)=-\frac{\delta(\beta H)}{\delta\sigma_i(t)}+\xi_i(t) , \tag{2.3a}$$

with Γ_0 a bare kinetic coefficient which sets the microscopic time scale and $\xi_i(t)$ is a Gaussian random noise with zero mean and variance,

$$\langle\xi_i(t)\xi_j(t')\rangle=\frac{2}{\Gamma_0}\delta_{ij}\delta(t-t') . \tag{2.3b}$$

Equations (2.3) ensure a correct equilibrium description. In the dynamical calculation the physical quantities of interest are the two-spin correlation function,

$$C_{ij}(t-t')=\langle\sigma_i(t)\sigma_j(t')\rangle , \tag{2.4a}$$

and the linear-response function,

$$G_{ij}(t-t')=\frac{\partial\langle\sigma_i(t)\rangle}{\partial h_j(t')},\ t>t' \tag{2.4b}$$

where $\langle\ \rangle$ means average over ξ. Causality yields the relation

$$G(t)=-\Theta(t)\partial_t C(t) , \tag{2.4c}$$

with $\Theta(t>0)=1$ and $\Theta(t<0)=0$.

To carry out the averaging over the quenched random interactions we use the dynamical functional-integral formulation of De Dominicis[18] and Janssen *et al.*[19] The same results can be obtained by a variety of other methods.[20] First, we define a generating functional for dynamical correlations and response functions.

$$Z\{J_{i_1\cdots i_p},l_i,\hat{l}_i\}=\int D\sigma\,D\hat{\sigma}\exp\left[\int dt[l_i(t)\sigma_i(t)+i\hat{l}_i(t)\hat{\sigma}_i(t)]+L(\sigma,\hat{\sigma})\right] , \tag{2.5a}$$

where

$$L(\sigma,\hat{\sigma})=\int dt\sum_i i\hat{\sigma}_i(t)\left[-\Gamma_0^{-1}\partial_t\sigma_i(t)-\frac{\delta(\beta H)}{\delta\sigma_i(t)}+\Gamma_0^{-1}i\hat{\sigma}_i(t)\right]+V\{\sigma\} . \tag{2.5b}$$

The functional Jacobian,

$$V=-\tfrac{1}{2}\int dt\sum_i\frac{\delta^2(\beta H)}{\delta\sigma_i^2} \tag{2.6a}$$

ensures the normalization,

$$Z\{J_{i_1\cdots i_p},\ l_i=\hat{l}_i=0\}=1 . \tag{2.6b}$$

Response and correlation functions can be obtained from Z by taking functional derivatives with respect to the sources $\hat{l}_i$ and l_i.

The correlation functions generated by Eq. (2.5a) depend on the random interactions $\{J_{i_1\cdots i_p}\}$. We are interested in average quantities. De Dominicis[21] was first to observe that the condition implied by Eq. (2.6b) is independent of the random variables and thus the quenched average can be done directly on Z. Performing the required Gaussian integration yields

$$[Z]_J=\int\prod dJ_{i_1\cdots i_p}P(J_{i_1\cdots i_p})Z\{J_{i_1\cdots i_p}\}=\int D[\sigma]D[\hat{\sigma}]\exp[L_0(\sigma,\hat{\sigma})+\Delta(\sigma,\hat{\sigma})] , \tag{2.7a}$$

with

$$L_0(\sigma,\hat{\sigma})=\int dt\sum_i[i\hat{\sigma}_i(-\Gamma_0^{-1}\partial_t\sigma_i-r_0\sigma_i-4u\sigma_i^3-h_i+i\Gamma_0^{-1}\hat{\sigma}_i)+i\hat{l}_i\hat{\sigma}_i+l_i\sigma_i]+V\{\sigma\} \tag{2.7b}$$

and

$$\Delta(\sigma,\hat{\sigma})=\frac{p\beta^2J^2}{4N^{p-1}}\sum_{i,j,i_3\cdots i_p}[i\hat{\sigma}_i(t)\sigma_j(t)i\hat{\sigma}_i(t')\sigma_j(t')\sigma_{i_3}(t)\sigma_{i_3}(t')\cdots\sigma_{i^p}(t)\sigma_{i_p}(t') +(p-1)i\hat{\sigma}_i(t)\sigma_j(t)i\hat{\sigma}_j(t')\sigma_i(t')\sigma_{i_3}(t)\sigma_{i_3}(t')\cdots\sigma_{i_p}(t)\sigma_{i_p}(t')] . \tag{2.7c}$$

In deriving Eqs. (2.7) we have assumed that the random interactions are symmetric with respect to interchange of all the site indices. The integration of the random interactions has generated $2p$-spin couplings which are nonlocal in time.

The path integrals in Eq. (2.7a) are evaluated using a generalization of the procedure devised by Gross and Mezard[3] for the equilibrium solution of the p-spin model. The basic idea is to decouple the $2p$-spin correlation function such that the effective Lagrangian is local in space (i.e., refers to a single site) but not in time. To achieve this we define the quantities

$$Q_1(\hat{\sigma}\,\hat{\sigma})=\frac{1}{N}\sum_{i=1}^{N} i\hat{\sigma}_i(t)i\hat{\sigma}_i(t') , \tag{2.8a}$$

$$Q_2(\sigma\sigma)=\frac{1}{N}\sum_{i=1}^{N} \sigma_i(t)\sigma_i(t') , \tag{2.8b}$$

$$Q_3(\hat{\sigma}\sigma)=\frac{1}{N}\sum_{i=1}^{N} i\hat{\sigma}_i(t)\sigma_i(t') , \tag{2.8c}$$

$$Q_4(\sigma\hat{\sigma})=\frac{1}{N}\sum_{i=1}^{N} \sigma_i(t)i\hat{\sigma}_i(t') \tag{2.8d}$$

with this $\Delta(\sigma,\hat{\sigma})$ becomes,

$$\Delta(\sigma,\hat{\sigma})=\frac{\mu N}{2}\int dt\,dt'[Q_1Q_2^{p-1}+(p-1)Q_3Q_4Q_2^{p-2}] , \tag{2.9}$$

where $\mu=p\beta^2J^2/2$ and the explicit dependence of the Q's on σ or $\hat{\sigma}$ has been suppressed. The path integrals in Eq. (2.7a) are evaluated by constraining $Q_\mu(\sigma)\equiv Q_\mu$ (a constant) and then integrating over all possible values of the constant. The constraint is introduced through the Lagrange multiplier λ_μ. This allows us to write Eq. (2.7a) as

$$[Z]_J=\int\prod_{\mu=1}^{4}D[Q_\mu]\int\prod_{\mu=1}^{4}\left[\frac{N}{2\pi i}\right]D[\lambda_\mu]\exp\left[-NG(Q_\mu,\lambda_\mu)+\ln\int D[\sigma]D[\hat{\sigma}]\exp L(\sigma,\hat{\sigma},\lambda_\mu,Q_\mu)\right] , \tag{2.10a}$$

where

$$G(Q_\mu,\lambda_\mu)=\int dt\,dt'\sum_{\mu=1}^{4}\lambda_\mu Q_\mu-\frac{\mu}{2}\int dt\,dt'[Q_1(t,t')Q_2^{p-1}(t,t')+(p-1)Q_3(t,t')Q_4(t,t')Q_2^{p-2}(t,t')] \tag{2.10b}$$

and the effective Lagrangian is

$$L(\sigma,\hat{\sigma},\lambda_\mu,Q_\mu)=L_0(\sigma,\hat{\sigma})+\int dt\,dt'\sum_i[\lambda_1(t,t')i\hat{\sigma}_i(t)i\hat{\sigma}_i(t')+\lambda_2(t,t')\sigma_i(t)\sigma_i(t')$$
$$+\lambda_3(t,t')i\hat{\sigma}_i(t)\sigma_i(t')+\lambda_4(t,t')\sigma_i(t)i\hat{\sigma}_i(t')] . \tag{2.10c}$$

In the limit of $N\to\infty$, the integrations over both Q_μ and λ_μ can be evaluated using saddle-point methods, which amounts to replacing Q_μ and λ_μ by their stationary values: This yields

$$Q_1^0=\frac{1}{N}\sum_{i=1}^{N}\langle i\hat{\sigma}_i(t)i\hat{\sigma}_i(t')\rangle , \tag{2.11a}$$

$$Q_2^0=\frac{1}{N}\sum_{i=1}^{N}\langle \sigma_i(t)\sigma_i(t')\rangle , \tag{2.11b}$$

$$Q_3^0=\frac{1}{N}\sum_{i=1}^{N}\langle i\hat{\sigma}_i(t)\sigma_i(t')\rangle , \tag{2.11c}$$

$$Q_4^0=\frac{1}{N}\sum_{i=1}^{N}\langle \sigma_i(t)i\hat{\sigma}_i(t')\rangle , \tag{2.11d}$$

where the average in Eqs. (2.11) is calculated with the Lagrangian given by Eq. (2.10c). The stationary values of λ are given by

$$\lambda_1^0=\frac{\mu}{2}(Q_2^0)^{p-1} , \tag{2.12a}$$

$$\lambda_2^0=\frac{\mu}{2}(p-1)[Q_1^0(Q_2^0)^{p-2}+(p-2)Q_3^0Q_4^0Q_2^{p-3}] , \tag{2.12b}$$

$$\lambda_3^0=\frac{\mu}{2}(p-1)Q_4^0(Q_2^0)^{p-2} , \tag{2.12c}$$

$$\lambda_4^0=\frac{\mu}{2}(p-1)Q_3^0(Q_2^0)^{p-2} . \tag{2.12d}$$

The structure of the field theory suggests that $Q_1^0\equiv 0$ is a self-consistent solution for all temperatures and this is also necessary for a causal theory. It can also be shown that for $p>2$, the functional Jacobian, $V\{\sigma\}$, gets renormalized to

$$V\{\sigma\}+\sum_i\int\lambda_2^0\sigma_i(t)\sigma_i(t')dt\,dt' .$$

Comparison of Eqs. (2.11) and (2.4) leads to

$$Q_2^0=C(t-t') , \tag{2.13a}$$

$$Q_3^0=Q_4^0=G(t-t') . \tag{2.13b}$$

Using Eqs. (2.12) and (2.13) in Eq. (2.10c), one can obtain an effective dynamic Lagrangian. The equation of motion for $\sigma_i(\omega)$, averaged over the quenched random interactions, is

$$\sigma_i(\omega)=G_0(\omega)[f_i(\omega)+h_i(\omega)]$$
$$-4uG_0(\omega)\int\frac{d\omega_1d\omega_2}{(2\pi)^2}\sigma_i(\omega_i)\sigma_i(\omega_2)$$
$$\times\sigma_i(\omega-\omega_1-\omega_2) , \tag{2.14a}$$

with $G_0(\omega)$ a renormalized bare propagator,

$$G_0^{-1}(\omega)=r_0-i\omega\Gamma_0^{-1} -\mu(p-1)\int_0^\infty dt\, e^{i\omega t}G(t)C^{p-2}(t)\ , \qquad (2.14b)$$

and $f_i(\omega)$ a renormalized noise term,

$$\langle f_i(\omega)f_j(\omega')\rangle=2\pi\delta(\omega+\omega')\delta_{i_j} \times\left[\frac{2}{\Gamma_0}+\mu\int_{-\infty}^{+\infty}dt\, e^{i\omega t}C^{p-1}(t)\right]. \qquad (2.14c)$$

The strategy employed to derive Eqs. (2.14) can also be used when $p=2$ and this would lead to the dynamical equations obtained by Sompolinsky and Zippelius.[22]

III. APPROXIMATE SOLUTION OF DYNAMICAL EQUATIONS

In this section we approximately solve the dynamical equations given by Eqs. (2.14), using the causal relation indicated in Eq. (2.4c), and discuss the dynamical glass transition predicted by them. Unlike the usual SK, or $p=2$, model, we are not able to make general statements, valid to all orders in u, about the critical behavior as T_g is approached from above. The difference is due to the discontinuous nature of the transition for spin models with $p>2$. We will work within a small-u theory. An important consequence of this approximation is that our treatment of the SG transition in these models is exact only for models with $p=2+\epsilon$ with ϵ small (an analytic continuation to fractional p values is assumed). The exactness of our procedure for small ϵ will be discussed further below and confirmed in Sec. IV, where we will give the equilibrium solution of these models in both the hard and soft limits. The basic technical idea is that for small ϵ the discontinuities at T_g are of $O(\epsilon)$ and an order parameter expansion is possible. We believe that even with this limitation, one can draw general conclusions.

We treat the u term in Eqs. (2.14) in the one-loop or mean-field approximation. Corrections will be discussed below. An equation for, Im>0,

$$\hat{C}(\omega)=\int_0^\infty dt\, e^{i\omega t}C(t)\ , \qquad (3.1a)$$

can be derived from Eqs. (2.14) and (2.4c). In the ergodic phase we obtain

$$\hat{C}(\omega)=\frac{C(t=0)}{-i\omega+\bar{r}_0\Gamma(\omega)}\ , \qquad (3.1b)$$

with the initial equal-time spin-correlation function given by

$$C(t=0)=\bar{r}_0^{-1}=[r_0-\mu C^{p-1}(t=0)+12uC(t=0)]^{-1}\ . \qquad (3.1c)$$

$\Gamma(\omega)$ is a renormalized kinetic coefficient,

$$\Gamma^{-1}(\omega)=\Gamma_0^{-1}+\mu\int_0^\infty dt\, e^{i\omega t}C^{p-1}(t)\ . \qquad (3.1d)$$

For use below we note that in the domain, $\phi(t)=C(t)/C(0)$ satisfies the equation

$$\nu_0^{-1}\dot{\phi}(t)+\phi(t)+\lambda\int_0^t dt_1\phi^{p-1}(t-t_1)\dot{\phi}(t_1)=0\ , \qquad (3.1e)$$

with $\phi(t=0)=1$, $\nu_0=\bar{r}_0\Gamma_0$, and the nonlinear coupling given by $\lambda=\mu\bar{r}_0^{2-p}$.

For $p=3$ these equations are mathematically identical to the dynamical equations for the structural glass problem. We argue below that these equations have similar critical properties for any $p>2$. Equation (3.1c) for $C(t=0)$ is an additional equation in the SG problem that self-consistently determines the equal-time spin correlations. In the structural glass problem the analogous quantity is the static structure factor, which is assumed to be insensitive to the transition from the liquid state to the glassy state. Assuming here that $C(t=0)$ is continuous at T_g, it can be easily shown that Eqs. (3.1) predict a glass transition at $T=T_g$, and a continuous slowing down of the dynamics as T_g is approached from above. The critical properties can be determined by generalizing (to any $p>2$) the arguments given for the structural glass problem.[12,13] First, we define the Edwards-Anderson order parameter,

$$q_{\rm EA}=q=\lim_{t\to\infty}C(t)\ . \qquad (3.2a)$$

Assuming that $\hat{C}(\omega)$ has a time-persistent part with a nonzero value of q and a decaying part, Eqs. (3.1) yield the equation of state,

$$q=\frac{C(t=0)\mu q^{p-1}}{(\bar{r}_0+\mu q^{p-1})}\ . \qquad (3.2b)$$

This equation leads to a physical q at a critical temperature given by Eq. (3.1c) and $\mu\to\mu_c$,

$$\mu_c=\bar{r}_{0c}^{p}\left[\frac{p-1}{p-2}\right]^{p-2}(p-1)\ . \qquad (3.3a)$$

The critical EA order parameter is

$$q_c=q\Big|_{T=T_g}=\frac{p-2}{\bar{r}_{0c}(p-1)}\ . \qquad (3.3b)$$

For $T=T_g$ the approach to Eq. (3.3b) is given by

$$C(t\to\infty)=\frac{p-2}{\bar{r}_{0c}(p-1)}+\frac{A}{t^{1-\alpha}}\ . \qquad (3.4a)$$

Here, A is a constant and α is an exponent that is between 0 and 1. An equation for α is obtained by inserting Eq. (3.4a) into Eq. (3.1). We obtain

$$2\Gamma^2(1-\alpha)=\Gamma(1-2\alpha)\ , \qquad (3.4b)$$

with Γ the gamma function. In deriving Eq. (3.4b) from Eqs. (3.1) it is clear that it is valid for any $p>2$ but not for $p=2$. Also, note that the EA order parameter at T_g is discontinuous for any $p>2$. We conclude that the usual SK model is an exceptional case. Solving Eq. (3.4b) yields $\alpha\simeq 0.395$. For $T\to T_g^+$ one also finds

$$\Gamma(T\to T_g^+)\sim|T-T_g|^{(1+\alpha)/2\alpha}\simeq|T-T_g|^{1.765} \qquad (3.5)$$

for any $p>2$.

The analytic results given by Eqs. (3.4) and (3.5) are quite surprising because they are independent of p for any $p>2$. For $p=2$ the exponent in Eq. (3.5) is clearly

unity. To check the asymptotic results presented above, we have numerically solved Eq. (3.1e) for several values of p. In Fig. 1 a plot of $\ln\phi(t)$ as a function of t for $p=2.01$ for various values of the coupling constants λ are given. For $\lambda<\lambda_c$ (~ 1.06) the decay is clearly exponential at long times. The predicted behavior of $\phi(t)$, as $t\to\infty$ for $\lambda=\lambda_c$ [cf. Eq. (3.4a)], was confirmed by plotting $\ln[\phi(t)-q_{\rm EA}]$ as a function of t for various values of p. This figure (not shown) yielded a straight line, the slope of which is $1-\alpha$. The value of α for $p=2.01$, 2.5, and 3.0 was equal to 0.39, in accord with the solution given by Eq. (3.4b). It should be pointed out that for p sufficiently close to 2, Eq. (3.4a) was satisfied, with $\alpha\simeq 0.395$, only for very long times.

From the results shown in Fig. 1, the value of the effective kinetic coefficient Γ can be obtained as a function of λ. In Fig. 2 we show a plot of Γ as a function of $\lambda_c-\lambda$ for $p=2.01$ and 2.5. Curves labeled (2) and (4) correspond to Γ obtained numerically for $p=2.5$ and 2.01, respectively. For comparison the asymptotic behavior predicted by Eqs. (3.5) is also shown. Curve (1) corresponds to $p=2.5$, while that labeled (3) is for $p=2.01$. The value of Γ is normalized so that, for $\lambda=0$, Γ is unity. Note that the value of Γ for $p=2.01$ have been divided by 10. It is clear that $\lambda_c-\lambda<0.008$, and the asymptotic result given by Eqs. (3.5) is in agreement with the numerical solution. This also suggests that Eqs. (3.5) are really only valid for T sufficiently close to T_g. Thus the numerical results shown in Figs. 1 and 2 support the asymptotic predictions of Eqs. (3.4) and (3.5).

For use later on we give Eqs. (3.3) to $O(\epsilon=p-2)$,

$$\mu_c\simeq\bar r_{0c}^{2+\epsilon}(1-\epsilon\ln\epsilon+\epsilon)\ , \tag{3.6a}$$

$$q_c\simeq\frac{\epsilon}{\bar r_{0c}}\ . \tag{3.6b}$$

Next, for small ϵ we discuss the two-loop or $O(u^2)$ corrections to Eqs. (3.1). At $T=T_g$, the equation for $\bar r_{0c}$ is modified by a term of $O[u^2C^3(t=0)]$. This term can be made arbitrarily small by a proper choice of u. The equation for q_c has an additional term that leads to $O(\epsilon^2)$ corrections to Eq. (3.6b). We conclude that at T_g the two-loop corrections do not change the structure of the theory to $O(\epsilon)$. This will also be confirmed in Sec. IV.

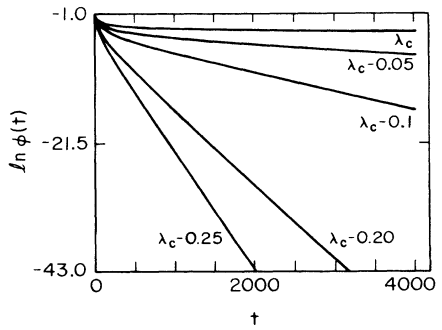

FIG. 1. Plot of $\ln\phi(t)$ for $p=2.01$ as a function of t [cf. Eq. (3.1e)] for various values of the coupling constants. The parameters correspond to $v_0=1.0$ and $\bar r_0=1.0$. The critical coupling constant λ_c is given by Eq. (3.3a).

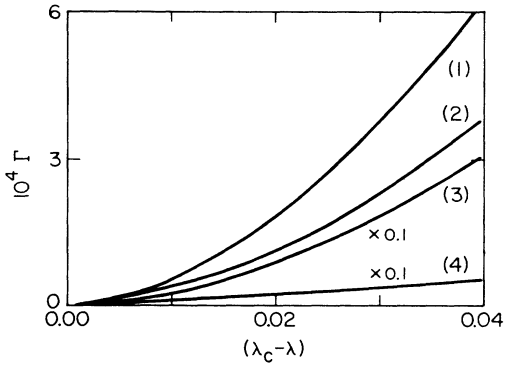

FIG. 2. Renormalized kinetic coefficient Γ as a function of $\lambda_c-\lambda$ for $p=2.01$ and 2.5. Curves (1) and (3) represent the asymptotic result, given by Eq. (3.5), for $p=2.5$ and 2.01, respectively. Curves (2) and (4) correspond to Γ obtained by numerically integrating Eq. (3.1e). The parameters are the same as for Fig. 1. The value of Γ for $p=2.01$ has been divided by 10.

Before continuing we wish to point out an important technical feature. The self-consistent equation for $C(t=0)$ given by Eq. (3.1c) ceases to have a physical solution for sufficiently large μ or low temperature. With the one-loop approximation given by Eq. (3.1c) the glass-transition temperature is in the physical range only for small ϵ and for $u>O(\epsilon\ln\epsilon)$. For larger ϵ, higher-order u terms are needed in Eq. (3.1c).

Finally, we discuss the assumption that $C(t=0)$ is continuous at T_g. This assumption seems physically well motivated for any glass transition.[23] Our discussion here, however, will lead to an interesting paradox which will be clarified in Sec. IV of this paper. We first argue that Eqs. (3.1) and (2.4) naively lead to a complete specification of the problem and that no assumption on $C(t=0)$ is needed. Assuming $C(t\to\infty)=q$, Eq. (2.4) yields

$$C(t=0)=\frac{1}{\bar r_0+\mu q^{p-1}}+q\ . \tag{3.7a}$$

Equations (2.14a) and (2.14c) give,

$$q=\frac{\mu q^{p-1}}{(\bar r_0+\mu q^{p-1})^2}\ . \tag{3.7b}$$

Equations (3.7a) and (3.7b) are not consistent with a continuous $C(t=0)$ at $T=T_g$, nor is Eq. (3.7b) consistent with Eq. (3.2b). It is straightforward to verify that the Eqs. (3.7) are identical to the equations (cf. Sec. IV obtained by a replica-symmetric (RS) equilibrium solution of the p-spin model. We conclude that a RS solution of the p-spin model does not lead to a T_g where the dynamical equations predict a glass transition. We solve this apparent paradox in Sec. IV (and the Appendix) by noting that a RS solution to the $(p>2)$-spin model is always unstable and cannot be used to locate T_g. We then relate the dynamical T_g to a broken RS solution of the p-spin model.

IV. EQUILIBRIUM DESCRIPTION OF THE SG TRANSITION IN $p=2+\epsilon$ MODELS

In this section we use equilibrium statistical mechanics and replica methods to discuss the SG transition in p-spin interaction SG models. In order to make our discussion as simple as possible, we will restrict ourselves to $p=2+\epsilon$, with $\epsilon \ll 1$. For this case we will show that the discontinuities at the equilibrium glass-transition temperature T_g', are of $O(\epsilon)$, and that the expansion of the free energy in terms of the order parameter is possible (at least near T_g'). In order to make connections with the dynamical calculations presented in Secs. II and III, we start with the soft-spin Hamiltonian given by Eq. (2.2). In our equilibrium calculation the hard or Ising limit is easily taken. In fact, the hard limit is algebraically simplier.

Generalizing the formalism of Gross and Mezard[8] for the hard p-spin model to the soft p-spin model,[7] we find that the free energy per spin is given by the maximum of the function ($J=1$ from now on),

$$\frac{\beta F}{N}=-\frac{\beta^2 C^p}{4}+\lambda_s C-\frac{\beta^2}{4n}\sum_{a\neq b}Q_{ab}^p+\frac{1}{2n}\sum_{a\neq b}\lambda_{ab}Q_{ab}$$
$$-\ln A_0-\frac{C_0^2}{4n}\sum_{a\neq b}\lambda_{ab}^2$$
$$-\frac{C_0^3}{6n}\sum_{\substack{a\neq b\neq c\\ c\neq a}}\lambda_{ab}\lambda_{bc}\lambda_{ca}+O(\lambda^4)\ , \tag{4.1a}$$

where $C_0=A_2/A_0$, with

$$A_n=\int d\sigma\,\sigma^n \exp[-\tfrac{1}{2}(r_0-2\lambda_s)\sigma^2-u\sigma^4]\ . \tag{4.1b}$$

Here, n ($\to 0$) is the number of replicas and a,b,c are replica indices. C gives the equal-time spin-correlation function ($C=C_0=1$, in the hard limit) which we have assumed to be replica independent, and Q_{ab} is the usual replica order parameter. λ_s and λ_{ab} are Lagrange multipliers related to C and Q_{ab} that have been introduced to carry out spin integrals in the mean-field limit.

We can discuss the equilibrium SG transition in the $p=2+\epsilon$ model in one of two equivalent ways. Earlier work on the Potts glass[5] and on the $p\to\infty$ model[3] suggest that only one replica-symmetry breaking (RSB) is needed to discuss the SG transition in the $p=2+\epsilon$ model. If the free energy is expressed in terms of one replica breaking,[3] which introduces the parameters λ_0, q_0, q_1 ($=q$), and λ_1 ($=\lambda$), it can be shown that the variational equations yield the solution $q_0=\lambda_0=0$. Thus one obtains a free energy that depends only on two SG order parameters, q and $y=1-m$. The replicas overlap with strength q or they do not, and the fraction of replicas that overlap is given by $y=1-m$. Both q and y are determined variationally from Eq. (4.1a). An equivalent procedure is to use the continuum of order parameters, $q(x)$ [and $\lambda(x)$], introduced by Parisi[24] to solve the SK model. Examination of the theory shows that the only physical order parameter must have the form, $0<x<1$,

$$q(x)=q\Theta(x-\bar{x})\ , \tag{4.2}$$

with q the strength of the overlap and $\bar{x}$ ($\to m$) related to the fraction of replicas that overlap. Once again, q and the break point $\bar{x}$ can be determined variationally. We use the second procedure here because it is easier to discuss (cf. the Appendix) the stability of the theory within this formulation.

With the Parisi ansatz, Eq. (4.1a) reads

$$\frac{\beta F}{N}=-\frac{\beta^2 C^p}{4}+\lambda_s C-\ln A_0+\int_0^1 dx\left[\frac{\beta^2 q^p(x)}{4}-\frac{q(x)\lambda(x)}{2}+\frac{C_0^2\lambda^2(x)}{4}\right]$$
$$-\frac{C_0^3}{6}\int_0^1 dx\left[x\lambda^3(x)+3\lambda(x)\int_0^x dy\,\lambda^2(y)\right]+O(\lambda^4)\ . \tag{4.3}$$

The saddle-point (SP) equation $\delta F/\delta q(x)=0$ yields,

$$\lambda(x)=\mu q^{p-1}(x)\ , \tag{4.4a}$$

and $\delta F/\delta\lambda(x)=0$ gives

$$q(x)=C_0^2\lambda(x)-C_0^3 x\lambda^2(x)-C_0^3\int_0^x dy\,\lambda^2(y)$$
$$-2C_0^3\lambda(x)\int_x^1 dy\,\lambda(y)+O(\lambda^3)\ . \tag{4.4b}$$

By repeated differentiation of Eq. (4.4b) with respect to x it is easy to show that if the $O(\lambda^3)$ term is neglected, then a physical, nondecreasing-function-of-x, continuous, Parisi order parameter is not possible. The conclusion is that the only possible Parisi order parameter is the discontinuous one given by Eq. (4.2). Using Eq. (4.2) in Eq. (4.3) yields

$$\frac{\beta F}{N}=-\frac{\beta^2 C^p}{4}+\lambda_s C-\ln A_0$$
$$+(1-\bar{x})\left[\frac{\beta^2 q^p}{4}-\frac{q\lambda}{2}+\frac{C_0^2\lambda^2}{4}\right]$$
$$-\frac{C_0^3}{6}(1-\bar{x})(2-\bar{x})\lambda^3\ . \tag{4.5}$$

The SP solution of Eq. (4.5) yields

$$1=C_0^2\mu q^{p-2}-C_0^3\mu^2 q^{2p-3}(2-\bar{x})\ , \tag{4.6a}$$

$$1-\bar{x}=\frac{3}{C_0^3\mu^2}q^{3-2p}\left[\frac{1-p}{2p}+\frac{\mu C_0^2}{4}q^{p-2}\frac{\mu^2 C_0^3}{6}q^{2p-3}\right]\ , \tag{4.6b}$$

$$\lambda_s = \frac{\mu}{2} C^{p-1} . \tag{4.6c}$$

A nontrivial dependence of Eqs. (4.2) and (4.6a) on $\bar{x}$ indicates that RS is broken. For $p=2$ it is easy to show that Eqs. (4.6a) and (4.6b) lead to $\bar{x}=0$. This is due to the fact that RS is broken in the SK model only if terms of $O(\lambda^4)$ are retained in Eqs. (4.1). We show here that RS is broken much more strongly, even in the λ^3 theory, in $p>2$ SG models. We also note that structure of the theory to $O(\lambda^3)$ is independent of the order in u considered since in Eqs. (4.6) only C_0 appears. We conclude that the SG transition in the hard and soft models are identical for small ϵ.

An equation for C follows from maximizing Eq. (4.5) with respect to λ_s. The equilibrium critical temperature T_g' is obtained when these equations first have a physical solution, $q>0$, $1-\bar{x}>0$. At critically, $\bar{x}=1$, and the self-consistent one-loop equation for $C(T=T_g')=C_c$ is given by

$$C_c \equiv C_{0c} = \frac{1}{\bar{r}_{0c}} = (r_0 - \mu_c C_c^{p-1} + 12uC_c)^{-1} . \tag{4.7a}$$

The critical parameters, to $O(\epsilon)$ are

$$q_c \simeq \frac{3\epsilon}{2\bar{r}_{0c}} , \tag{4.7b}$$

$$\mu_c \simeq \bar{r}_{0c}^{2+\epsilon}[1-\epsilon \ln(\tfrac{3}{2}\epsilon)+\tfrac{3}{2}\epsilon] . \tag{4.7c}$$

Comparing Eqs. (3.6a) and (4.7c) and using $\mu=\beta^2 p/2$, we see that the dynamical transition occurs before the equilibrium transition. This will be discussed further below. To $O(t=1-T/T_g')$ the soft model is cumbersome. In the hard limit one obtains

$$q \simeq \tfrac{3}{2}(\epsilon+t) , \tag{4.8a}$$

$$\bar{x} \simeq 1-t/\epsilon , \tag{4.8b}$$

$$(T_g')^2 \simeq 1+\epsilon \ln(\tfrac{3}{2}-\epsilon)-\epsilon , \tag{4.8c}$$

and the free energy given by

$$\frac{\beta F}{N} = -\frac{\beta^2}{4} - \ln 2 + (1-\bar{x})^2 \frac{\mu^3 q^{3p-3}}{6} . \tag{4.8d}$$

Note that $F_{SG} > F_{PM}$, as is usual for SG. From Eq. (4.8d) it follows that there is no latent heat at T_g' and that the specific heat is discontinuous at T_g'. In the Appendix we show that this SG phase is stable (within the Parisi subspace) according to a local replica-based stability analysis.

Next, we discuss the connection between this equilibrium SG transition and the dynamical transition discussed in Sec. III. First we expand Eq.(3.2b) in powers of q to an order consistent with Eqs. (4.1a) and (4.6a),

$$1 = C_0^2 \mu q^{p-2} - C_0^3 \mu q^{2p-3} , \tag{4.9a}$$

with C_0 given by Eq. (4.7a) or (3.1c). For comparison we also expand Eq. (3.7b),

$$1 = C_0^2 \mu q^{p-2} - 2C_0^3 \mu^2 q^{3p-3} . \tag{4.9b}$$

We next compare these expansions with Eq. (4.6a). Equation (4.9b) corresponds to Eq. (4.6a) with $\bar{x}=0$ and, as discussed in Sec. III, it is the RS solution of the p-spin model. The dynamical theory leads to Eq. (4.9a), which corresponds to Eq. (4.6b) with $\bar{x}=1$, and it corresponds to the BRS solution of the p-spin model at critically, where $\bar{x}=1$. However, at T_g' the static theory also requires that Eq. (4.6b) be satisfied, and this additional equation leads to $T_g' < T_g$, with T_g the transition temperature according to the dynamical theory. The important point is that the dynamical theories for this class of models lead to a T_g that is apparently *greater* than that predicted by the equilibrium theory. We interpret this as the free energy being maximized at T_g by the value $\bar{x}$ at its physical endpoint $\bar{x}=1$ in the temperature $T_g > T > T_g'$. The variational equation for $\bar{x}$ given by Eq. (4.6b) is not a relevant equation if F as a function of $\bar{x}$ is not maximized in the physical region, $0<\bar{x}<1$. We discuss this point further in Sec. V. Finally, we point out that the BRS solution gives a continuous C at T_g since $\bar{x}=1$. This is in accord with the dynamical theory.

Technically, the freezing predicted by the dynamical theory is easy to understand using local stability theory. For $T>T_g$ the dynamical equation for $q=C(t\to\infty)$ has the stable trivial solutions and unphysical and unstable complex solutions. At T_g two of the complex solutions become degenerate real solutions. At T_g there are three real critical points x_i $(i=1,2,3)$ of the dynamical equation that satisfy $x_1=0<x_2<x_3=q_c<C(t=0)$. The fixed points x_1 and $x_3=q_c$ are stable and x_2 is an unstable fixed point. We then have a situation where $C(t\to\infty)=0$ and $C(t\to\infty)=q_c$ are both stable solutions, but where it is impossible to reach $C(t\to\infty)=0$ due to the intervening unstable fixed point. The only possible conclusion is that the system freezes into a SG state because it cannot reach the equilibrium state defined by $C(t\to\infty)=0$. It is interesting to point out that in the SK model the situation is quite different. The dynamical equation has only two fixed points and at T_g there is an exchange of stability between the fixed points.

V. DISCUSSION

We conclude this paper with a discussion of our results.

(1) We have emphasized the discontinuous nature of the SG transition in the $p>2$ SG models. It is interesting to point out that the transition is only partly discontinuous. In the thermodynamic transition q_{EA} is discontinuous, but there is no latent heat at T_g' and the specific heat is discontinuous at T_g'. These last two features are common to standard mean-field continuous transitions. In the dynamical transition, q_{EA} is also discontinuous, but there is a continuous slowing down as T_g is approached from above.

It is also interesting to point out the close formal connections with the dynamical structural glass theories. The analog of q_{EA} is the long-time limit of the density correlation function (DCF), which is discontinuous at the (mean-field) dynamical transition temperature T_g. Physically, the discontinuity seems unavoidable. It reflects the fact that the static elastic constants, or Debye-Waller factors, in the glassy phase are also

nonzero in the liquid phase at finite frequency. At T_g the finite-frequency elastic constants in the liquid phase cross over smoothly to the static elastic constants in the glass. (2) The SG transition in the usual SK model is a blocking transition. A continuation, to $T < T_g$, of the paramagnetic free energy, $F_{\rm PM}$ is a lower than the SG free energy $F_{\rm SG}$. The physical reason for the occurrence of the SG transition is the occurrence of a shattered spin-glass susceptability in the paramagnetic phase at T_g. The occurrence of the SG phase prevents a catastrophe for all $T < T_g$.

The situation in the $p > 2$ SG model is less clear. The thermodynamic transition discussed in Sec. IV does not appear to be a blocking transition. It has been conjectured[5] that in the Potts model the physical mechanism for the transition is related to finite-N fluctuations that become divergent as $T \to T_g'$. This is in agreement with results for the $p \to \infty$ SG model.[3] It is interesting to point out that in the dynamical transition discussed in Sec. III, the effective SG susceptability is singular (cf. the Appendix) as $T \to T_g$.

(3) It is interesting to consider the free-energy surface as a function of q and $\bar{x}$. At T_g, $F_{\rm SG}(q,\bar{x})$ is a maximum in the physical region, $0 \le \bar{x} \le 1$, the $\bar{x} = 1$ and q given by Eq. (4.6a). For $T_g > T > T_g'$, this solution continues to be a maximum as long as q is given in Eq. (4.6a) and $\bar{x}$ is fixed to be unity. For $T < T_g'$, $F_{\rm SG}$ develops a maximum for q and $\bar{x}$ given by Eqs. (4.6a) and (4.6b). This suggests that the break point $\bar{x}$ has a step-function structure and that in reality there are two transitions. The first occurs at T_g and corresponds to the dynamical transition where there is critical slowing down as $T \to T_g^+$. The free energy is simply $F_{\rm PM}$ since $\bar{x}$ is fixed to be unity and, as a consequence, there are no obvious thermodynamic anomalies. The second transition is the thermodynamic one at T_g', which was discussed in Sec. IV. There is no latent heat, but the specific heat is discontinuous at T_g'.

(4) Spin glasses are characterized by many macroscopically equivalent free-energy states. In mean-field SG models, the barriers between these states are macroscopically large. It is unclear whether the barriers are macroscopic in finite-range spin glasses. It seems likely that in structural glasses some of the barriers are not macroscopic.[25] The transitions between the states will be responsible for the slow transport processes[26] and hysterisis phenomena, which occur in the glass-transition region. Using this and the discussion in point (3) above, we can present a scenario for the glass transition and an interpretation of the dynamical theories of the structural glass transition.

We first assume that the dynamical structural glass theories are at the level of the mean-field SG theory presented here and that glass transitions with discontinuous EA order parameters are all similar. The predicted transition temperature should be identified with a temperature where there is a dramatic change (at least in fragile glass formers) in the slope of the Arrhenius plot of the transport properties.[27] Owing to transitions between different free-energy states, transport processes are not frozen below this temperature, but they are extremely slow. At lower temperature, at the analog of T_g' here, there is a smeared-out thermodynamic transition where there is a rapid change in the heat capacity. This temperature would correspond to the conventional glass-transition temperature. In agreement with the picture given above, it is interesting to point out that in structural glasses transport changes smoothly as one goes through the region where the specific heat changes markedly.

(5) The role of frustration (in the sense of competing interactions) in dictating the dynamics in supercooled liquids, as the structural glass transition is approached, is not obvious. In fact, noting that a system of particles interacting by purely repulsive potential (in the extreme-limit hard spheres) can form structures resembling those of metallic glasses[28] suggests that frustration in the sense used in the context of spin glasses is not required. If frustration is evident then it is of a different type. Furthermore, the equations containing the nonlinear feedback mechanism,[29] which lead to the continuous slowing down of the transport coefficients in the structural glass problem, were obtained without any obvious appeal to frustration.

It is interesting to point out that in spin models with randomness but no frustration the dynamics in the ergodic phase at low temperatures is similar to that observed in a frustrated and disordered spin system as the spin-glass–transition temperature is approached.[30,31] This is because in both cases the free-energy surface has a large number of local minima (metastable states) separated by barriers which can be surmounted by dynamics involving a small number of spins. The only effect of frustration seems to be in producing many symmetry-unrelated ground states. This probably has an insignificant consequence on the continuous slowing down at temperatures where the system does not explore the topology of the free-energy surface near the ground state.

Finally, we point out that, in the dynamical theories of the spin-glass transition, the role of frustration in the ergodic phase is not obvious after the quenched average over the random interactions is carried out. What seems essential in causing the critical slowing down as $T \to T_g^+$ is the form of the nonlinearities in the dynamical equations. As has already been emphasized, the dynamical theory for the $p > 2$ spin-glass models (at least in the ergodic phase) are actually very similar to the dynamical equations in the structural glass theories.

(6) In the analysis presented in Sec. IV, we neglected the $O(\lambda^4)$ term in Eq. (4.3). Including this term leads to another SG transition at temperatures less than T_g'.[3] The new SG phase has a more complicated (there are many RS breakings) Parisi order parameter. The resulting spin-glass transition can be investigated by retaining higher-order λ terms in the expansion of the free energy.[24] It is interesting to note that in structural glasses there are often additional transitions below the conventional glass-transition temperature.

(7) We caution the reader that our discussion at the end of Sec. III does not imply a violation of fluctuation dissipation theorem in our solution. Rather, it implies that Eqs. (2.14) are not valid for $T = T_g$ if $C(t=0)$ is

continuous at T_g and if q_{EA} is discontinuous at T_g. For a discussion of this point in a different context, see the Appendix of the paper by Göetze and Sjögren.[32]

(8) It should be pointed out that if there is an equilibrium transition at T_g, with $\bar{x}$ fixed to be 1, then it seems problematic to describe it with the replica approach, as is done here. The reason is because in the replica approach there are factors of $1-\bar{x}$ that are assumed to be nonzero in obtaining the equation of state given by Eq. (4.6a). This technical problem is due to the fact that $\bar{x}=1$ implies that there is no overlap between replicas for the discontinuous Parisi solution of the form given in Eq. (4.2). On the other hand, Eqs. (4.1) arise from a formalism which involves an order-parameter expansion of what is essentially a replica overlap. Even with this problem we feel that the procedure used in Sec. IV is reasonable. If $\bar{x}=1$, then there is only self-overlap, $q_{\alpha\alpha}$. However, it is known that $q(x=1)=q_{EA}$ in the Parisi approach.[34]

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APPENDIX

In this Appendix we consider the stability of the equilibrium SG phase discussed in Sec. IV. Here we restrict ourselves to a stability analysis in the Parisi subspace. We also comment on the local stability of the dynamic transition discussed in Sec. III. For algebraic simplicity, we consider the hard p $(=2+\epsilon)$ model for which the free energy may be written as

$$\frac{\beta F}{N}=-\ln 2-\frac{\beta^2}{4}+\max\left[\frac{1}{n}\left[-\frac{\beta^2}{4}\sum_{a,b}Q_{ab}^p+\frac{1}{2}\sum_{a,b}\lambda_{ab}Q_{ab}-\frac{1}{4}\sum_{a,b}\lambda_{ab}^2-\frac{1}{6}\sum_{a\neq b\neq c}\lambda_{ab}\lambda_{bc}\lambda_{ca}\right]\right]+O(\lambda^4)\ . \tag{A1}$$

Using the Parisi ansatz for Q_{ab}, the continuum limit of Eq. (A1) becomes

$$\frac{\beta F}{N}=-\ln 2-\frac{\beta^2}{4}-\int_0^1 dx\left[\frac{\mu q^p(x)}{2p}-\frac{1}{2}\lambda(x)q(x)+\frac{1}{4}\lambda^2(x)\right]-\frac{1}{6}\int_0^1 dx\left[x\lambda^3(x)+3\lambda(x)\int_0^x\lambda^2(x)dx\right]\ . \tag{A2}$$

Thus, in order to ascertain the stability of the solution presented in Sec. IV, we have to analyze the second variation of $\beta F/N$ with respect to both $\lambda(x)$ and $q(x)$. This would yield a 2×2 matrix whose eigenvalue spectrum will indicate the stability of the variational solution. Here we perform a restricted analysis;[33] i.e., in the free energy given by Eq. (A2) we use $\lambda(x)=\mu q^{p-1}(x)$ [cf. Eq. (4.4a)] to eliminate $\lambda(x)$. Therefore, fluctuations in $\lambda(x)$ are manifested only through fluctuations in $q(x)$.

With this the crucial quantity that determines the stability of the SG phase is

$$F_2=\frac{1}{2}\int_0^1 dx\int_0^1 dy\left.\frac{\delta^2\beta F/N}{\delta q(x)\delta q(y)}\right|_{eq}\delta q(x)\delta q(y)\ . \tag{A3}$$

Here, $\delta q(x)$ is the fluctuation in the order parameter $q(x)$, and Eq. (A3) denotes that the second variational of $\beta F/N$ is to be evaluated with the equilibrium $q(x)$ given by Eqs. (4.2) and (4.8). The SG phase is locally stable if $F_2<0$ for any $\delta q(x)$. Substituting Eq. (4.2) into (A3) yields

$$F_2=F_{21}+F_{22}\ , \tag{A4a}$$

with

$$F_{21}=\frac{1}{2}\int_{\bar{x}}^1 dx[\delta q(x)]^2\Delta \tag{A4b}$$

and

$$F_{22}=-\frac{\mu^3(p-1)^2q^{3p-5}}{2}\left[\int_{\bar{x}}^1 dx\ \delta q(x)\right]^2\ , \tag{A4c}$$

where

$$\Delta=\frac{\mu}{2}(p-1)[-(p-1)q^{p-2}+\mu(2p-3)q^{2p-4}-2\mu^2(2p-3)q^{3p-5}+\mu^2(p-2)q^{3p-5}\bar{x}]\ . \tag{A4d}$$

Since F_{22} is a negative semidefinite, we concentrate here on F_{21}. Letting $p=2+\epsilon$ and using Eq. (4.8a), we obtain, at $t=0$,

$$\Delta=-\frac{\mu_c}{4}(p-1)q^{\epsilon}[\epsilon+O(\epsilon^2\ln\epsilon)]\ . \tag{A5}$$

Since $\Delta<0$, we conclude that the variational solution of the SG phase discussed in Sec. IV is locally stable.

We conclude with two comments. First, the SG transition predicted by the RS solution ($\bar{x}=0$) can be easily shown to be unstable even at $t=0$. This implies that RS methods cannot be used to locate the glass-transition temperature if the transition is discontinuous. Second, if we use the critical parameters implied by the dynamical theory,

$$q_c\simeq\epsilon\ , \tag{A6a}$$

$$\mu_c\simeq 1-\epsilon\ln\epsilon+\epsilon\ , \tag{A6b}$$

then this phase transition can be shown to be marginally stable at T_g according to Eq. (A4d).

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