Nearest-neighbor frustrated random-bond model in $d=2$: Some exact results

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We study a $d=2$ Ising model where the veritcal bonds are fixed and ferromagnetic and the horizontal bonds can vary randomly in sign and in magnitude (within some limits) but are same within each now. The model therefore generalizes that of McCoy and Wu since it allows for the interesting case of frustration. We use the transfer matrix to map our problem to a collection of random field $d=1$ problems about which a lot is known. We find generally three transitions: a Griffiths transition, its dual version, and one with infinite correlation length and index $\nu=1$. In all cases the free energy has infinitely differentiable singularities. In addition there are some zero-temperature transitions.

I. INTRODUCTION

This paper is devoted to a study of a certain class of random-bond Ising models in $d=2$. The reasons for studying random systems are the usual ones: they are a theoretical challenge and of considerable practical relevence since the truly homogeneous model is only an idealization (though there are enough real life systems that resemble the latter to make their study worthwhile). The choice of $d=2$ came from a desire to obtain some exact information without the introduction of long-range interactions (e.g., following the procedure of Sherrington and Kirkpatrick[1]) which make the problem tractable in $d=3$. Here we allow only nearest-neighbor interactions, avoid the replica trick, and obtain several exact results.

The models we study are described as follows: The vertical coupling is ferromagnetic and equals K in the whole lattice. The horizontal couplings are random, can have either sign, but are required to be same in each row. We will also assume that they are bound in their variation by some K_1 and K_2. Since this model is closely related to the one studied by McCoy and Wu[2] (MW), let us first clarify the relation between the two. In the MW case the horizontal bonds are fixed and the vertical bonds are random but equal within each row. This makes the two models inequivalent: our model can have frustration if two adjacent horizontal rows have opposite signs for the horizontal bonds while in their case the product of the signs of the bonds around any plaquette is always positive. Indeed this is why they restricted their random interactions to be ferromagnetic. (One could relate our model to theirs by a duality transformation; however, this can become awkward when ferromagnetic and antiferromagnetic bonds coexist, since the dual temperatures will then become complex.) The original MW papers, which specialized to a very narrow distribution of randomness, were eventually generalized by McCoy[3] to a broader class for which he obtained the transition temperature and showed that at the phase transition from paramagnetism to ferromagnetism, the free energy had an infinitely differentiable singularity.

In this paper we show that for the case where K_1 and K_2 are both positive there are three phase transitions: as we lower T, there is a Griffiths[4] transition, a paramagnetic to ferromagnetic transition with correlation length index $\nu=1$, and finally a new transition at finite correlation length which can only be described as the dual of the Griffith's transition in the McCoy-Wu model. If K_1 is negative, there are generically three transitions, but the order parameter (if any) of the low-T phase is unclear. If $K_1=K_2$, the two finite correlation length transitions coincide and there is a transition at $T=0$ with $\nu=\infty$. In all cases the free energy remains infinitely differentiable at the transition.

Not only is our problem different from that of MW as explained above, so is our approach to the solution. While they deal directly with the determinant that occurs in the Pfaffian approach (but this time with random elements) we work with the row-to-row transfer matrix. We exploit the translation invariance of each row to express the free energy of the $d=2$ *random-bond* model as a sum of the free energies of a collection of $d=1$ *random-field* models. Of course such a sum must and eventually does appear in the MW formula, but they do not rely on this mapping as we do. We find the mapping to $d=1$ Ising models very useful since one has a lot of intuition for the $(d=1)$ problem as well as many exact results,[5-7] especially due to Derrida and Hillhorst[8] and Nieuwenhuizen and Luck.[9] We are also able to make some progress in the calculation of correlations because of this mapping. Overall, we find that this approach makes our research easier in many ways.

In the next section we set up the notation and as an introduction to our approach, provide a solution of the usual nonrandom problem via our mapping. In Sec. III we consider the case where the horizontal coupling takes on two values K_1 and K_2, both ferromagnetic, with equal probability. We argue that nothing of substance changes if we fill in the region between K_1 and K_2 with some smooth distribution. We briefly sketch the proof that $\nu=1$ and leave some details for the appendixes. Section IV deals with the problem where K_1 can be negative, i.e., there is frustration. Surprisingly, the computation of the free energy is no more complicated than in the unfrustrat-

ed case though the nature of the low-temperature phase changes from being ferromagnetic to something unknown. We summarily treat the special case $K_1=-K_2$. The last section contains a summary and discussion of this work.

II. A NEW LOOK AT THE HOMOGENEOUS MODEL

In this section we explain what we mean by the $d=1$ mapping by solving the homogeneous (Onsager) problem using it. Let us begin with all couplings equal to K so that

$$Z=\sum_{S_i}\exp\left[K\sum_{\langle ij\rangle}S_iS_j\right]. \tag{2.1}$$

We can write Z as $\mathrm{Tr}T^N$ (where N is the number of rows) if we choose as the row-to-row transfer matrix

$$T=(\mathcal{I})\exp\left[K^*\sum_m\sigma_1(m)\right]\exp\left[K\sum_m\sigma_3(m)\sigma_3(m+1)\right], \tag{2.2}$$

where $\sigma(m)$ are Pauli matrices at site m in a row, $K^*=-\frac{1}{2}\ln\tanh K$, and $\mathcal{I}$ denotes a factor that can be ignored hereafter. We then introduce two Majorana (Hermitian) fields at each site:

$$\psi_1(m)=\frac{1}{\sqrt{2}}(\overleftarrow{\pi}\sigma_1)\sigma_2(m),\quad \psi_2(m)=\frac{1}{\sqrt{2}}(\overleftarrow{\pi}\sigma_1)\sigma_3(m), \tag{2.3}$$

where $\overleftarrow{\pi}$ denotes a product from site $m-1$ to the left end. It is readily verified that

$$\{\psi_i(m),\psi_j(n)\}_+=\delta_{ij}\delta_{mn} \tag{2.4}$$

and that

$$T=\exp\left[-2iK^*\sum\psi_1(m)\psi_2(m)\right]\times\exp\left[2iK\sum\psi_1(m)\psi_2(m+1)\right]. \tag{2.5}$$

(The last few steps differ from Schultz, Mattis, and Lieb[10] only in that we use Majorana and not Dirac spinors, the former being the more natural ones for this problem.) Finally, let us expand ψ_i in plane waves, making manifest $\psi_i^\dagger=\psi_i$:

$$\psi_i(n)=\int_0^\pi\frac{dq}{2\pi}[e^{iqn}C_i(q)+e^{-iqn}C_i^\dagger(q)]. \tag{2.6}$$

The C's obey the usual fermion anticommutation rules. In terms of these,

$$T=\exp\left[\int_0^\pi[C_1(q)C_2^\dagger(q)+C_1^\dagger(q)C_2(q)](-2iK^*)dq\right]\exp\left[\int_0^\pi[C_1(q)C_2^\dagger(q)e^{-iq}+C_1^\dagger(q)C_2(q)e^{iq}](2iK)dq\right] \tag{2.7a}$$

$$=\bigotimes_q T(q), \tag{2.7b}$$

where

$$T(q)=\exp\{-2iK^*[C_1(q)C_2^\dagger(q)+C_1^\dagger(q)C_2(q)]\}\times\exp\{2iK[C_1(q)C_2^\dagger(q)e^{-iq}+C_1^\dagger(q)C_2(q)e^{iq}]\} \tag{2.8}$$

In going from (2.7a) to (2.7b) we have used the fact that fermion bilinear operators at different q commute.

Let us next note that each $T(q)$ acts on a four-dimensional Fock space labeled by the eigenvalues n_1 and n_2 (=0 or 1) of $C_1^\dagger C_1$ and $C_2^\dagger C_2$. Now

$$T(q)|n_1=0,\ n_2=0\rangle=|n_1=0,\ n_2=0\rangle,\qquad T(q)|11\rangle=|11\rangle \tag{2.9}$$

for all K, K^*, and q, a result that follows if we expand the exponentials in Eq. (2.8) and note that these states are annihilated by every operator in the exponents and hence by every term in the expansion except the identity. Let us therefore focus on $\hat{T}(q)$, the restriction of $T(q)$ to the subspace $|-\rangle\equiv|01\rangle$ and $|10\rangle\equiv|+\rangle$. Since

$$-C_1C_2^\dagger|+\rangle=0,\qquad -C_1C_2^\dagger|-\rangle=|+\rangle, \tag{2.10}$$

we may represent $-C_1C_2^\dagger$ as $\tau_+=\frac{1}{2}(\tau_1+i\tau_2)$ (where τ_1 and τ_2 are again Pauli matrices), and $\hat{T}(q)$ as

$$\hat{T}(q)=\exp[2iK^*(\tau_+-\tau_-)]\exp[-2iK(\tau_+e^{-iq}-\tau_-e^{iq})] \tag{2.11}$$

$$=\exp[-2K^*\tau_2]\exp[2K(\tau_2\cos q-\tau_1\sin q)]. \tag{2.12}$$

If we now switch to τ matrices "rotated" by an angle q, we get

$$\hat{T}(q)=\exp[-2K^*(\tau_2\cos q+\tau_1\sin q)]\exp(2K\tau_2). \tag{2.13}$$

Finally, let us reverse the sign of τ_1 and τ_2 (this does not affect their anticommutation relations $\{\tau_i,\tau_j\}=2\delta_{ij}$) and rename τ_2 as τ_3 to get

$$\hat{T}(q)=\exp[2K^*(\tau_3\cos q+\tau_1\sin q)]\exp(-2K\tau_3). \tag{2.14}$$

Let us now observe that $\hat{T}(q)$ is the transfer matrix for the following $d=1$ Ising model in a magnetic field:

$$Z=\sum_{S_i}\exp\left[\sum_i[J(q)(S_iS_{i+1}-1)+h(q)S_i+f_0(q)]\right] \tag{2.15}$$

with

$$e^{-2J(q)}=\frac{S^*\sin q}{C^*+S^*\cos q}\ , \tag{2.16}$$

$$h(q)=\frac{1}{2}\ln\left[\frac{C^*+S^*\cos q}{C^*-S^*\cos q}\right]-2K\equiv 2[K^*(q)-K]\ , \tag{2.17}$$

$$f_0(q)=\tfrac{1}{2}\ln[1+(S^*)^2\sin^2 q] \tag{2.18}$$

with

$$C^*=\cosh 2K^*,\quad S^*=\sinh 2K^*\ . \tag{2.19}$$

Note that J is always ferromagnetic. Let us denote the free energy per site of this model (in the thermodynamic limit) by $f(J(q,K), h(q,K))\equiv f(q,K)$. Now the partition function of the original $d=2$ model is

$$\begin{aligned} Z=\mathrm{Tr}T^N&=\mathrm{Tr}\left[\otimes_q T(q)\right]^N\\ &=\prod_q \mathrm{Tr}[T(q)^N]\\ &=\prod_q [2+\mathrm{Tr}\hat{T}(q)^N]\ . \end{aligned} \tag{2.20}$$

In reaching the last line we have used the fact that the trace of a tensor product is the product of the traces in each 4×4 space and that T equals 1 in the $|00\rangle$ and $|11\rangle$ subspace and equals $\hat{T}$ in remaining subspace. Next, we observe that $\hat{T}(q)$ is the matrix of determinant 1 [see Eq. (2.12)] being a product of two matrices, both of which are exponentials of traceless matrices. Consequently, its eigenvalues will be of the form $\exp[\pm\lambda(q)]$. Evidently, the positive root will dominate in Eq. (2.20) and the free energy per site of the $d=2$ model in the thermodynamic limit will be

$$\begin{aligned} F(k)&=\int_0^\pi \frac{dq}{2\pi}\lambda(q)\\ &=\int_0^\pi \frac{dq}{2\pi}f(q,K)\ , \end{aligned} \tag{2.21}$$

since $\lambda(q)$ is none other than the free energy $f(q,K)$ of the $d=1$ model defined in Eqs. (2.14)–(2.19).

Now, it is well known that $F(K)$ has the form of a q integral due to translational invariance. However, the fact that the integrand is the free energy of a $d=1$ model in a field has not been exploited before to our knowledge though Derrida and Hillhorst (DH) made a passing reference to it.[8] Let us now begin this exploitation with this homogeneous case as preparation for the random case.

In the complex q and K variables, $F(K)$ is defined by performing the line integral of $f(q,K)$ in the q plane between $q=0$ and π. The integral can be singular only if the integrand is singular. When will that happen? We know that the $d=1$ model has a transition only at $J=\infty$, $h=0$, i.e., $q=0$, $K=K^*$ according to Eqs. (2.16)–(2.19). For future use let us recall why this is so. Since $\exp f$ is an eigenvalue of a 2×2 matrix $\hat{T}(q)$, it can be singular only if it is degenerate with the other eigenvalue $\exp(-f)$, i.e., when $\exp(\pm f)=1$ or $\hat{T}=I$. From inspecting Eq. (2.14) we see $\hat{T}(q)=I$ at $q=0$, $K=K^*$. In fact, we can find $f(0,K)$ simply by inspecting Eq. (2.14): it is given by $2|K-K^*|$. Thus the $q=0$ model has a *first-order* transition in $t=K-K^*$. However, models with $q\simeq 0$, although not singular, are arbitrarily close to being so and we must integrate over q, to get the singularity in F. To do this we expand, ignoring f_0,

$$f(J,h)=\ln\{\cosh h+[\sinh h+\exp(-4J)]^{1/2}\} \tag{2.22}$$

for small h and e^{-4J}:

$$f\simeq \ln[1+(h^2+e^{-4J})^{1/2}]\simeq (h^2+e^{-4J})^{1/2}\ .$$

In our problem

$$\begin{aligned} h=h(q)&=2[K^*(q)-K]=2[K^*(0)-K]+O(q^2)\\ &\equiv 2(K^*-K)+O(q^2)\\ &\equiv t+O(q^2)\ , \end{aligned} \tag{2.23a}$$

$$\begin{aligned} e^{-4J(q)}&=(S^*)^2q^2(1+O[q^2,t)]\\ &=q^2[1+O(q^2,t)]\ \text{ since } S^*(K=K^*)=1\ , \end{aligned} \tag{2.23b}$$

$$f(q\simeq 0,\ K\simeq K^*)=(t^2+q^2)^{1/2}[1+O(q^2,t,)]$$

$$F(t)=\int dq(t^2+q^2)^{1/2}=t^2\ln t+\text{less singular terms} \tag{2.24}$$

which is the familiar result. One point of this exercise that we will recall later is that whenever the $q=0$ mode (which is always trivial to study) goes singular, so will F, but the singularity in F will be different (softer).

Our analysis reveals also that $q=\pi$ is just as easy to study. Upon inspecting Eq. (2.14) we see that $f(\pi,K)=|K+K^*|$. This becomes singular if K is antiferromagnetic and the phase transition in $d=2$ corresponds to the change from the paramagnetic state to the striped one with the sign of magnetization alternating with the columns.

We conclude with a pictorial description of all this in preparation for the random case. Figure 1 shows, at low temperatures, the range of $K^*(q)$ as a shaded region between $-K^*=K^*(\pi)$ and $K^*=K^*(0)$. Also shown is K. Each point in the shaded region corresponds to a $d=1$ model: its coupling $J(q)$ is given by Eq. (2.16), and is finite except at 0 and π. Its $h(q)$ is given (to a factor of -2) by the distance between that point $K^*(q)$ and K (see Fig. 1). At the low temperature shown, all models are subject to negative h's. As we raise the temperature the shaded region will expand out symmetrically and meet K

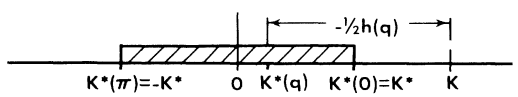

FIG. 1. Schematic description of the homogeneous $d=2$ model as a collection of $d=1$ models. Each point in the shaded region is a $d=1$ model labeled by some q between 0 and π. The coupling is infinite at 0 and π. The field $h(q)$ is as shown. As T rises, the shaded area expands out to meet the left moving spike marked K.

(which is moving inwards) at $K=K^*$. The picture tells us something should happen now and of course it does: the $q=0$ model has a sign reversal in its field. Since all spins are rigidly locked by the infinite J, they flip over at this point and give a free energy $f=|h|=|t|$. Models with q small but not zero [i.e., $J(q)$ large but not ∞] are having their h's reversed as we raise the temperature further. In fact for all $K<K^*$, there is some model with $h(q)=0$, i.e., $h(q)$ changing sign, but only the $J=\infty$ model responds with a singularity. However, models at arbitrarily small q are close to being singular and their effect is to produce a $t^2 \ln t$ singularity in $F(t)$. (Had we chosen K negative, the $q=\pi$ front of the shaded region, expanding leftwards would have collided with the negative K that is moving in.) After this preparation we are ready for the first random case.

III. RANDOM FERROMAGNETIC BONDS

Let us begin with the simple case where all vertical bonds equal K and the horizontal bonds, constrained to be same within a row, can vary from row to row assuming the values K_1 or K_2 (both >0) with equal probability. Physical arguments or Furstenberg's theorem[11] can be invoked to show that the free energy per site will, with probability 1, reach a sample-independent limit in the thermodynamic limit. Consider the evaluation of this limit for a given random sample. For illustrative purposes we will assume that in the rows numbered 1 to 6 in this sample the bonds have the values K_1, K_1, K_2, K_1, K_2, and K_1. How far can we go with our analysis from Sec. II? First, we can always write a row-to-row transfer matrix, but it can be either T_1 or T_2 obtained by setting $K=K_1$ or K_2 in Eq. (2.2). We can then write

$$Z=\mathrm{Tr}(T_1T_1T_2T_1T_2T_1\cdots) \tag{3.1}$$

for our sample. Next we can go to fermion operators and using the translation invariance of T perform a Fourier analysis to obtain

$$\begin{aligned} Z&=\mathrm{Tr}\bigotimes_q [T_1(q)T_1(q)T_2(q)T_1(q)\cdots] \\ &=\prod_q \mathrm{Tr}[T_1(q)T_1(q)T_2(q)\cdots] . \end{aligned} \tag{3.2}$$

Since $T|00\rangle=|00\rangle$ and $T|11\rangle=|11\rangle$ for all values of K, K^*, and q, we can write as before

$$Z=\prod_q [2+2\cosh Nf(q,K,N)] , \tag{3.3}$$

where $\exp(\pm Nf)$ are the eigenvalues of the N-fold random product of $\hat{T}_{1,2}(q)$ which also has $\det=1$. Furstenberg's theorem assures us that as $N\to\infty$, $f(q,K,N)\to f(q,K)$ with probability 1, so that just as in the homogeneous case we have, in the thermodynamic limit,

$$F(K)=\int_0^\pi \frac{dq}{2\pi} f(q,K) \tag{3.4}$$

independent of sample. From Eqs. (2.16) and (2.17) we see that $f(q,K)$ is the limiting free energy per site of a $d=1$ Ising model with a nonrandom ferromagnetic $J(q)$ and a random magnetic field $h(q)=2[K^*(q)-K_{1,2}]\equiv h_{1,2}$. The site index for these $d=1$ models is just the row index for $d=2$ models. As a result, if the model at $q=0.1$ has the field $h_1(0.1)$ at site 4, the one at $q=0.2$ (or another q) will also have $h_1(0.2)$, reflecting the fact that the horizontal coupling in row 4 was K_1 in the $d=2$ model. This correlation between *noninteracting* $d=1$ models is of no consequence. The main point is that within each model, as we move from site to site, h will be h_1 or h_2 in a random way. That is all we need to use Furstenberg's theorem and to be assured a sample independent $f(q,K)$.

Let us now use this mapping to study the phase structure as T, the $d=2$ temperature, is raised. In Fig. 2(a), to be interpreted like Fig. 1, we see two spikes at K_1 and K_2 that move leftwards (since $K_1,K_2\propto 1/T$), while the shaded region moves outwards. In the low-temperature region depicted in Fig. 2(a), the $d=2$ models at each q are subject to a field random in magnitude [$h_1(q)$ or $h_2(q)$] but fixed in sign (negative). There will be three special points as T rises: (i) K^* meets K_1, (ii) K^* meets $\langle K\rangle=\frac{1}{2}(K_1+K_2)$, and (iii) K^* meets K_2, which correspond to singular points in $F(K)$. Let us now study them in turn.

Case (i) K^* meets K_1. In the $d=2$ description this is the T at which the system would have demagnetized had all horizontal bonds been K_1. Of course we now have rows of K_2 in between rows of K_1. However, there can be arbitrarily large islands of pure K_1 that occur with nonzero probability and come arbitrarily close to being critical. These will be seen to produce an essential singu-

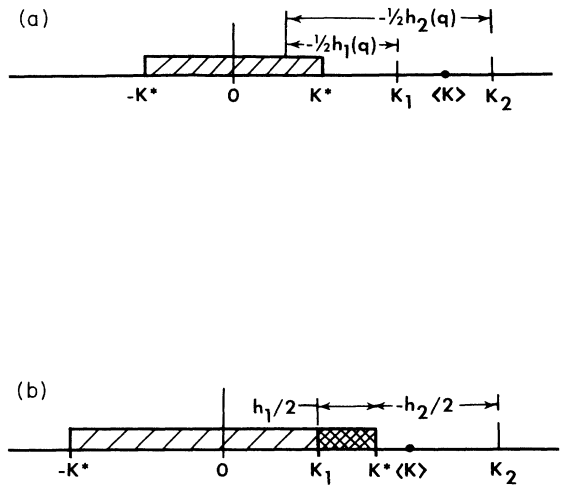

FIG. 2. (a) Low-T picture for the case with horizontal bond equal to K_1 or K_2 both positive. Note that $h(q)$ is random in magnitude but not sign. $\langle K\rangle$ denotes the mean horizontal bond. (b) The situation after the transition at $K^*=K_1$. The doubly shaded region has magnetic frustration. [In the figure h_1 and h_2 stand for $h_1(0)$ and $h_2(0)$.] When the shaded front hits $\langle K\rangle$, we have an infinite correlation length ferromagnetic-paramagnetic transition. Finally, there will be a Griffiths transition at $K^*=K_2$.

larity in *temperature* T. A similar singularity will occur at $K^*=K_2$. The relationship between these and the singularities briefly discussed by Griffiths[4] in his paper on *magnetic* singularities will be discussed later in this section.

To proceed further, it helps to think in the $d=1$ description. As mentioned earlier, for $K^*<K_1$, all the models are subject to fields random in magnitude, but not in sign: $h_1,h_2<0$. The spins, ferromagnetically coupled, follow $\langle h\rangle$, the average field. In our case, since $\langle h\rangle<0$, the total spin will be negative. However, when K^* crosses K_1 the spins in the doubly hatched region of Fig. 2(b) are frustrated: there is a conflict between the tendency to order, induced by J, and the random sign of h. (This *magnetic* frustration is not to be confused with $d=2$ *bond* frustration, which is absent in this case with K, K_1, K_2 positive.) We will now show that this onset of frustration leads to a nonanalyticity in $f(q,K)$ at $q=0$ $K^*=K_1$. Since $q=0$ is an end point of integration, the singularity will be reflected in $F(K)$.[12] The result from Ref. 12 that we use is this: In an integral representation like Eq. (3.4), when a *moving* singularity $q(K)$ of $f(q,K)$ hits the end point of integration it will lead to a singularity in the integral at that value of K.

It is not difficult to see why $q=0$ is a nonanalytic point for $K^*>K_1$, i.e., in the presence of frustration. From our mapping we see $q=0$ implies $J=\infty$. When $J=\infty$, the spins are insensitive to the randomness in h; they are chained to each other by an infinite coupling and react to $\langle h\rangle$, which is still negative. However, for any other J, large but finite, there is a genuine conflict or frustration between alignment induced by the ferromagnetic J and a magnetic field of random sign which gives conflicting signals on the direction of alignment. In other words, if there is a large cluster of length L where $h=h_1$ (>0), all the spins there can flip over and align with h_1 rather than with $\langle h\rangle$ which is negative. The energy cost is $4J$ (from the ends) $-2Lh_1$ (from within the cluster). Since the probability for the cluster is 2^{-L}, we will have an unstable situation when

$$2^{-L}e^{2h_1L}e^{-4J}>1\ ,$$

i.e., for (3.5)

$$h_1>\frac{\ln 2}{2}$$

large overturns are unsuppressed. Actually instability sets in as soon as h_1 turns positive. To see this we follow the argument of Derrida and Hillhorst. Consider $f(J,h)$, with h and J as independent variables instead as functions of q and K. In the *unfrustrated* region, $f(J,h)$ can be expanded in a low-temperature series:

$$f(J,h)=-\langle h\rangle+\frac{\langle Z\rangle}{1-\langle Z\rangle}x+O(x^2)\ , \tag{3.6}$$

where $\langle Z\rangle=\exp(2h)\rangle$, $x=\exp(-4J)$, and $\langle\ \rangle$ denotes the average over the random field distribution.

The first term indicates that at $J=\infty$, i.e., $x=0$, all spins align with $\langle h\rangle$, and produce a free energy $f=|\langle h\rangle|=-\langle h\rangle$ for our case. The next term describes the overturn of a cluster of *adjacent* spins. The overturn produces a factor [besides the $\exp(-4J)$ from the ends] of $\exp(2h_i)\exp(2h_{i+1})\cdots(\exp 2h_{i+L})$ if spins from i to $i+L$ are flipped. The average penality factor is $\langle Z_i\rangle\langle Z_{i+1}\rangle\cdots\langle Z_{i+L}\rangle=\langle Z\rangle^L$. When summed over L, we get $\langle Z\rangle/(1-\langle Z\rangle)$. Specializing to our problem where $h=h_1$ or h_2 (both negative),

$$\langle Z\rangle=\frac{e^{2h_1}+e^{2h_2}}{2}<1 \tag{3.7}$$

so that the sum over L converges. If however, h_1 turns positive, $\langle Z\rangle$ can exceed 1. Assuming $\exp(2h_2)=\exp(-2|h_2|)\ll 1$, the condition for the diverging first derivative at $x=0$ becomes

$$\frac{e^{2h_1}}{2}\simeq 1\ , \tag{3.8}$$

which agrees with our crude estimate Eq. (3.5). Thus what we were seeing there was the divergence of the first derivative. But as DH point out, higher derivatives diverge sooner. For example, they show explicitly that the second derivative varies as $(1-\langle Z^2\rangle)^{-1}$. It is reasonable to believe that the nth derivative goes as $(1-\langle Z^n\rangle)^{-1}$ and diverges at

$$h_1\simeq\frac{\ln 2}{2n}\ . \tag{3.9}$$

As $h_1\to 0$, from above, the diverging derivative is pushed further out in the series, but the limits $h_1\to 0$ and $n\to\infty$ cannot be exchanged, i.e., the function is nonanalytic at $x=0$, $h_1=0$, though infinitely differentiable. We will now see exactly how it behaves for $h_1\to 0^+$.

Let us look at the series, Eq. (3.6) for $h_1>0$. It does not describe $f(J,h)$ in the sense of a power series for $h_1>0$ since it is not a convergent series. However, any derivative *that exists* for the true f is indeed given by the coefficient of x^n in that series. Our preceding analysis tells us that for a small positive h_1, f will have up to $\alpha(h)$ derivatives, where $\langle Z^\alpha\rangle=1$, but no more. *One* function that has this feature is

$$f(x,h)=x^{\alpha(h)}+\text{analytic pieces}\ ,$$

where α is defined by

$$\langle Z^\alpha\rangle=1\ . \tag{3.10}$$

We will take this to be *the* function in the frustrated region because DH show that in the case $\langle Z\rangle>1$, i.e., where the *first* derivative is divergent, Eq. (3.10) is indeed correct.

Since we are now considering the region K^* just past K_1, i.e., h_1 small positive and h_2 large negative let us use the approximate solution,

$$\alpha=\frac{\ln 2}{2h_1}$$

and write

$$f_{\text{sing}}(x,h)\simeq x^{a/h_1},\quad a=\tfrac{1}{2}\ln 2\ . \tag{3.11}$$

To apply this result to our problem where x and h are controlled by K and q, we write [see Eqs. (2.16) and (2.17)]

$$x(q)=(K^*q)^2+O(q^4) ,$$

$$h_1(q)=h_1(0)-\tfrac{1}{2}S^*q^2 \tag{3.12}$$

$$=2(K^*-K_1)-\tfrac{1}{2}S^*q^2$$

$$\equiv t-cq^2 \quad (c=\tfrac{1}{2}S^*) . \tag{3.13}$$

In Fig. 2(b) we must integrate $f_{\rm sing}$ over the doubly shaded region, i.e., between $q^2=0$ and $q^2=t/c$ where $h_2=0$. We ignore the analytic contribution that comes from higher values of q, i.e., from the unfrustrated models. Thus

$$F_{\rm sing}(t)=\begin{cases}\int_0^{\sqrt{t}}(q^2)^{1/(t-q^2)}dq & (t>0) , \quad (3.14)\\ 0 & (t\le 0) , \quad (3.15)\end{cases}$$

where we have set constants like S^*, $\frac{1}{2}\ln 2$, etc. equal to unity since they do not affect the singularity structure.

The evaluation of Eq. (3.14) by saddle-point methods is described in Appendix A. The result is

$$F_{\rm sing}(t)\sim t^{1/t} \quad , t>0$$

$$=0, \quad t<0 . \tag{3.16}$$

We now turn to the next transition as T is raised.

Case (ii) $K^*=\langle K\rangle$. Figure 2 makes it clear that there are two points where we can expect something singular: $K^*=K_1$, which we have studied, and $K^*=K_2$, which we will study next. In the $d=1$ description these are the points at which $q=0$ becomes singular and regular again. We will see that these are transitions at finite correlation length. We do, however, expect one more transition between these two, namely the ferromagnetic-to-paramagnetic transition. It is clear it must happen after $K^*=K_1$ (due to the admixture of stronger bonds K_2) and before $K^*=K_2$ (due to the admixture of weaker bonds K_2). To locate the transition we turn to the $q=0$ mode, which is trivial since $\hat{T}(0)$ is diagonal:

$$f(0,K)=2\,|K^*-\langle K\rangle|=2\,|\langle h(0)\rangle|\equiv|t| . \tag{3.17}$$

Had $F(t)$ been a sum over q of $f(q,K)$, it too would have had a $|t|$ singularity. However, it is an integral, and to find its singular part, we need f for small q and small t. This is precisely the region studied by DH. For a wide class of distributions, they showed that as $x\to 0$,

$$f_{\rm sing}(x,h)=|\langle h\rangle|+Cx^{|\alpha(h)|} , \tag{3.18}$$

where C is x independent and $\alpha(h)$ is given by the solution to

$$\langle Z^\alpha\rangle=1 \tag{3.19}$$

as long as $0<|\alpha|<1$. [In this region the singular piece dominates the analytic piece in numerical importance, which is what permitted DH to extract their result by asymptotic methods. In this paper we have assumed that Eqs. (3.18) and (3.19) are valid in the entire frustrated region based on continuity and our derivative counting argument at the onset of frustration.]

For the transition at hand α is very small. This is because $\langle h\rangle=0$ for $q=0$ at $K^*=\langle K\rangle$ and $\langle h\rangle$ is small for small q and small $t=K^*-\langle K\rangle$. A small $\langle h\rangle$ in turn implies small α. For example, if $\langle h\rangle=0$, so that $h_1=-h_2$,

$$\langle Z^\alpha\rangle=\cosh\alpha h_1=1\to\alpha=0 . \tag{3.20}$$

It is easy to go on to show that α is proportional to $-\langle h\rangle$, but the results of DH are not enough in the critical region since they exclude $\alpha=0$. Here we turn to Nieuwenhuizen and Luck,[9] who show that for a special, solvable distribution,

$$C\propto|\langle h\rangle|$$

in Eq. (3.18) and

$$f(x,\langle h\rangle=0)\approx\frac{1}{\ln(1/x)} . \tag{3.21}$$

We shall take these two to be general features. We also know exactly that at $x=0$, $f=|\langle h\rangle|$. However, to analyze the critical region we need a *single* function and not piecewise information of the above kind. We find that the following function is valid throughout:

$$f_{\rm sing}(x,h)=\langle h\rangle\coth(\tfrac{1}{2}\alpha\ln x) . \tag{3.22}$$

One can consider various limits, such as α fixed, $x\to 0$; $x=0$, α anything; $x\to 0$ with $\langle h\rangle\gtrless 0$, etc., and verify its correctness. We were also gratified to discover subsequently that the same function enters the analysis of McCoy in his study of a general ferromagnetic MW model. Turning to our model where x and h are functions of q and K:

$$x=aq^2+\cdots, \quad a=S^*e^{-2K^*} \tag{3.23}$$

$$\langle h\rangle=2[K^*(q)-\langle K\rangle]\equiv t+bq^2 , \tag{3.24}$$

$$\alpha=-c(t+bq^2) , \tag{3.25}$$

so that

$$F_{\rm sing}(t)=\int_0^\epsilon(t+q^2)\{\coth[(t+q^2)\ln(1/q)]\}dq , \tag{3.26}$$

where we have set constants like a, b, etc. to unity as they do not control the form of the singularity, and where ϵ is some small fixed number. In the t plane, the integrand has poles at

$$t=-q^2\pm\frac{in\pi}{\ln q} . \tag{3.27}$$

As $q\to 0$, these poles pinch the origin, the direction of approach being the imaginary axis, up to exponentially small terms (q^2 compared to $\ln q$). Ignoring the q^2 terms on this basis, we get

$$F_{\rm sing}(t)=\int_0^\epsilon t\coth(-t\ln q)dq . \tag{3.28}$$

Using the series for coth and performing the q integration, we get

$$F_{\rm sing}(t)=\sum_k 2^{2k}t^{2k}B_{2k}\Gamma(2k)/(2k!) . \tag{3.29}$$

Since the Bernoulli numbers B_{2k} grow as $(2k!)/(2\pi)^{2k}$ with k, we see that the series representing $F_{\text{sing}}(t)$ has zero radius of convergence. The same singularity was discovered years ago by McCoy in his study of a general ferromagnetic MW model. Since in the ferromagnetic case, there exists a real duality relation between our models, it follows that they should have the same singularity structure for this transition. We now turn to the third and final transition as T is raised.

Case (iii) $K^*=K_2$. Figure 2(b) suggests that singular behavior is to be expected when the expanding shaded region hits the second spike at K_2. Our analysis of the frustrated models near $q=0$ tells us that at $K^*=K_2-t$, $F_{\text{sing}}(t)$ takes the form (setting all irrelevant factors to unity):

$$F_{\text{sing}}(t)=\int_0^{\epsilon}(q^2)^{1/(t+q^2)}dq \ . \tag{3.30}$$

The integrand is just the $x^{|\alpha|}$ term with $|\alpha|=\ln 2/|h_2|$. To see how we get α in this region, we write

$$\langle Z^{\alpha}\rangle=\tfrac{1}{2}[\exp(h_1\alpha)+\exp(-|h_2|\alpha)]$$

as

$$\langle Z^{\alpha}\rangle=\tfrac{1}{2}[\exp(-h_1|\alpha|)+\exp(|\alpha|\ |h_2|)] \ ,$$

which is identical to what we saw at $K^*=K_1$ with the roles of h_1 and h_2 exchanged, and α replaced by $|\alpha|$, i.e.,

$$|\alpha|=\frac{\ln 2}{h_2}+O[\exp(-h_1|\alpha|)] \ . \tag{3.31}$$

From Fig. 2(b) it is clear that $h_2(q,t)=t+q^2$, in our reduced units.

The integrand has a moving singularity at $q^2=-t$ which hits the lower limit of integration at $t=0$. It is easy to check that $F_{\text{sing}}(t)$ is infinitely differentiable at $t=0$. We cannot, however, evaluate $F_{\text{sing}}(t)$ by saddle-point methods. We can, however, advance a plausibility argument that it also behaves as $t^{1/t}$.

Physically, it is clear that we are witnessing here a Griffiths singularity due to arbitrarily large islands of pure K_2 that are coming arbitrarily close to a phase transition. Presumably, a similar thing is happening at $K^*=K_1$. There are, however, differences. The former is the usual Griffiths singularity associated with the temperature above which no bond in the ensemble can become critical. On the other hand, no such interpretation exists for the transition at $K^*=K_1$ where the *weakest* bonds are involved in the virtual transition. The correct way to interpret it (pointed out by Chayes[13]) is as the dual of the Griffiths transition of the MW model. In other words, the dual of our model is MW model (as stated in the Introduction) with fixed horizontal bonds K^* and random vertical bonds which can be K_1^* or K_2^*. The highest temperature at which any member of the ensemble can go critical is given by $(K_1^*)^*=K^*$, which coincides with our condition $K_1=K^*$ upon dropping a pair of stars. *Thus the $t^{1/t}$ singularity we found is the Griffiths singularity for the MW model.* If we believe that the singularity is the same for both kinds of randomness, then the one at $K^*=K_2$ must also be $t^{1/t}$.

Let us now note that if the two spikes are replaced by a smooth weight function between them, the transitions will occur at exactly the same points. For example, $K^*=K_1$, when $q=0$ turns nonanalytic, will be a singular point. As we cross this point, $q=0$ will continue to be singular whether we have just two spikes or a smooth density. For the middle transition, all one has to note is that the $q=0$ mode has a $|t|$ singularity when $K^*=\langle K\rangle$, where $\langle K\rangle$ is now the average horizontal bond [given in the two-spike case by $(K_1+K_2)/2$].

We now pause to investigate another by-product of our $d=1$ mapping.

Correlation functions. An unexpected bonus in our approach is that it is possible to express the correlation length in the vertical direction in terms of $f(q,K)$. Let O_1 and O_2 be two translationally invariant operators, bilinear in the fermions (so as to be local in the spins) and associated with rows R_1 and R_2 separated by a distance L in the vertical direction. The connected correlation function is

$$\langle O_2O_1\rangle_c=\left[\frac{\mathrm{Tr}[T(R_1R_2)O_2T(R_2R_1)O_1]}{\mathrm{Tr}[T(R_1R_2)T(R_2R_1)]}\right] - \left[\frac{\mathrm{Tr}[T(R_1R_2)O_2T(R_2R_1)]}{\mathrm{Tr}[T(R_1R_2)T(R_2R_1)]}\times(2\leftrightarrow 1)\right] , \tag{3.32}$$

where $T(R_iR_j)$ is the product of transfer matrices between rows R_i and R_j and where $(2\leftrightarrow 1)$ means that the subscripts 2 and 1 are interchanged. (It is assumed that $R_{N+1}=R_1$.) The above correlation function will depend on R_1 and R_2 separately and will also be sample dependent. We fix both up by performing an average over all values of R_1. The resulting object, denoted by an overbar, will define a correlation length ξ if it falls as $\exp(-L/\xi)$.

Each operator will have the form

$$O=\int_0^{\pi}O(q)dq \ , \tag{3.33}$$

where the fermion bilinear operator $O(q)$ will be chosen to be nonzero only in the $|10\rangle$ and $|01\rangle$ subspace [e.g. $C_1^{\dagger}(q)C_2(q)$]. Thus we can write

$$\langle O_2O_1\rangle_c=\int_0^{\pi}dq\,\langle O_2(q)O_1(q)\rangle_c \ , \tag{3.34}$$

where the integrand is computed using a formula like (3.31) with T replaced by $\hat{T}(q)$ everywhere. Let us denote by A and B the random products of such matrices from R_1 to R_2 back to R_1, respectively. Then

$$\langle O_2(q)O_1(q)\rangle_c=\frac{\mathrm{Tr}(BO_2AO_1)}{\mathrm{Tr}(AB)}-\frac{\mathrm{Tr}(BO_2A)\mathrm{Tr}(AO_1B)}{\mathrm{Tr}(BA)\mathrm{Tr}(AB)} \ . \tag{3.35}$$

Let us write

$$A=|\theta_+\rangle\langle\theta_+|e^{L\lambda(L)}+|\theta_-\rangle\langle\theta_-|e^{-L\lambda(L)} \ , \tag{3.36}$$

$$B=|\theta_0\rangle\langle\theta_0|e^{(N-L)f}+O(e^{-2N}) \ . \tag{3.37}$$

In the above equations, $|\theta_+\rangle$ and $|\theta_-\rangle$ are the two

eigenkets of the L-fold products of the $\hat{T}(q)$ matrices. The real ket $|\theta_+\rangle$ points along a direction θ_+ in the first quadrant, and $|\theta_-\rangle$ will be at an angle $\pi/2$ with respect to it. [Here we are assuming $\hat{T}(q)$ is Hermitian to simplify the discussion. A Hermitian $\hat{T}(q)$ can always be chosen without affecting the free energy. Alternately, one can repeat this analysis using left and right eigenvectors.] Since $\det A=1$, its two eigenvalues will be inverses of each other, as implied in Eq. (3.36). We also know that as $L\to\infty$, $\lambda(L)\to f$, the free energy per site of the random field model at that q. As for B, we keep only its leading ket, since we can let N be arbitrarily large early in the game: We are still careful with terms down by $\exp(-L)$ but not $\exp(-N)$. In this case we can also use f in the exponent in Eq. (3.37). If one inserts the above forms for A and B into Eq. (3.35), one finds

$$\langle O_2(q)O_1(q)\rangle_c = e^{-2\lambda(L)L}S(\theta_0,\theta_\pm,q)[1+O(e^{-2L})] . \tag{3.38}$$

The answer is very much sample dependent since S contains a variety of scalars like $\langle\theta_+|O_1|\theta_0\rangle$, etc. That is, even though the eigenvalues of B stabilize to $\exp(\pm Nf)$, the eigenkets are drawn randomly from a distribution and depend on the sample. Similarly, even though we may replace $\lambda(L)$ by f in A (the errors being corrections to the leading behavior we are trying to extract) its eigenkets are sample dependent. Suppose now that we average over R_1 and R_2, keeping the separation fixed at L. Then the random products occurring in A and B will vary over the ensemble and the various angles θ will occur with the limiting probability distribution arising in Furstenberg's theorem. In other words, a spatial average will lead to an integration over the densities $\rho(\theta_0)$ and $\rho(\theta_+)$, where $\rho(\theta)$ is the limiting distribution for the leading eigenvector. (The distribution of θ_- is fixed, given that of θ_+, by orthogonality.) Thus we get

$$\overline{\langle O_2O_1\rangle_c} = \int_0^\pi e^{-2Lf(q)}R(q)dq , \tag{3.39}$$

where R has no L dependence. Let us now use what we know about $f(q)$ in the region $K^*\approx\langle K\rangle$:

$$f(q,t)=t\coth(-t\ln q)+\text{analytic term} . \tag{3.40}$$

The analytic term varies as $A+Bq^2$ while the singular piece varies as $|t|(1+2q^{|t|})$. However, $A=0$ since $f(0,t)=|t|$. Thus the singular part is also numerically dominant in the critical region. Inserting it into (3.39) and evaluating it by saddle point (Appendix B), we get

$$\overline{\langle O_2O_1\rangle_c}\approx e^{-m(L)L}$$

where

$$m(L)=2\left[t^2+\frac{1}{2L}\right]^{1/2}+\frac{1}{tL}\sinh^{-1}(t\sqrt{2L}) . \tag{3.41}$$

If we keep t fixed and let $L\to\infty$, we get $m=2|t|$ so that $\nu=1$. *At $t=0$, we find*

$$m(0,L)=\left[\frac{8}{L}\right]^{1/2} , \tag{3.42}$$

so that

$$\langle O_2O_1\rangle_c\approx e^{-\sqrt{8L}} . \tag{3.43}$$

Thus we find that at criticality, correlations fall faster than any power but slower than exponentially.

At the other two singular points, the correlation length is finite and equal to $2|\langle h(0)\rangle|$. [In calculating m, it is this *analytic* numerically important part of f which is to be used in Eq. (3.39).] Let us recall what made it possible for us to calculate the bulk correlations using Furstenberg's theorem, which only tells us something about the leading eigenvalue and eigenvector of the random product of $\hat{T}(q)$'s at each q, whereas the correlation length requires knowledge of the nonleading eigenvalue and eigenvector. The answer of course was the fact that the random product of $\hat{T}(q)$'s, like each of its factors, is *unimodular and* 2×2. Thus, given the leading eigenvalue, the next one, which is also the only other one, follows. Likewise, given a limiting distribution for the leading eigenvector, that of the other one (shifted by a $\pi/2$ rotation) follows from orthogonality. For example if $\hat{T}(q)$ had been 3×3, we could not have carried out the calculation of the mass gap m.

IV. RANDOM BONDS WITH FRUSTRATION

We now consider the very interesting case where K_1 has turned negative (Fig. 3). Now the $d=2$ model faces frustration whenever a row of K_1's is adjacent to a row of K_2's. It turns out that with respect to the free energy, things are no more complicated than in the unfrustrated case depicted in Fig. 2. (This is because the $d=1$ models are frustrated even in the unfrustrated $d=2$ problem. Making the $d=2$ model frustrated does not cause complications.) From Fig. 3 we see that there will be three transitions: $K^*=K_2$, $K^*=\langle K\rangle$, and $K^*(\pi)(=-K^*)=K_1$. The sequence of these three transitions depends on which is larger, $|K_1|$ or $\langle K\rangle$. (Figure 3 corresponds to the case $|K_1|>\langle K\rangle$.) From Fig. 3 it is clear that near $K^*=K_2$, the free-energy singularity is the same as in Fig. 2 near $K^*=K_2$, since the neighborhood of $q=0$, which controls the singularity, is locally the same. Next consider $K^*(\pi)$ $(=-K^*)=K_1$. This is the mirror image of the transition at $K^*=K_2$, as $J(q\approx\pi)$ is the same as $J(q\approx0)$. It is also clear from the figure that the transition at $K^*=\langle K\rangle$ has the same singularity as before for similar reasons.

If we turn away from the free energy and ask what the spins are doing we get a completely different picture. For example, the singularity at $K^*(\pi)=K_1$, now comes from

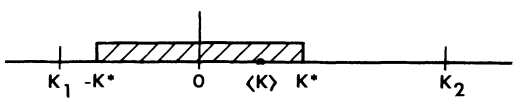

FIG. 3. The model with $d=2$ frustration. The ferromagnetic-paramagnetic transition has already occurred at $K^*=\langle K\rangle$. This will be followed by Griffiths transitions at $-K^*=K_1$ and $K^*=K_2$.

arbitrarily large strips of pure K_1 undergoing a virtual transition from a state of striped order to disorder. Since K_1 is the strongest *anti*ferromagnetic bond, this is the usual Griffiths transition. The transition at $K^*=K_2$ is once again the Griffiths transition and has the same interpretation as before.

The situation at $K^*=K_1$ poses a greater challenge: It is not clear what the order parameter is at low T. The following arguments suggest that both the magnetization M and the Edwards-Anderson (EA) order parameter[14] $q_{\rm EA}$ vanish. As a prelude to $K_1<0$, let us consider the borderline case $K_1=0$. Consider L rows of zero K_1 separating two clusters of pure K_2. Since spins in these rows have no horizontal interactions, they can be summed exactly using the $d=1$ relation: $\tanh K\rightarrow(\tanh k)^L$. In other words, we now have a MW system with fixed horizontal bonds K_2 and vertical bonds whose tanh can be $(\tanh k)^L$ with probability 2^{-L}. Since arbitrarily weak vertical bonds occur here, the system comes arbitrarily close to being quasi-one-dimensional. We conjecture that β is large, and possibly infinite [i.e., $M(t)\approx t^{1/t}$] in the low-T phase and that $M=0$ at $K_1=0$.

Let K_1 now turn negative. Let us start with $T=0$ and think in terms of the bond energies J_1, J_2, and J where $K_1=J_1/T$, etc. Assume for convenience that $|J_1|>J$. Then the fate of each spin is controlled by the horizontal coupling and the lattice will contain layers of ferromagnetically ordered spins separated by layers of striped spins. The ground state is, however, infinitely degenerate since global sign reversal of any cluster costs no extra energy: half the vertical bonds separating the clusters are broken in either case. We can put the system in a unique state by applying an infinitesimal positive field to every other column. Now $M=\frac{1}{2}$ and $\hat{M}=\frac{1}{2}$ where $\hat{M}$ is striped magnetization. However, this order will be destroyed at any $T\neq 0$, i.e., $T=0$ will be critical point for $K_1\leqq 0$. Consider, for example, a ferromagnetic cluster of L rows. If there is a thermal fluctuation in which a vertical seam of L flipped spins occurs, this seam can spread out with no resistance either from the bonds within the cluster or from the vertical bonds linking it to the striped clusters on either side. This situation wherein the perfect order of each type of cluster (ferromagnetic and striped) annihilates order of the other was observed earlier by Wolf and Zittartz.[15] Given the one-dimensional nature of the system it seems that M, $\hat{M}$, and $q_{\rm EA}$ are all zero in the low-T phase. The physics of the infinite correlation length transition at $K^*=\langle K\rangle$ is an open question that may be resolved by computer simulation or furter cerebration.

We conclude this section with a few words on the special case $K_1=-K_2$. First, the two Griffiths transitions coalesce. Next, the transition at $K^*=\langle K\rangle$ moves to $T=0$. The mass gap $m\propto K^*\approx\exp(-1/T)$ i.e., $\nu=\infty$.

V. CONCLUSIONS

In this paper we obtained several exact results for a random-bond Ising model without introducing the replica trick or long-range interactions. The model was more general than that of McCoy and Wu since it allowed for frustration. It was more general than that of Wolf and Zittartz, who took ν layers of randomly chosen bonds (same within a row) and repeated them indefinitely. (In doing so they missed the transitions at $K^*=K_1$ and K_2 since the arbitrarily large islands of pure K_1 or K_2 do not occur at any fixed ν.) However, our model needs to be generalized in one very important way: removal of bond correlations within each row. There *is* one problem where correlated bonds will appear naturally.[16] Suppose we take a fermion Hamiltonian H with random on-site or hopping terms and consider the quantum partition function, $\mathrm{Tr}\exp(-\beta H)$. If we evaluate this trace as a partition function using intermediate states of Ising spins after a Jordan-Wigner transformation, we will get a problem like ours except for a 90° rotation of the lattice.

Our strategy of mapping the random-bond problem to a random-field problem allowed us to exploit our intuition for the latter as well as to use existing results on this subject. We were able to evaluate the transition points exactly and to obtain explicit forms for many of the singularities. We were also able to evaluate bulk correlation functions by exploiting the special properties of the $\hat{T}(q)$ matrices. We also learned that in addition to the Griffiths singularity associated with the strongest bond in the distribution there is its dual version associated with the weakest bond. We were able to show that it went as $t^{1/t}$.

We were not so successful with the order parameter in the low-T region for the problem with frustration. It is an open problem to see what happens below the transition at $K^*=\langle K\rangle$. In summary, it seems that despite its simplicity our model has yielded many interesting results and promises a few more.

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APPENDIX A

Here we consider

$$F_{\rm sing}(t)=\int_0^{\sqrt{t}}(q^2)^{1/(t-q^2)}dq\ . \tag{A1}$$

The integrand is zero at both limits and has a maximum in between. The saddle point $\bar{q}$ is given by

$$\ln\bar{q}^2=1-\frac{t}{\bar{q}^2}\ , \tag{A2}$$

which leads to

$$F(t)\approx e^{-(1/\bar{q}^2)} \tag{A3}$$

We must now express $\bar{q}^2$ as a function of t. Let x stand for $1/\bar{q}^2$. Then Eq. (A2) reveals that

$$\ln x=tx-1\ . \tag{A4}$$

As $t\rightarrow 0$, x must be large so that $tx-1$ can equal $\ln x$, which is at least as large as $\ln(1/t)$. (See the limits of the integral.) Since $\ln x$ itself is large, it follows from above

that xt is also large. Ignoring the -1 in the equation and taking logarithms, we get

$$\ln x = \ln(1/t) + \ln\ln x \ . \tag{A5}$$

This equation can be solved iteratively since $\ln x \gg \ln\ln x$. At the first level we find $\ln x$ equals $\ln(1/t)$. Inserting this, we get

$$\ln x = \ln(1/t) + \ln\ln(1/t) + \cdots \ , \tag{A6}$$

which in turn leads to $F(t) \approx t^{1/t}$.

APPENDIX B

Consider the correlation function

$$G(L) = \overline{\langle O_1 O_2 \rangle_c} = \int_0^{\epsilon} \exp[-2tL \coth(-t\ln q)]dq \ , \tag{B1}$$

which we get by combining Eqs. (3.9) and (3.40), and setting $R(q) \approx R(0)$ and ignoring the latter. Since L tends to ∞, we can evaluate the integral at the saddle point. But first we use a new variable $q = e^{-x}$ in terms of which

$$G(L) = \int_{-\ln\epsilon}^{\infty} \exp[-2tL \coth(xt)] \exp(-x)dx \ . \tag{B2}$$

The saddle point X obeys

$$2t^2 L = \sinh^2(Xt) \ , \tag{B3}$$

so that

$$G(L) \approx \exp\left[-2tL\left[\frac{1+2t^2L}{2Lt^2}\right]^{1/2} - \frac{1}{t}\sinh^{-1}(t\sqrt{2L}\,)\right]$$

$$\equiv \exp(-mL) \ , \tag{B4}$$

where

$$m = \left[2\left[t^2 + \frac{1}{2L}\right]^{1/2} + \frac{1}{tL}\sinh^{-1}(t\sqrt{2L}\,)\right] \ . \tag{B5}$$

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