Integrable narrow-band model with possible relevance to heavy-fermion systems

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A lattice model consisting of a single narrow band is introduced to describe some aspects of heavy electrons. The model excludes double occupancy of the sites and electrons on nearest-neighbor sites interact via a charge interaction and spin exchange. The model is integrable in one dimension for some special values of the coupling constants. These cases are related to the SU(3) invariance. The Bethe-ansatz equations are obtained and ground-state and thermodynamic properties are discussed and solved in some limiting cases.

I. INTRODUCTION

Heavy-electron metals[1-3] have received a large amount of attention in recent years, in particular because of their unusual low-temperature properties. Characteristic to heavy fermions is a very large electronic specific heat at low temperatures, $C=\gamma T$, where γ corresponds to a very high density of states at the Fermi level, or, equivalently, to an effective electron mass of 10^2-10^3 times that of the free-electron mass. As a consequence of the large density of states, these systems typically have a large Pauli susceptibility or order antiferromagnetically. The temperature dependence of C and χ can be explained in terms of narrow resonant levels or a narrow band with a typical width of a few meV or less. The narrow peak in the density of states has been attributed to a Kondo resonance[2,3] arising from the screening of the magnetic moment of the quasilocalized f electrons by the conduction electrons. Heavy-fermion behavior occurs in a variety of Ce-, U-, and Yb-based alloys and compounds.

The resistivity of stoichiometric heavy-fermion compounds initially increases as one lowers T (which can be attributed to the Kondo effect), then goes through a large maximum and shows a sharp decrease at very low T. Both features, the existence of a maximum and the high resistivity at the maximum, are uncommon to normal metals. The rapid decrease of $\rho(T)$ at low T is caused by a transition from incoherent to coherent scattering of the conduction electrons by the rare-earth (actinide) ions.[1-3]

Compounds may become antiferromagnetic or superconducting at low T as a consequence of the coherence of the low-energy excitation spectrum. Anomalous superconductivity has been discovered in some U compounds with highly unusual properties which may be due to triplet pairing.[4,5] The parameters of interactions determining the low-T phase, i.e., superconducting, magnetically ordered, or a Pauli paramagnet, are still to be understood.

The systems are usually described within the framework of the Kondo and Anderson lattices. Numerous approximate treatments have been applied to these models, which have been extensively reviewed in Refs. 2, 3, 6, and 7. In particular, the $1/N$ approaches (diagrammatic[8,9] and functional integral method[10-13]), variational methods,[14-16] and local Fermi-liquid theories[17-19] gave rise to important results and contributed to a preliminary understanding of heavy-fermion compounds. In a stoichiometric compound at low T, the Kondo resonances of the different rare-earth (actinide) sites superimpose coherently and form a narrow band at the Fermi level of width T_K. The low-temperature properties of the system, and in particular the coherence effects, are governed by the low-energy excitations of this narrow band.

Exact results, even for a simplified model which does not have all the features of the Anderson on Kondo lattices, are always useful and provide a testing ground for approaches intended for the full problem. The condition of exact diagonalization imposes limitations on the choice of the Hamiltonian. First, the integrability by means of a Bethe ansatz, i.e., the existence of an exact solution, restricts the model to one space dimension. The integrability requires further that in a scattering process the momenta of the outgoing particles are the same as those of the incoming particles. This restricts the model to have only one bandwidth or Fermi velocity. Second, since the Kondo resonance in a lattice consists of a narrow band at the Fermi level and its width is a fundamental energy scale, we describe the dynamics of the electrons by a nearest-neighbor tight-binding hopping model. A continuum model with a parabolic or linear dispersion would not provide a natural bandwidth parameter. Third, since f electrons are highly correlated, and, in particular, Ce compounds have only one f electron per Ce ion, it is reasonable to exclude the multiple occupancy of the sites. Fourth, since the f electrons are spin compensated at low temperatures, in part by the conduction electrons but possibly mainly by antiferromagnetic correlations among the rare-earth moments themselves, it is necessary to assume that the scattering is different if the two electrons involved form a singlet or a triplet state. The model is then considerably different from the traditional Hubbard model.[20]

The rest of the paper is organized as follows. In Sec. II we explicitly introduce the model, derive the two-particle scattering matrix, obtain the conditions for the integrability, and state the discrete Bethe-ansatz equations. In Sec. III we obtain the ground-state integral equations for the most important case, i.e., when two electrons in the singlet state are scattered, but they are

not scattered if in a triplet state. Some ground-state properties are derived in Sec. IV. The thermodynamics of the model is obtained in Sec. V. A summary and discussion follows in Sec. VI.

II. MODEL, SCATTERING MATRIX, AND CONDITIONS FOR INTEGRABILITY

A. The model

We consider electrons with spin $\frac{1}{2}$ on a one-dimensional lattice with nearest-neighbor hopping. We assume a large on-site Coulomb repulsion that *excludes* the *double-occupancy* of every site. In other words each lattice site is constrained to have either one electron (with spin up or down) or none. Two types of interactions are considered between electrons on nearest-neighbor sites: A charge interaction independent of the spin of strength V and a spin exchange interaction J. The Hamiltonian is then given by

$$H=-\sum_{i,\sigma}(c_{i\sigma}^{\dagger}c_{i+1\sigma}+c_{i+1\sigma}^{\dagger}c_{i\sigma}) +J\sum_{i,\sigma,\sigma'}c_{i\sigma}^{\dagger}\mathbf{S}_{\sigma\sigma'}c_{i\sigma'}\cdot c_{i+1\sigma'}^{\dagger}\mathbf{S}_{\sigma'\sigma}c_{i+1\sigma} +V\sum_{i,\sigma,\sigma'}c_{i\sigma}^{\dagger}c_{i\sigma}c_{i+1\sigma'}^{\dagger}c_{i+1\sigma'}\ , \tag{2.1}$$

where $c_{i\sigma}$ is the annihilation operator for an electron with spin σ at the site i and $\mathbf{S}_{\sigma\sigma'}$ represents the $S=\frac{1}{2}$ spin matrices. The hopping matrix element has been set equal to -1, such that the bandwidth is 2.

The above model is not integrable for arbitrary values of J and V. We next derive the two-electron scattering matrix to obtain the conditions for the integrability of the model.

B. Two-electron scattering matrix

Consider the linear chain described by (2.1) with only two electrons. Let $|0\rangle$ be the empty-lattice state, i.e., without electrons, then the two-particle wave function can be written in the following form:

$$\Psi_{\sigma_1\sigma_2}=\sum_{n_1,n_2}[a_{\sigma_1\sigma_2}(n_1,n_2)c_{n_1\sigma_1}^{\dagger}c_{n_2\sigma_2}^{\dagger}|0\rangle +a_{\sigma_2\sigma_1}(n_1,n_2)c_{n_1\sigma_2}^{\dagger}c_{n_2\sigma_1}^{\dagger}|0\rangle]\ . \tag{2.2}$$

If $\sigma_1=\sigma_2$ the two terms are identical. The wave function obeys Schrödinger's equation $H\Psi_{\sigma_1\sigma_2}=E\Psi_{\sigma_1\sigma_2}$, giving rise to the following relations for the coefficients $a_{\sigma_1\sigma_2}(n_1,n_2)$:

$$-a_{\sigma_1\sigma_2}(n_1+1,n_2)-a_{\sigma_1\sigma_2}(n_1-1,n_2)-a_{\sigma_1\sigma_2}(n_1,n_2+1)-a_{\sigma_1\sigma_2}(n_1,n_2-1) +\left[\left[V-\frac{J}{4}\right]a_{\sigma_1\sigma_2}(n_1,n_2)+\frac{J}{2}a_{\sigma_2\sigma_1}(n_1,n_2)\right]\delta_{n_1+1,n_2}=Ea_{\sigma_1\sigma_2}(n_1,n_2)\ . \tag{2.3}$$

The solution of this recursion relation is of the form

$$a_{\sigma_1\sigma_2}(n_1,n_2)=C_{\sigma_1\sigma_2}^{(1)}\exp(ik_1n_1+ik_2n_2) +C_{\sigma_1\sigma_2}^{(2)}\exp(ik_2n_1+ik_1n_2) \tag{2.4}$$

with

$$E=-2\cos k_1-2\cos k_2\ , \tag{2.5}$$

where k_1 and k_2 are the momenta of the particles. The coefficients $C_{\sigma_1\sigma_2}^{(l)}$ are not all independent but related by

$$(C_{\sigma_1\sigma_2}^{(2)}+C_{\sigma_2\sigma_1}^{(2)})/(C_{\sigma_1\sigma_2}^{(1)}+C_{\sigma_2\sigma_1}^{(1)})=\exp(-2i\Psi_{k_1k_2}^{t})\ , \tag{2.6a}$$

$$(C_{\sigma_1\sigma_2}^{(2)}-C_{\sigma_2\sigma_1}^{(2)})/(C_{\sigma_1\sigma_2}^{(1)}-C_{\sigma_2\sigma_1}^{(1)})=\exp(-2i\Psi_{k_1k_2}^{s})\ , \tag{2.6b}$$

where $\Psi_{k_1k_2}^{t,s}$ is given by

$$\cot\Psi_{k_1k_2}^{t,s}=x^{t,s}\frac{\cot\frac{1}{2}k_2-\cot\frac{1}{2}k_1}{(1-x^{t,s})\cot\frac{1}{2}k_1\cot\frac{1}{2}k_2-(1+x^{t,s})} \tag{2.7}$$

with $x^{t}=(V/2)+J/8$ and $x^{s}=(V/2)-3J/8$. If the pair of electrons forms a singlet state, then $\Psi^{t}=0$ and conversely if they are in a triplet state, $\Psi^{s}=0$. The expression (2.7) for the phases Ψ is analogous to the one for the anisotropic spin-$\frac{1}{2}$ Heisenberg chain,[21] where x is the anisotropy parameter. Taking into account the antisymmetry of a fermion wave function we obtain the following two-particle scattering matrix

$$\hat{S}(k_1,k_2)=-\tfrac{1}{2}[\exp(-2i\Psi_{k_1k_2}^{t})-\exp(-2i\Psi_{k_1k_2}^{s})]\hat{I} -\tfrac{1}{2}[\exp(-2i\Psi_{k_1k_2}^{t})+\exp(-2i\Psi_{k_1k_2}^{s})]\hat{P}\ , \tag{2.8}$$

where $\hat{I}$ is the identity matrix and $\hat{P}$ permutes the spin indices.

C. Condition for integrability

Let us now consider N electrons in the lattice arranged such that $n_1<n_2<\cdots<n_N$. The wave function can then be written in the form

$$\Psi_{\sigma_1,\sigma_2,\ldots,\sigma_N}=\sum_{n_1<n_2<n_3<\cdots<n_N}\sum_{P}a_{\sigma_{P1},\sigma_{P2},\ldots,\sigma_{PN}}(n_1,n_2,\ldots,n_N)c_{n_1\sigma_{P1}}^{\dagger}c_{n_2\sigma_{P2}}^{\dagger}\cdots c_{n_N\sigma_{PN}}^{\dagger}|0\rangle\ , \tag{2.9}$$

where $P=(P1,\ldots,PN)$ is a permutation of the integers $1,\ldots,N$. There are $N!$ such permutations. Other arrangements than $n_1<n_2<\cdots<n_N$ follow from the antisymmetry of the wave function with respect to the permutation of two particles.

In analogy to the two-particle problem, the coefficients in (2.9) are not all independent, but related by a scattering matrix. The condition for the applicability of Bethe's hypothesis is that the scattering matrix can be factorized into a product of two-particle scattering matrices. The single-valuedness of the wave function is ensured by Yang's triangular relation,[22,23] which is a sufficient condition for the integrability of the model,

$$S^{\sigma_1\sigma_1'}_{\sigma_2\sigma_2'}(k_1,k_2)S^{\sigma_1'\sigma_1''}_{\sigma_3\sigma_3'}(k_1,k_3)S^{\sigma_2'\sigma_2''}_{\sigma_3'\sigma_3''}(k_2,k_3)$$
$$=S^{\sigma_2\sigma_2'}_{\sigma_3\sigma_3'}(k_2,k_3)S^{\sigma_1\sigma_1'}_{\sigma_3'\sigma_3''}(k_1,k_3)S^{\sigma_1'\sigma_1''}_{\sigma_2'\sigma_2''}(k_1,k_2)\ . \quad (2.10)$$

The two-particle scattering matrix, (2.8), however, does not satisfy the triangular relation but for special values of the couplings J and V. These special cases are as follows:

(a) $x^t=0$, $x^s=\pm1$ or equivalently $V=\pm\frac{1}{2}$ and $J=\mp2$,

(b) $x^s=0$, $x^t=\pm1$ or equivalently $V=\pm\frac{3}{2}$ and $J=\pm2$, and

(c) $x^s=x^t$, i.e., $J=0$ and V is arbitrary (or $J\to\pm\infty$).

Below we discuss the scattering matrices for the special cases (a) and (b) and by imposing periodic boundary conditions to the system we obtain the discrete Bethe-ansatz equations. Case (c) can be mapped onto the XXZ Heisenberg chain.

D. Two-particle scattering matrix for the integrable cases

In the special case (a), electrons forming a triplet state are not scattered. On the other hand, if their spins are in a singlet state the scattering depends on their crystal momenta k_1 and k_2. The scattering matrix (2.8) reduces to

$$\hat{S}(k_1,k_2)=\frac{p_1-p_2}{p_1-p_2\pm i}\hat{I}+\frac{\pm i}{p_1-p_2\pm i}\hat{P}\ , \quad (2.11)$$

where $\pm$ refers to the sign of x^s and $p=\frac{1}{2}\cot\frac{1}{2}k$ if $x^s=+1$ and $p=\frac{1}{2}\tan\frac{1}{2}k$ if $x^s=-1$.

In the special case (b), on the other hand, electrons in a triplet state scatter, while they do not scatter if their spins form a singlet state. The scattering matrix for this case is

$$\hat{S}(k_1,k_2)=-\frac{p_1-p_2}{p_1-p_2\pm i}\hat{I}+\frac{\pm i}{p_1+p_2\pm i}\hat{P}\ , \quad (2.12)$$

where $\pm$ refers to the sign of x^t and $p=\frac{1}{2}\cot\frac{1}{2}k$ if $x^t=+1$ and $p=\frac{1}{2}\tan\frac{1}{2}k$ if $x^t=-1$.

In case (c) when $J=0$, the scattering is independent of the spin. The model then basically reduces to the anisotropic $S=\frac{1}{2}$ (XXZ) Heisenberg chain,[21] where the total number of electrons plays the role of the magnetization and the chemical potential is the magnetic field. The spin degree of freedom of the electrons introduces an additional constant term to the entropy given by the number of electrons times ln2.

E. Bethe-Ansatz equations

The integrable cases (a) and (b) discussed above can now be solved by a standard procedure.[20,22–24] Imposing periodic boundary conditions the problem reduces to the simultaneous solution of N eigenvalue equations. For N-M electrons with spin up, and M electrons with spin down we obtain by means of a second Bethe Ansatz the following set of coupled algebraic equations (N_a is the number of lattice sites and assumed to be even). For case (a),

$$\left[\frac{p_j+i/2}{p_j-i/2}\right]^{N_a}=\prod_{\beta=1}^{M}\frac{p_j-\Lambda_\beta+i/2}{p_j-\Lambda_\beta-i/2}\ ,\quad j=1,\ldots,N \quad (2.13)$$
$$\prod_{j=1}^{N}\frac{\Lambda_\alpha-p_j+i/2}{\Lambda_\alpha-p_j-i/2}=-\prod_{\beta=1}^{M}\frac{\Lambda_\alpha-\Lambda_\beta+i}{\Lambda_\alpha-\Lambda_\beta-i},\quad \alpha=1,\ldots,M\ .$$

For case (b),

$$\left[\frac{p_j+i/2}{p_j-i/2}\right]^{N_a}=(-1)^N\prod_{l=1}^{N}\frac{p_j-p_l+i}{p_j-p_l-i}\prod_{\beta=1}^{M}\frac{p_j-\Lambda_\beta-i/2}{p_j-\Lambda_\beta+i/2}\ ,$$
$$j=1,\ldots,N \quad (2.14)$$
$$\prod_{j=1}^{N}\frac{\Lambda_\alpha-p_j+i/2}{\Lambda_\alpha-p_j-i/2}=-\prod_{\beta=1}^{M}\frac{\Lambda_\alpha-\Lambda_\beta+i}{\Lambda_\alpha-\Lambda_\beta-i},\quad \alpha=1,\ldots,M\ .$$

Here Λ_α are rapidities related to the spin degrees of freedom. The total energy of the system is, in both cases, given by

$$E=\mp2N\pm2\sum_{i=1}^{N}\frac{\frac{1}{2}}{p_i^2+\frac{1}{4}}\ , \quad (2.15)$$

where the $\pm$ refers to the sign of x^s or x^t, respectively.

Since in a Kondo system the spins are compensated into a singlet state, case (a), i.e., when electrons paired in a singlet state are scattered, is the physically more relevant situation. In the remaining sections we restrict ourselves to derive the properties of case (a). Note that the Bethe-ansatz equations for this case, (2.13), are closely related to those of the SU(3)-invariant $S=1$ Heisenberg chain with ferromagnetic and antiferromagnetic coupling, respectively (see Sec. VI).

III. GROUND-STATE INTEGRAL EQUATIONS

In order to obtain the ground-state properties in the case of singlet scattering only [case (a)] we have to find the solution to Eqs. (2.13). The sets of rapidities $\{p_j\}$ and $\{\Lambda_\alpha\}$ have real and complex solutions. Complex solutions for the Λ_α corresponds to excited states and are discussed in Sec. V. The rapidities p_j may be real or complex.

Consider (2.13) for large N_a. If p_j has a positive imaginary part, there exists Λ_β such that to order $\exp(-N_a)$

$$p_j=\Lambda_\beta+i/2\ . \quad (3.1a)$$

Similarly, if p_j has a negative imaginary part, there exists Λ'_β such that to order $\exp(-N_a)$

$$p'_j=\Lambda'_\beta-i/2 \ . \tag{3.1b}$$

Since the energy must be real, it follows that the set $\{p_j\}$ consists of real p_j values (unpaired charge modes) and pairs of complex conjugated p_j values (spin-paired electrons). Depending on the sign of x^s spin-paired electrons are energetically favorable compared to unpaired particles or vice versa.

The set $\{p_j\}$ consists of N-$2M$ real rapidities and M pairs of complex conjugated p_j-values related to the real Λ_α by

$$p^{\pm}_\alpha=\Lambda_\alpha\pm i/2 \ . \tag{3.2}$$

Inserting (2.2) into (2.13) we obtain

$$\left[\frac{p_j+i/2}{p_j-i/2}\right]^{N_a}=\prod_{\beta=1}^{M}\frac{p_j-\Lambda_\beta+i/2}{p_j-\Lambda_\beta-i/2}, \quad j=1,\ldots,N\text{-}2M$$
$$\left[\frac{\Lambda_\alpha+i}{\Lambda_\alpha-i}\right]^{N_a}=-\prod_{j=1}^{N-2M}\frac{\Lambda_\alpha-p_j+i/2}{\Lambda_\alpha-p_j-i/2}\times\prod_{\beta=1}^{M}\frac{\Lambda_\alpha-\Lambda_\beta+i}{\Lambda_\alpha-\Lambda_\beta-i}, \quad \alpha=1,\ldots,M \tag{3.3}$$

and the energy and the spin are given by

$$E=\mp 2N\pm 2\sum_{j=1}^{N-2M}\frac{\frac{1}{2}}{p_j^2+\frac{1}{4}}\pm 2\sum_{\alpha=1}^{M}\frac{1}{\Lambda_\alpha^2+1} \ , \tag{3.4}$$

$$S_z=\frac{N}{2}-M \ . \tag{3.5}$$

The sign of the energy refers to $x^s=\pm 1$.

Logarithmizing Eqs. (3.3) and in the thermodynamic limit ($N_a\to\infty$ with M/N_a and N/N_a being fixed) we obtain the following integral equations

$$\frac{1}{\pi}\frac{1}{\Lambda^2+1}=\sigma'_h(\Lambda)+\sigma'(\Lambda)+\frac{1}{\pi}\int d\Lambda'\sigma'(\Lambda')\frac{1}{(\Lambda-\Lambda')^2+1}+\frac{1}{\pi}\int dp\,\rho(p)\frac{\frac{1}{2}}{(\Lambda-p)^2+\frac{1}{4}} \ , \tag{3.6}$$

$$\frac{1}{\pi}\frac{\frac{1}{2}}{p^2+\frac{1}{4}}=\rho_h(p)+\rho(p)+\frac{1}{\pi}\int d\Lambda\,\sigma'(\Lambda)\frac{\frac{1}{2}}{(p-\Lambda)^2+\frac{1}{4}} \ , \tag{3.7}$$

where $\rho(p)$ and $\sigma'(\Lambda)$ are the distribution density functions for the p and Λ rapidities and $\rho_h(p)$ and $\sigma'_h(\Lambda)$ are the respective hole distribution functions. The intervals in which ρ and σ are nonvanishing depend on the total number of particles, the total spin and the energy

$$S_z/N_a=\tfrac{1}{2}\int dp\,\rho(\rho) \ , \tag{3.8}$$

$$N/N_a=\int dp\,\rho(p)+2\int d\Lambda\,\sigma'(\Lambda) \ , \tag{3.9}$$

$$E/N_a=\mp 2N/N_a\pm 2\int dp\,\rho(p)\frac{\frac{1}{2}}{p^2+\frac{1}{4}}\pm 2\int d\Lambda\,\sigma'(\Lambda)\frac{1}{\Lambda^2+1} \ . \tag{3.10}$$

These equations are solved in Sec. IV for some situations.

IV. GROUND-STATE PROPERTIES

A. Filled-band solution

We assume that there is one electron per lattice site, i.e., $N/N_a=1$. Fourier transforming (3.6) and using (3.9) it follows that the hole-distribution function $\sigma'_h(\Lambda)$ vanishes identically. Equation (3.7) then becomes

$$\rho_h(\xi)+\rho(\xi)-\int d\xi'\,p(\xi')G_1(\xi-\xi')=G_0(\xi) \ , \tag{4.1}$$

where

$$G_l(\xi)=\frac{1}{2\pi}\int_{-\infty}^{\infty}d\omega\,e^{-i\omega\xi}\frac{e^{-(l/2)|\omega|}}{2\cosh(\omega/2)} \ . \tag{4.2}$$

The solution of this integral equation depends on the magnetization of the system. In the absence of an external magnetic field two solutions are of interest: the nonmagnetic and the ferromagnetic states.

If the state is nonmagnetic $\rho(\xi)\equiv 0$ and the solution of (3.6) is $\sigma'(\Lambda)=G_1(\Lambda)$. The energy of the system is straightforwardly obtained, $E/N_a=\mp 2\ln 2$. For the ferromagnetic state, on the other hand, we have $\sigma'(\Lambda)\equiv 0$ and $\rho_h(\xi)\equiv 0$, such that

$$\rho(\xi)=\frac{1}{\pi}\frac{\frac{1}{2}}{\xi^2+\frac{1}{4}} \ . \tag{4.3}$$

In this case the spins of all the electrons are parallel and the total energy of the system is zero.

Hence, if $x^s=+1$ the ground state is nonmagnetic, while if $x^s=-1$ the ground state is the ferromagnetic state. For $x^s=+1$ the states with lowest energy correspond to large $|\Lambda|$ and $|p|$ values (long-wavelength states), such that $\sigma'(\Lambda)$ is nonzero only in the intervals $|\Lambda|>Q$ and $\rho(p)$ in the intervals $|p|>B$. $\sigma'_h(\Lambda)$ and $\rho_h(p)$ are complementary functions. For $x^s=-1$ we have that states with small $|\Lambda|$ and $|p|$ values have lowest energy (again long-wavelength states), such that $\sigma'(\Lambda)$ and $\rho(p)$ are nonvanishing in the intervals $|\Lambda|<Q$ and $|p|<B$. The interval limits are determined from the total number of particles and the magnetization.

B. Magnetic susceptibility ($x^s=+1$)

For $x^s=+1$ the magnetization vanishes in the absence of an external magnetic field. If we apply an arbitrarily small magnetic field the integration limit B can be made much larger than any given Q. By Fourier transformation of Eqs. (3.6) and (3.7) we obtain the following integral equation for ρ:

$$\rho_h(\xi)+\rho(\xi)-\left[\int_{-\infty}^{-B}+\int_{B}^{\infty}\right]d\xi'\rho(\xi')G_1(\xi-\xi')=G_0(\xi)+\int_{-Q}^{Q}d\xi'\sigma'_h(\xi')G_0(\xi-\xi') \ . \tag{4.4}$$

Since $B\gg Q$ it is convenient to define $y(\xi)=\rho(\xi+B)$, such that

$$y_h(\xi)+y(\xi)-\int_0^\infty d\xi' y(\xi')G_1(\xi-\xi')=G_0(\xi+B)+\int_{-Q}^{Q} d\xi'\sigma_h'(\xi')G_0(\xi+B-\xi')+\int_0^\infty d\xi' y(\xi')G_1(\xi+\xi'+2B)\ , \tag{4.5}$$

where for the last term we have assumed that $\rho(\xi)$ is an even function. This integral equation can be solved by iteration,[25] by writing $y=y_1+y_2+\cdots$ with y_1 and y_2 satisfying

$$y_{1h}(\xi)+y_1(\xi)-\int_0^\infty d\xi' y_1(\xi')G_1(\xi-\xi')=RG_0(\xi+B)\ , \tag{4.6a}$$

$$y_{2h}(\xi)+y_2(\xi)-\int_0^\infty d\xi' y_2(\xi')G_1(\xi-\xi')=\int_0^\infty d\xi' y_1(\xi')G_1(\xi+\xi'+2B)\ . \tag{4.6b}$$

Here we used $B\gg Q$ such that for $\xi>0$

$$\int_{-Q}^{Q} d\xi'\sigma_h'(\xi')G_0(\xi+B-\xi') \simeq G_0(\xi+B)\int_{-\infty}^{\infty} d\xi'\sigma_h'(\xi')\exp(\pi\xi')$$

and

$$R=1+\int_{-Q}^{Q} d\xi'\sigma_h'(\xi')\exp(\pi\xi')\ . \tag{4.7}$$

In (4.6), $y_1(\xi)$ is the leading contribution to $y(\xi)$ if the field is small. Since $G_1(\xi)$ falls off like $1/\xi^2$ for large ξ, $y_2(\xi)$ is the next leading correction. Higher-order contributions can be obtained in a similar way. Both, (4.6a) and (4.6b), are standard integral equations of the Wiener-Hopf type;[25,26] the solution for $y_1(\xi)$ is

$$y_1(\xi)=\frac{1}{2\pi i}\int_{-\infty}^{\infty} dx\frac{e^{-i\xi x}}{g^+(x)}\int_{-\infty}^{\infty}\frac{dy}{2\pi}\frac{Re^{-iyB}}{y-x-io}\times\left[g^-(y)2\cosh\frac{y}{2}\right]^{-1} \tag{4.8}$$

and the leading contribution to the magnetization is given by

$$\int d\xi\, y_1(\xi)=[2R/(2\pi e)^{1/2}]\exp(-\pi B)+O(\exp(-2\pi B))\ . \tag{4.9}$$

Here the magnetic field is proportional to $\exp(-\pi B)$ and

$$g^+(x)=g^-(-x)=(2\pi)^{-1/2}\left[\frac{-ix+o}{2\pi e}\right]^{i(x/2\pi)}\times\Gamma\left[\frac{1}{2}-i\frac{x}{2\pi}\right] \tag{4.10}$$

with o being a positive infinitesimal and Γ the gamma function.

In order to solve (4.6b) we insert $y_1(\xi)$, (4.8), into the driving term. After some algebra we obtain to leading order in the field

$$\int d\xi\, y_2(\xi)=\frac{1}{\pi}\frac{R}{(2\pi e)^{1/2}}\frac{\exp(-\pi B)}{2B}\times\left[1-\frac{\ln 2B}{2\pi B}+\cdots\right], \tag{4.11}$$

such that the low-field magnetization is given by

$$S_z=\frac{2R}{(2\pi e)^{1/2}}e^{-\pi B}\left[1+\frac{1}{4\pi B}-2\left[\frac{1}{4\pi B}\right]^2\ln 2B+\cdots\right]. \tag{4.12}$$

Since H is proportional to $\exp(-\pi B)$ we have that the susceptibility has logarithmic corrections, in analogy to the isotropic SU(2) Heisenberg antiferromagnet.[25,27] The leading-order relation between the field H and B will be given at the end of Sec. V.

C. Charge fluctuations ($x^s=+1$)

Since the magnetization vanishes in the absence of an external magnetic field, the integral equation to be solved in this case is

$$\sigma_h'(\xi)+\sigma'(\xi)+\frac{1}{\pi}\int_{-\infty}^{\infty} d\xi'\sigma'(\xi')\frac{1}{(\xi-\xi')^2+1}=\frac{1}{\pi}\frac{1}{\xi^2+1}\ , \tag{4.13}$$

where $\sigma'(\xi)$ vanishes for $|\xi|<Q$ and $\sigma'_h(\xi)$ is the complementary function. This integral equation has a simple solution only if the band is almost half-filled or almost empty.

If the band is almost half-filled Q is small and (4.13) is more conveniently written as

$$\sigma'(\xi)+\sigma_h'(\xi)-\int_{-Q}^{Q} d\xi'\sigma_h'(\xi')G_1(\xi-\xi')=G_1(\xi)\ . \tag{4.14}$$

This equation can now be solved by iteration by writing $\sigma'(\xi)=\sigma_0'(\xi)+\sigma_1'(\xi)+\cdots$, where $\sigma_0'(\xi)$ and $\sigma_1'(\xi)$ satisfy

$$\sigma_0'(\xi)+\sigma_{0h}'(\xi)=G_1(\xi)=\frac{1}{2\pi}\mathrm{Re}\left[\Psi\left[1+i\frac{\xi}{2}\right]-\Psi\left[\frac{1}{2}+i\frac{\xi}{2}\right]\right], \tag{4.15}$$

$$\sigma_1'(\xi)+\sigma_{1h}'(\xi)=\int_{-Q}^{Q} d\xi'\sigma_{0h}'(\xi')G_1(\xi-\xi')\ . \tag{4.16}$$

Here ψ is the digamma function. To first order in Q we obtain

$$\sigma'(\xi)=\begin{cases}\left[1+2Q\dfrac{\ln 2}{\pi}\right]G_1(\xi), & |\xi|>Q\\ 0, & |\xi|<Q\end{cases} \tag{4.17}$$

and the occupation of the band is given by

$$N/N_a = 2\int_{-\infty}^{\infty} d\xi \sigma'(\xi) = 1 - 2Q\frac{\ln 2}{\pi} . \tag{4.18}$$

On the other hand, if the band is almost empty Q is very large and a similar procedure as for the spin susceptibility can be used. Since $\sigma'(\xi)$ is nonvanishing only for $|\xi| > Q$, we define $y(\xi) = \sigma'(\xi + Q)$, such that (assuming that $\sigma'(\xi)$ is an even function)

$$y_h(\xi) + y(\xi) + \frac{1}{\pi}\int_0^{\infty} d\xi' y(\xi') \frac{1}{(\xi-\xi')^2+1} = \frac{1}{\pi}\frac{1}{(\xi+Q)^2+1} - \frac{1}{\pi}\int_0^{\infty} d\xi' y(\xi') \frac{1}{(\xi+\xi'+2Q)^2+1} . \tag{4.19}$$

This equation is solved by iteration; we write $y(\xi) = y_1(\xi) + y_2(\xi) + \cdots$, where $y_1(\xi)$ and $y_2(\xi)$ obey

$$y_{1h}(\xi) + y_1(\xi) + \frac{1}{\pi}\int_0^{\infty} d\xi' y_1(\xi') \frac{1}{(\xi-\xi')^2+1} = \frac{1}{\pi}\frac{1}{(\xi+Q^2)+1} , \tag{4.20a}$$

$$y_{2h}(\xi) + y_2(\xi) + \frac{1}{\pi}\int_0^{\infty} d\xi' y_2(\xi') \frac{1}{(\xi-\xi')^2+1} = -\frac{1}{\pi}\int_0^{\infty} d\xi' y_1(\xi') \frac{1}{(\xi+\xi'+2Q)^2+1} . \tag{4.20b}$$

Both equations are of the Wiener-Hopf type[26] and can straightforwardly by solved. We obtain for $y_1(\xi)$

$$y_1(\xi) = \frac{1}{2\pi i}\int_{-\infty}^{\infty} dx\, e^{-i\xi x} g^+(x) \times \int_{-\infty}^{\infty} \frac{dy}{2\pi} \frac{e^{-iyQ-|y|}}{y-x-i0} g^-(y) \tag{4.21}$$

and the leading contribution to the occupation number is given by

$$N/N_a = 2\int_0^{\infty} d\xi y_1(\xi) = \frac{1}{\pi Q} + \frac{1}{2}\left[\frac{1}{\pi Q}\right]^2 \ln Q + \cdots . \tag{4.22}$$

Note that the leading contribution of $y_2(\xi)$ is of the order of $(1/\pi Q)^2$. The relation between Q and the Fermi energy is obtained at the end of Sec. V. Similarly, it is possible to obtain the number of particles for the ferromagnetic case, i.e., for $x^s = -1$, if the band is nearly empty or nearly full.

V. THERMODYNAMIC EQUATIONS

Below we derive the thermodynamic Bethe-ansatz equations for case (a) ($x^s = \pm 1$), i.e., when electrons in a singlet state are scattered.

A. Excitations

The excitations of the system are given by the solutions of (2.13). The structure of the solutions is similar as for the fermion gas with attractive δ-function potential, solved by Lai[28] and Takahashi,[29] and for the $j = \frac{1}{2}$ Anderson impurity in the $U \to \infty$ limit.[30] The rapidities can be classified according to (i) $(N-2M')$ real charge rapidities, which correspond to unpaired propagating electrons; (ii) $2M'$ complex charge rapidities, which correspond to bound or paired electron states, of the form

$$p_\alpha^{\pm} = \Lambda'_\alpha \pm i/2 , \tag{5.1}$$

where Λ'_α is a real spin rapidity, $\alpha = 1, \ldots, M'$; (iii) M_n strings of complex spin rapidities of length n, $n = 1, \ldots, \infty$ which correspond to bound spin states and are of the form

$$\Lambda_{\alpha,n}^{\mu} = \Lambda_{\alpha,n} + i\frac{\mu}{2} , \quad \mu = -(n-1), -(n-3), \ldots, (n-3), (n-1) , \tag{5.2}$$

where $\Lambda_{\alpha,n}$ is a real parameter.

The integers M' and M_n satisfy the relation

$$M' + \sum_{n=1}^{\infty} nM_n = M . \tag{5.3}$$

The magnetization and the energy are given by

$$S_z = \frac{N}{2} - M' - \sum_{n=1}^{\infty} nM_n , \tag{5.4}$$

$$E = \mp 2N \pm 2\sum_{i=1}^{N-2M'} \frac{\frac{1}{2}}{p_i^2 + \frac{1}{4}} \pm 2\sum_{\alpha=1}^{M'} \frac{1}{\Lambda_\alpha'^2 + 1} . \tag{5.5}$$

The above rapidities are inserted into Eqs. (2.13) and the resulting coupled equations for p_j, Λ'_α, and $\Lambda_{\alpha,n}$ are logarithmized. We define the usual distribution functions for the rapidities as $\rho(p)$ for the real p's, $\sigma'(\Lambda)$ for the Λ'_α and $\sigma_n(\Lambda)$ for the $\Lambda_{\alpha,n}$. In the thermodynamic limit we obtain a set of linearly coupled integral equations for the distribution functions. Introducing the corresponding "hole"-distribution functions and Fourier-transforming the equations, we have, after some algebra,

$$\begin{aligned}
&\hat\sigma_{m+1,h}(\omega) + \hat\sigma_{m-1,h}(\omega) = 2\cosh\frac{\omega}{2}[\hat\sigma_m(\omega) + \hat\sigma_{m,h}(\omega)], \quad m \geq 2 \\
&\hat\sigma_{2,h}(\omega) + \hat\rho(\omega) = 2\cosh\frac{\omega}{2}[\hat\sigma_1(\omega) + \hat\sigma_{1,h}(\omega)] , \\
&\hat\sigma_{1,h}(\omega) + \hat\sigma'_h(\omega) + 1 = 2\cosh\frac{\omega}{2}[\hat\rho(\omega) + \hat\rho_h(\omega)] , \\
&e^{-(1/2)|\omega|}\hat\sigma'_h(\omega) - \hat\rho(\omega) + e^{-(1/2)|\omega|} = 2\cosh\frac{\omega}{2}[\hat\sigma'(\omega) + \hat\sigma'_h(\omega)] ,
\end{aligned} \tag{5.6}$$

where the caret denotes a Fourier transform. These equations differ only by the driving terms (independent terms) from the corresponding ones for the one-dimensional fermion gas with attractive δ-function interaction[28,29] and the $j = \frac{1}{2}$ Anderson impurity in the $U \to \infty$ limit.[30]

B. Minimization of the free energy

The distribution functions, ρ, σ', and σ_n are actually determined by minimizing the free-energy functional, $F=E-TS$, where

$$E/N_a = \mp 2\int_{-\infty}^{\infty} dp\,\rho(p) \mp 4\int_{-\infty}^{\infty} d\Lambda\,\sigma'(\Lambda) \pm 2\int_{-\infty}^{\infty} dp\,\rho(p)\frac{\frac{1}{2}}{p^2+\frac{1}{4}} \pm 2\int_{-\infty}^{\infty} d\Lambda\,\sigma'(\Lambda)\frac{1}{\Lambda^2+1}\ , \tag{5.7}$$

T is the temperature, and S is the distribution entropy, which, e.g., for $\rho(p)$ is given by

$$S_\rho = \int dp[(\rho+\rho_h)\ln(\rho+\rho_h)-\rho\ln\rho-\rho_h\ln\rho_h]\ . \tag{5.8}$$

The minimization must be carried out subject to the constraints (5.6) and the conservation of electrons and the total spin, i.e.,

$$N/N_a = \int_{-\infty}^{\infty} dp\,\rho(p)+2\int_{-\infty}^{\infty} d\Lambda\sigma'(\Lambda)\ , \tag{5.9}$$

$$S_z/N_a = \tfrac{1}{2}\int_{-\infty}^{\infty} dp\,\rho(p)-\sum_{n=1}^{\infty} n\int_{-\infty}^{\infty} d\Lambda\sigma_n(\Lambda)\ . \tag{5.10}$$

The corresponding Lagrange multipliers are the chemical potential (Fermi-energy) and the magnetic field.

Introducing the following functions

$$\rho_h/\rho=\exp(\varepsilon/T),\quad \sigma_h'/\sigma'=\exp(\Psi/T)\ ,\qquad \sigma_{n,h}/\sigma_n=\exp(\varphi_n/T)\ , \tag{5.11}$$

we obtain, if we consider $\sigma_{n,h}$, ρ, and σ_h' as independent functions,

$$\begin{aligned}
\varepsilon &= \pm 2\pi G_0(\xi)+TG_0*\ln[(1+e^{\Psi/T})/(1+e^{\varphi_1/T})]\ ,\\
\Psi &= \mp 2-A\pm 2\pi G_1(\xi)+TG_0*\ln(1+e^{-\varepsilon/T})\\
&\quad +TG_1*\ln(1+e^{\Psi/T})\ ,\\
\varphi_1 &= TG_0*\ln[(1+e^{-\varepsilon/T})(1+e^{\varphi_2/T})]\ ,\\
\varphi_n &= TG_0*\ln[(1+e^{\varphi_{n-1}/T})(1+e^{\varphi_{n+1}/T})],\quad n\geq 2
\end{aligned} \tag{5.12}$$

where G_1 is defined by (4.2) and $*$ denotes convolution. Here A is the chemical potential and $A'=A\pm 2$ is the energy from the bottom and top of the band, respectively. The field boundary condition is given by

$$\lim_{n\to\infty}\frac{\varphi_n(\xi)}{n}=H\ . \tag{5.13}$$

It is convenient to replace $\pm 2\pi G_l(\xi)$ in (5.12) by $2\pi x^s G_l(\xi)$. Differentiating (5.12) with respect to x^s, we obtain from similarity with Eqs. (5.6) that

$$\begin{aligned}
\sigma_{n,h} &= -\frac{1}{2\pi}\frac{\partial\varphi_n}{\partial x^s}/(1+e^{-\varphi_n/T})\ ,\\
\rho &= \frac{1}{2\pi}\frac{\partial\varepsilon}{\partial x^s}/(1+e^{\varepsilon/T})\ ,\\
\sigma_h' &= \frac{1}{2\pi}\frac{\partial\Psi}{\partial x^s}/(1+e^{-\psi/T})\ .
\end{aligned} \tag{5.14}$$

and similarly the complementary functions.

The free energy of the system is given by

$$F=-\Psi(0)-2A\mp 2\ . \tag{5.15}$$

If, on the other hand, we consider σ_n, ρ, and σ' as independent functions and eliminate all the "hole"-distribution functions by means of (5.6), we obtain another set of integral equations, which is equivalent to (5.12),

$$\begin{aligned}
\varepsilon &= \mp 2\pm 2\frac{\frac{1}{2}}{\xi^2+\frac{1}{4}}-\frac{H}{2}-A+T\frac{1}{\pi}\frac{\frac{1}{2}}{\xi^2+\frac{1}{4}}*\ln(1+e^{-\Psi/T})-T\sum_{n=1}^{\infty}\frac{1}{\pi}\frac{n/2}{\xi^2+(n/2)^2}*\ln(1+e^{-\varphi_n/T})\ ,\\
\Psi &= \mp 4\pm 2\frac{1}{\xi^2+1}-2A+T\frac{1}{\pi}\frac{\frac{1}{2}}{\xi^2+\frac{1}{4}}*\ln(1+e^{-\varepsilon/T})+T\frac{1}{\pi}\frac{1}{\xi^2+1}*\ln(1+e^{-\Psi/T})\ ,\\
\varphi_n &= nH-T\ln(1+e^{-\varphi_n/T})+T\frac{1}{\pi}\frac{n/2}{\xi^2+(n/2)^2}*\ln(1+e^{-\varepsilon/T})+T\sum_{m=1}^{\infty}\Theta_{m,n}(\xi)*\ln(1+e^{-\varphi_m/T})\ ,
\end{aligned} \tag{5.16}$$

where

$$\Theta_{m,n}(\xi)=\int\frac{d\omega}{2\pi}e^{-i\xi\omega}\coth\frac{|\omega|}{2}\times\left[e^{-(|\omega|/2)|m-n|}-e^{-(|\omega|/2)(m+n)}\right]\ . \tag{5.17}$$

Rewriting

$$\pm 2\frac{l/2}{\xi^2+(l/2)^2}\to 2x^s\frac{l/2}{\xi^2+(l/2)^2}$$

in the first two equations of (5.16) for $l=1$ and 2, respectively, and differentiating with respect to x^s, we obtain, after some algebra and making use of (5.14), the original set of relations (5.6).

An alternative expression for the free energy is

$$F/N_a=-\frac{T}{\pi}\int_{-\infty}^{\infty} d\xi\ln(1+e^{-\varepsilon/T})\frac{\frac{1}{2}}{\xi^2+\frac{1}{4}}-\frac{T}{\pi}\int_{-\infty}^{\infty} d\xi\ln(1+e^{-\Psi/T})\frac{1}{\xi^2+1}\ . \tag{5.18}$$

The equivalence between (5.15) and (5.18) is straightforwardly shown via the second equation of (5.16).

C. Special limits

In this subsection we discuss the high temperature and $T\to 0$ limits of the thermodynamic equations. If the temperature is much larger than the bandwidth, i.e., $T\gg 2$, we can neglect the independent terms in (5.12). In the absence of driving terms the potentials ε, Ψ, and φ_n do not depend on ξ and (5.23) can be reduced to an algebraic system of equations, similar to the one discussed by Takakashi[29] for the fermion gas with attractive δ-function potential,

$$
\begin{aligned}
&e^{2\varepsilon/T}=(1+e^{\Psi/T})/(1+e^{\varphi_1/T})\ ,\\
&e^{2(\Psi+A)/T}=(1+e^{-\varepsilon/T})(1+e^{\Psi/T})\ ,\\
&e^{2\varphi_1/T}=(1+e^{-\varepsilon/T})(1+e^{\varphi_2/T})\ ,\\
&e^{2\varphi_n/T}=(1+e^{\varphi_{n+1}/T})(1+e^{\varphi_{n-1}/T}),\quad n\geq 2\ .
\end{aligned}
\tag{5.19}
$$

The general solution is given by

$$
1+e^{\varphi_n/T}=\left[\sinh\left[\frac{nH}{2T}+\mu\right]/\sinh\left[\frac{H}{2T}\right]\right]^2\ ,
$$

where

$$
e^{\mu}=e^{H/T}\left\{\left[1+\exp\left[-\frac{H}{2T}-\frac{A}{T}\right]\right]\Big/\left[1+\exp\left[\frac{H}{2T}-\frac{A}{T}\right]\right]\right\}^{1/2}
$$

and the free energy is the one corresponding to three degrees of freedom per site

$$
F/N_a=-T\ln\left[e^{A/T}2\cosh\frac{H}{2T}+1\right]\ . \tag{5.20}
$$

Let us now analyze the thermodynamic equations in the limit $T\to 0$. From the last two equations it follows that $\varphi_n>0$ for $n=1,\ldots,\infty$, such that $\sigma_n\equiv 0$ for all n as $T\to 0$. The functions ε and Ψ, on the other hand, change sign as a function of ξ. As $T\to 0$ the first two equations of (5.16) yield

$$
\begin{aligned}
\varepsilon(\xi)=&\mp 2+2x^s\frac{\frac{1}{2}}{\xi^2+\frac{1}{4}}-\frac{H}{2}-A\\
&-\frac{1}{\pi}\int_{-\infty}^{\infty}d\xi'\frac{\frac{1}{2}}{(\xi-\xi')^2+\frac{1}{4}}\Psi^-(\xi')\ ,\\
\Psi(\xi)=&\mp 4+2x^s\frac{1}{\xi^2+1}-2A\\
&-\frac{1}{\pi}\int_{-\infty}^{\infty}d\xi'\frac{1}{(\xi-\xi')^2+1}\Psi^-(\xi')\\
&-\frac{1}{\pi}\int_{-\infty}^{\infty}d\xi'\frac{\frac{1}{2}}{(\xi-\xi')^2+\frac{1}{4}}\varepsilon^-(\xi')\ ,
\end{aligned}
\tag{5.21}
$$

where $\Psi=\Psi^+ +\Psi^-$ and $\varepsilon=\varepsilon^+ +\varepsilon^-$ with Ψ^+, $\varepsilon^+>0$, and Ψ^-, $\varepsilon^-<0$. Differentiating with respect to x^s and using (5.14) we recover the ground-state equations.

D. Integration limits $B(H)$ and $Q(A)$.

Equations (5.21) can be used to determine the integration limits $B(H)$ and $Q(A)$ introduced in Sec. IV A. In (4.12) we expressed the magnetization for small fields as a function of B and in (4.22) the number of particles as a function of Q for $x^s=+1$.

We Fourier transform (5.21) and express Ψ^- as a function of ε^- and Ψ^+ in the second equation and insert it into the first equation. We obtain after Fourier transforming back

$$
\begin{aligned}
&\varepsilon^+(\xi)+\varepsilon^-(\xi)-\left[\int_{-\infty}^{-B}+\int_{B}^{\infty}\right]d\xi'\varepsilon^-(\xi')G_1(\xi-\xi')\\
&\quad=\int_{-Q}^{Q}d\xi'\Psi^+(\xi')G_0(\xi-\xi')+2\pi G_0(\xi)-\frac{H}{2}\ .
\end{aligned}
\tag{5.22}
$$

Equation (5.22) is a linear integral equation for ε. It is convenient to separate $\varepsilon^{\pm}=\varepsilon_a^{\pm}+\varepsilon_b^{\pm}$ with

$$
\begin{aligned}
&\varepsilon_a^+(\xi)+\varepsilon_a^-(\xi)-\int_{-\infty}^{\infty}d\xi'\varepsilon_a^-(\xi')G_1(\xi-\xi')\\
&\qquad=\int_{-Q}^{Q}d\xi'\Psi^+(\xi')G_0(\xi-\xi')+2\pi G_0(\xi)\ ,\\
&\varepsilon_b^+(\xi)+\varepsilon_b^-(\xi)-\int_{-\infty}^{\infty}d\xi'\varepsilon_b^-(\xi')G_1(\xi-\xi')=-\frac{H}{2}\ .
\end{aligned}
$$

Comparing with (4.4) we have that $\varepsilon_a^+(\xi)=2\pi\rho_h(\xi)$ and $\varepsilon_a^-(\xi)=2\pi\rho(\xi)$, as well as $\Psi^+(\xi)=2\pi\sigma_h(\xi)$. The procedure to obtain ε_b is to rewrite the integral equation in the Wiener-Hopf from and solve it iteratively for large B, in analogy to Sec. IV B. To leading order in the field we obtain

$$
\varepsilon_b^-(\xi+B)=-i\frac{H}{\sqrt{2}}\int_{-\infty}^{\infty}\frac{d\omega}{2\pi}\frac{e^{-i\xi\omega}}{\omega+i0}[g^+(\omega)]^{-1}\ . \tag{5.23}
$$

The condition $\varepsilon(B)=0$ determines B. Using that

$$
\varepsilon^-(B)=\lim_{\xi\to+0}\varepsilon^-(\xi+B)=-\lim_{\omega\to\infty}i\omega\hat{\varepsilon}^-(\omega)=0
$$

we obtain

$$
B=-\frac{1}{\pi}\ln\left[H/2\pi R\left[\frac{2\pi}{e}\right]^{1/2}\right]\ . \tag{5.24}
$$

Using (5.24) in (4.12) the susceptibility is straightforwardly obtained.

Similarly, $Q(A)$ can be obtained. In the absence of a field $\varepsilon^-\equiv 0$, such that

$$
\begin{aligned}
&\hat{\Psi}^+(\omega)+\hat{\Psi}^-(\omega)(1+e^{-|\omega|})\\
&\qquad=2\pi e^{-|\omega|}-4\pi(A+2)\delta(\omega)\ .
\end{aligned}
\tag{5.25}
$$

We assume the band is almost empty (Q is very large) and write $\Psi=\Psi_1+\Psi_2$ with

$$\hat{\Psi}_1^+(\omega)+\hat{\Psi}_1^-(\omega)(1+e^{-|\omega|})=2\pi e^{-|\omega|} ,$$

$$\hat{\Psi}_2^+(\omega)+\hat{\Psi}_2^-(\omega)(1+e^{-|\omega|})=-4\pi(A+2)\delta(\omega) .$$

In comparison with (4.13) we have that $\Psi_1^-(\xi)=2\pi\sigma'(\xi)$. The integral equation for Ψ_2 is put into the Wiener-Hopf form and solved iteratively. Q is then obtained from $\Psi(Q)=0$

$$Q=(A+2)^{-1/2} . \tag{5.26}$$

Hence $A\geq -2$, i.e., the Fermi level must be above the bottom of the band, and Q diverges with a square root singularity as expected for a one-dimensional system.

VI. CONCLUDING REMARKS

We have introduced a narrow-band model which includes some of the main features of heavy-fermion compounds. We considered a one-dimensional lattice of spin-$\frac{1}{2}$ electrons hopping between nearest-neighbor sites. Double occupancy of every site has been excluded and nearest-neighbor electrons interact via a charge interaction and a spin exchange. The integrability of the model imposes restrictions on the dimensionality and the interaction parameters. We obtain the Bethe-ansatz solution for the integrable cases. Of particular interest is the case of singlet scattering, since, in heavy-fermion systems, the f-electron spins are compensated in part by the conduction electrons and in part by antiferromagnetic spin fluctuations. We treated this case explicitly, obtained the ground-state and thermodynamic integral equations, and solved them for several special limits.

Although the model for general parameters seems to be nonintegrable and approximate methods have to be employed to solve it, the exact solution of special cases is of interest and provides an important testing ground for the approximations used. The model can be generalized to more internal degrees of freedom of the electrons, e.g., N components. These components may arise from combined spin and orbital degrees of freedom of the electrons. The model seems to be integrable for arbitrary N under similarly restrictive conditions as for $N=2$. The $1/N$ expansions applied previously to the Kondo and Anderson lattices, as well as to the Hubbard model, are adequate approximation schemes to solve the model for general parameters.

The integrable cases of our model can be mapped onto the multicomponent quantum lattice gas introduced by Sutherland.[31,32] Each lattice site can be in three possible states, namely empty or occupied with an electron with spin up or down. The problem can then be rewritten in terms of spin operators corresponding to $S=1$. Our four cases $x^s=\pm 1$ with $x^t=0$ and $x^t=\pm 1$ with $x^s=0$ correspond then to the examples discussed in Ref. 31. The logarithmic dependences obtained for the magnetization and the occupation number in Sec. IV are possibly a consequence of the SU(3) invariance of the model. If the coupling is weaker, e.g., $|x^s|<1$, we expect an analytic behavior, while if $|x^s|>1$, we expect an exponential activation in analogy to the anisotropic Heisenberg chain.

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