Nonuniversal critical dynamics of the alternating-bond Ising chain: Relaxational and diffusive kinetics

James H. Luscombe
Ames Laboratory, Iowa State University, Ames, Iowa 50011
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Nonuniversal dynamic critical exponents are obtained for both the Glauber and Kawasaki dynamics of the Ising chain with alternating near-neighbor interactions, J_1, J_2, with the exponents related to the ratio of the two interaction strengths. We expand upon the recent work of Droz *et al.* [Phys. Lett. **115A**, 448 (1986)] for the ferromagnetic system ($J_1>0, J_2>0$). For Glauber (spin-flip) dynamics we obtain the exact exponent $z=1+\rho$ ($\rho \equiv |J_1/J_2|$) which generalizes the result of Droz *et al.* and is valid irrespective of the signs of J_1 and J_2, where it is assumed that $|J_1|>|J_2|$ ($\rho \geq 1$). For Kawasaki (spin-exchange) dynamics, we obtain dynamic critical exponents from conventional theory which provides a rigorous lower bound for the exponent z. For the case of the conserved (ferromagnetic) order parameter, however, we present arguments that the conventional exponent is exact. We obtain $z=4+\rho$ in this case. For $J_1>0, J_2<0$ we derive the conventional exponent $z=1+\rho$, whereas for $J_1<0$ we find $z=2$ irrespective of J_2. A key aspect of this system is the narrowing of the dynamic critical region as compared with the isotropic system ($J_1=J_2$). The extra bond periodicity splits the isotropic order parameter into components such that only comparatively closer to criticality does the order parameter become the dominant slow mode. The nonuniversal critical dynamics is intrinsically linked to the nonuniform bond distribution, being shown to arise from a kinetic coefficient which vanishes with a nonuniversal critical exponent. Nonuniversality of the dynamic exponent for Ising systems with inhomogeneous couplings is argued to be specific to zero-temperature critical points.

I. INTRODUCTION AND SUMMARY

Recently, Droz, Kamphorst Leal da Silva, and Malaspinas (DKM) have investigated the critical dynamics of an Ising chain with two different near-neighbor interaction strengths, J_1, J_2 alternating in succession.[1] As these authors show, the dynamic critical exponents of this system are *nonuniversal* and can assume a continuum of values given by the ratio of the two coupling constants. For relaxational, spin-flip Glauber dynamics,[2] in which the order parameter is not conserved, DKM obtain the exact exponent $z=1+J_1/J_2$ where it is assumed $J_1>J_2>0$ (ferromagnetic interactions). Thus, z can assume any value greater than two for the spin-flip dynamics of this system. For diffusive, spin-exchange Kawasaki dynamics,[3] which conserves the total spin, DKM obtain $z=3+2J_1/J_2$ from an analysis of the motion of domain boundaries, also for the ferromagnetic chain. The larger exponent here reflects the slower dynamics of the conserved order parameter. In the isotropic limit, $J_1=J_2$, these expressions reduce to previously known values: $z=2$ for Glauber dynamics and $z=5$ for Kawasaki dynamics.[4] The findings of DKM are noteworthy in that the nonuniversality occurs within a *fixed* dynamical model and hence is directly associated with the spatial nonuniformity of coupling strengths. This is to be contrasted with prior dynamic universality classes in one dimension,[5] which are prescribed essentially through the arbitrariness which remains in these models once detailed balance has been satisfied. Nonuniversal factors in the transition probability, which are irrelevant for detailed balance, can nonetheless control the dynamic critical exponent. As discussed below, this anomalous situation is related to the zero-temperature critical point. The latter occurrence of nonuniversality, therefore, is *not* related to an intrinsic property of the system, e.g., as specified by the Hamiltonian, but rather to an *ambiguity* of kinetic Ising models whereby the transition probability is not uniquely determined by detailed balance.

In this article, we examine at greater length the critical dynamics of the alternating-bond Ising chain. In particular, we address in detail the mechanism whereby dynamic nonuniversality occurs for this system, which we find to be directly tied to the vanishing of the transition temperature. We also discuss, as contrasted with the isotropic system, the *narrowing* of the dynamic critical region which results from the extra bond-periodicity. We obtain expressions for the dynamic critical exponents for both spin-flip and spin-exchange dynamics for arbitrary interaction parameters, i.e., where J_1, J_2 are not restricted to be positive. Thus, in addition to the ferromagnetic system, we investigate the critical dynamics of the alternating antiferromagnetic chain ($J_1<0, J_2<0$) and also the mixed ferro- and antiferromagnetic cases ($J_1 J_2<0$). We assume throughout that $|J_1|>|J_2|$. Our results can be summarized as follows. For Glauber dynamics we obtain the exact relaxation spectrum from an explicit time-dependent solution. The extra periodicity splits the "unperturbed" spectrum of the isotropic Glauber model into two branches. From the low-lying branch, we obtain quite

generally $z=1+\rho(\rho\equiv|J_1/J_2|,\geq 1)$ which generalizes the DKM result, and applies for all ground states of the system. For Kawasaki dynamics, however, it is necessary to consider the separate cases of J_1 and J_2 as regards their sign. This is because only for the ferromagnetic system is the order parameter conserved. For reasons discussed below, we study the Zwerger model[4] adapted to the alternating chain. Unable to solve the spin-exchange equations of motion (EOM) in general, we obtain information about the critical dynamics from the *initial* response rate of the order parameter in the critical region (conventional theory[6]), which, when combined with Kawasaki's inequality,[7] provides a rigorous *lower bound* for z.[8] In this way, for $J_1>0$, and hence when the *dominant* interaction is ferromagnetic, we obtain the lower bounds $z=4+\rho$ for $J_2>0$ and $z=1+\rho$ for $J_2<0$. The large change in exponent here is related to the difference in ground state structures which occur for the two signs of the weaker bond J_2. For the case of the conserved order parameter, however, we present evidence that the dominant contribution to the long-wavelength critical response is given by conventional theory. Hence, we conclude that $z=4+\rho$ is the exact dynamic exponent for spin-exchange kinetics on the ferromagnetic alternating chain. This differs from DKM's exponent and is in fact smaller, as would be expected of the lower bound. Our result would suggest there is a faster mechanism for domain diffusion than that considered by DKM. Finally, for the case of $J_1<0$ (dominant interaction antiferromagnetic), we obtain the conventional exponent $z=2$, irrespective of J_2.

In contrast to the Glauber model, spin-exchange dynamics cannot be solved without approximation, even in one dimension. The single-spin EOM contains three-spin terms, and in principle one is faced with an infinite hierarchy of coupled dynamical equations. Some analytic results, however, are known for the long-wavelength limit. It can be shown,[9] for the *particular* case of the (isotropic) Zwerger model, that while the order parameter EOM contains three-spin nonlinearities, the projection of these nonlinearities orthogonal to the order parameter contributes to the associated memory function only at the leading order $O(q^4)$ for small wave vector q. The significance of this is that the projected, linear EOM is therefore exact through $O(q^2)$, and thus the diffusion coefficient can be obtained exactly in the isotropic limit. We discuss the anisotropic case below. Neglecting the "orthogonal nonlinearities" in the EOM is equivalent to retaining only the static, frequency-independent part of the memory function,[8] which prescribes the initial response rate. The frequency dependence of the memory function therefore begins at $O(q^4)$ for the isotropic Zwerger model.[9] Since the initial response rate for kinetic Ising systems forms an upper bound to the subsequent response,[10] dynamic scaling of the static memory function provides a rigorous *lower* bound to the dynamic critical exponent.[8] Thus, while Haake and Thol[5] have shown for general one-dimensional spin-exchange models that $z\geq 5$, the (isotropic) Zwerger model is in fact described by the lower bound $z=5$. In Sec. IV we utilize the static memory function for the anisotropic exchange dynamics to find the concomitant lower bounds for z. We will argue that likewise for the anisotropic dynamics, the lower bound is exact for the conserved order parameter. Retaining only the static part of the memory function is tantamount to the conventional approximation of dynamic critical phenomena. A naive application of conventional theory to one dimension, however, leads to erroneous results: $z=4-\eta=3$ for Kawasaki dynamics and $z=2-\eta=1$ for Glauber dynamics. The new wrinkle is a kinetic coefficient which vanishes at the zero-temperature critical point.[11] As discussed in Sec. III, it is in fact the *nonuniversal* vanishing of the kinetic coefficient, which depends on short-range correlations, which is responsible for dynamic nonuniversality.

For the anisotropic exchange dynamics we find the following. The anisotropic interactions result in anisotropic transition rates, which we find leads to a frequency dependence in the memory function at the lower order $O(q^2)$. Thus, conventional theory for this system does not readily lead to an exact exponent as in the isotropic case. Still, the possibility of an exact dynamic critical exponent cannot be *a priori* excluded from this level of approximation. As shown by Mazenko and Valls,[8] to obtain a nonconventional exponent requires, in the zero-frequency limit, a delicate cancellation between the frequency-dependent part of the memory function and the static part in the critical region. We show in Sec. IV that the many-body contributions to the response rate at $O(q^2)$ in fact vanish in the critical region *faster* than the conventional response, and thus there is no evidence for this cancellation through second order.

The article is organized as follows. We first present the equilibrium properties of the alternating-bond Ising chain which will be useful in our discussion of the dynamics. In Sec. III we solve for the explicit time dependence of the spin-flip dynamics. The exact exponent for arbitrary interaction strengths is obtained and the mechanism for dynamic nonuniversality, via the vanishing kinetic coefficient, is discussed. In Sec. IV we set up the spin-exchange equations of motion for the alternating system. The critical dynamics is then examined in the frequency domain via the associated memory function. In Sec. V we close with some remarks.

II. EQUILIBRIUM PROPERTIES OF THE ALTERNATING CHAIN

In this section we present the equilibrium properties of the alternating system which will be required in our discussion of the dynamics. The system in equilibrium is governed by the alternating-bond nearest-neighbor Ising Hamiltonian:

$$H[\sigma]=-\sum_i (J_1\sigma_{2i-1}+J_2\sigma_{2i+1})\sigma_{2i}\ , \tag{2.1}$$

where we have $2N$ spins with N unit cells. There is a critical point at $T_c=0$ for the infinite system ($N\to\infty$), which we have assumed. The equilibrium probability distribution is given by

$$P[\sigma]=Z^{-1}\exp(-\beta H[\sigma]) \tag{2.2}$$

where $\beta \equiv 1/k_B T$, with T the temperature and where Z is the one-dimensional partition function. All equilibrium averages can be calculated exactly for this system. For example, the correlation functions are of the form[12]

$$\langle \sigma_i \sigma_j \rangle = \prod_{n=i}^{j-1} u_n \quad (j > i) , \tag{2.3}$$

where the brackets denote an average with respect to $P[\sigma]$, and where $u_n \equiv \tanh(K_n)$ with $K_n \equiv \beta J_n$ and J_n is the bond coupling spins $\sigma_n \sigma_{n+1}$. From Eq. (2.3) one obtains the correlation length,

$$\xi = \frac{-2}{\ln | u_1 u_2|} \simeq \frac{\exp(2 | K_2 |)}{1+\exp[-2(| K_1 | - | K_2 |)]} \quad (T \to 0) . \tag{2.4}$$

Thus, since $| K_1 | > | K_2 |$, $\xi = \exp(2 | K_2 |)$ as $T \to 0$, i.e., ξ is controlled by the *weaker* bond. We note here that $\exp(2 | K_1 |) = \xi^p$.

The sequence of alternating bonds creates an extra periodicity, $J_{n+2} = J_n$, and can be thought of as creating two interpenetrating sublattices which, respectively, "see" J_1 (J_2) to the left and J_2 (J_1) to the right, the even (odd) sites. It will be useful therefore to introduce the spin Fourier transforms at alternating sites, which act as separate sublattice order parameters,

$$R_q = \frac{1}{\sqrt{N}} \sum_n \exp(iqna) \sigma_{2n} , \tag{2.5a}$$

$$S_q = \frac{1}{\sqrt{N}} \sum_n \exp[iq(2n+1)a/2] \sigma_{2n+1} , \tag{2.5b}$$

where **a** is the length of the unit cell, i.e., *twice* the lattice spacing. An order parameter for the entire system is then given by

$$\psi_q = \frac{1}{\sqrt{2}} (R_q + S_q) . \tag{2.6}$$

Of importance in what follows is the structure factor χ_q, which prescribes the spectrum of equilibrium fluctuations,

$$\chi_q = \langle \psi_{-q} \psi_q \rangle . \tag{2.7}$$

We find, in the limit $N \to \infty$,

$$\chi_q = \frac{(1-u_1 u_2)[1+u_1 u_2 + (u_1+u_2)\cos(qa/2)]}{(1-2u_1 u_2 \cos(qa) + u_1^2 u_2^2)} . \tag{2.8}$$

Equation (2.8) reduces to the isotropic expression when the doubling of the Brillouin zone is taken into account. χ_q demonstrates the critical fluctuations of this system. When both $u_1, u_2 \to 1$, the system is dominated by long-wavelength fluctuations. When both $u_1, u_2 \to -1$ (antiferromagnetic chain) the ground state of alternately aligned spins develops for $q \to 2\pi/a$. Finally, for the mixed ferro- and antiferromagnetic cases ($u_1 u_2 \to -1$), a dimerized ground state of alternately oriented spin pairs develops for $q \to \pi/a$. We note that for each of the three ground-state structures, the susceptibility at the critical wavevector diverges linearly with ξ, $\chi \sim \xi$. This is a manifestation of the fact that the static critical exponents of this system ($\gamma/\nu = 1$) are independent of J_1 and J_2, i.e., universal.[13]

III. SPIN-FLIP DYNAMICS

In this section we derive the exact spectrum of relaxation rates for the alternating system under spin-flip dynamics, and from this we extract the dynamic exponent. We then examine the mechanism of nonuniversality.

A. Dynamical equations

We first require the basic dynamical equations obeyed by Ising spins. The time development of Ising spins[2,7] is generated stochastically such that the time-dependent probability distribution satisfies a master equation, the action of which we denote as an operator

$$\frac{dP[\sigma,t]}{dt} = D_\sigma P[\sigma,t] . \tag{3.1}$$

D_σ is an operator in the space of system configurations, which, via the spin dynamics, evolves an arbitrary vector in this space to the equilibrium governed by Eq. (2.2), about which detailed balance is satisfied. Analytic expressions and further properties of spin-flip operators can be found in Refs. 14 and 15. For our purposes, we note that the time rate of change for a single spin under spin-flip dynamics is generated via

$$\tilde{D}_\sigma \sigma_i = -\alpha \sigma_i W_i(\sigma) , \tag{3.2}$$

where $\tilde{D}_\sigma$ is the adjoint of D_σ and is the evolution operator for spin functions. α is the basic flip rate in the absence of interactions and $W_i(\sigma)$ is the transition probability (see below) which contains the effects of local correlations. The simplest form of $W_i(\sigma)$ is that due to Glauber,[2] which is readily adapted to the alternating chain.

The alternating spins satisfy separate EOM of the form given by Eq. (3.2), with transition rates

$$W_{2n}(\sigma) = 1 - \sigma_{2n}(\gamma_2 \sigma_{2n+1} + \gamma_1 \sigma_{2n-1}) \tag{3.3a}$$

$$W_{2n+1}(\sigma) = 1 - \sigma_{2n+1}(\gamma_1 \sigma_{2n+2} + \gamma_2 \sigma_{2n}) , \tag{3.3b}$$

where, to satisfy detailed balance,

$$\gamma_1 = \frac{u_1(1-u_2^2)}{1-u_1^2 u_2^2} , \tag{3.4a}$$

$$\gamma_2 = \frac{u_2(1-u_1^2)}{1-u_1^2 u_2^2} . \tag{3.4b}$$

The above transition rates have been derived previously in a real-space dynamic renormalization-group analysis of the Glauber chain.[14] In that context it was necessary to develop an anisotropic Glauber dynamics in which, for coarse-graining cells of two spins, the alternating bonds were referred to as "intra-cell" and "inter-cell" couplings.

The sublattice degrees of freedom form separate, but coupled, dynamical systems. It is therefore convenient to first obtain the evolution of an auxiliary column vector $\Psi_q \equiv (R_q S_q)^T$ (T denotes transpose). The order parameter ψ_q is ultimately the dynamical variable of interest, the time dependence of which will be obtained from that of Ψ_q. We find from Eqs. (2.5), (3.2), and (3.3), the linear system of dynamical equations,

$$\tilde{D}_\sigma \Psi_q = -\alpha M_q \Psi_q \ , \tag{3.5}$$

where M_q is the Hermitian matrix

$$M_q = \begin{bmatrix} 1 & -\Lambda_q e^{i\delta} \\ -\Lambda_q e^{-i\delta} & 1 \end{bmatrix} \tag{3.6}$$

with

$$\Lambda_q = [(\gamma_1+\gamma_2)^2 - 4\gamma_1\gamma_2 \sin^2(qa/2)]^{1/2} \tag{3.7a}$$

and phase angle δ defined via

$$\tan(\delta) = \frac{\tanh(K_1-K_2)}{\tanh(K_1+K_2)} \tan(qa/2) \ . \tag{3.7b}$$

Note that a direct EOM for ψ_q does not couple to itself, but rather separately to its components R_q, S_q, i.e., the order parameter is *not* an eigenmode of the dynamics as occurs in the isotropic case. As we will see, it is only in the critical region that the components of ψ_q order together such that the order parameter is the characteristic dynamical mode.

M_q has eigenvalues and eigenvectors

$$\lambda_q^{\pm} = 1 \pm \Lambda_q \ , \tag{3.8a}$$

$$\phi_q^{\pm} = \frac{1}{\sqrt{2}} \begin{bmatrix} \mp e^{i\delta} \\ 1 \end{bmatrix} , \tag{3.8b}$$

in terms of which the solution to Eq. (3.5) is expressed as

$$\Psi_q(t) = \sum_{\nu=\pm} A_q^\nu \phi_q^\nu e^{-\lambda_q^\nu \alpha t} \ , \tag{3.9}$$

where the expansion coefficients

$$A_q^{\pm} = \frac{1}{\sqrt{2}} (\mp e^{-i\delta} R_q + S_q) \ . \tag{3.10}$$

The time-dependent order parameter then readily follows via $\psi_q(t) = U\Psi_q(t)$, where the row vector $U = 1/\sqrt{2}(1 \quad 1)$. Thus, the auto-correlation function of the order parameter is given by

$$\langle \psi_{-q} \psi_q(t) \rangle = \sum_{\nu=\pm} \chi_q^\nu e^{-\lambda_q^\nu \alpha t} \ , \tag{3.11}$$

where

$$\chi_q^\nu = (U\phi_q^\nu) \langle \psi_{-q} A_q^\nu \rangle \ . \tag{3.12}$$

Equation (3.11) is the time-dependent structure factor; it contains all information concerning the time-dependent correlations of fluctuations for this system under spin-flip dynamics. Of particular interest, however, are the dynamics of critical fluctuations to which we now turn.

B. Critical dynamics

The eigenvalues $\lambda_q^{\pm}$ prescribe the relaxation spectrum for the alternating-bond Ising chain, of which there are two branches separated by a gap at the zone boundary. The gap size is, in all cases, given by $2\tanh(|K_1| - |K_2|)$, which has a sensitive temperature dependence: For the slightest anisotropy, a large gap eventually develops for $T \to 0$. The low-temperature dynamics is therefore dominated by the lower branch. Note that λ_q^- is nonzero for all q, which reflects the absence of conserved modes for spin-flip dynamics. For temperatures approaching criticality, however, the minimum of λ_q^- vanishes like ξ^{-z}, where z is the dynamic critical exponent.

The dynamics in the critical region is characterized by asymptotically long-lived critical fluctuations, i.e., λ_q^- vanishes at the same ordering points where χ_q diverges. This can be seen as follows: from Eq. (3.8a),

$$\lambda_q^- = 1 - \{ (|\gamma_1| + |\gamma_2|)^2 - 4|\gamma_1\gamma_2| \sin^2[(q-q_0)a/2] \}^{1/2} \tag{3.13}$$

where q_0 is the wave vector describing the ground state structure (see Sec. II), i.e., $q_0 a = 0, 2\pi$, for $(J_1 J_2) > 0$ (ferro- and antiferromagnetic systems, respectively) and $q_0 a = \pi$, for $(J_1 J_2) < 0$ (mixed cases). We easily find for $T \to 0$ that in each case for $q \to q_0$,

$$\lambda_q^- \simeq 2\xi^{-(1+\rho)} \{ 1 + \tfrac{1}{4}[(q-q_0)\xi]^2 + \cdots \} \ . \tag{3.14}$$

This expression is of the dynamic scaling form[6] and thus we readily identify $z = 1+\rho$, which is nonuniversal, i.e., dependent upon J_1 and J_2. The nonuniversality mechanism is discussed below. This result generalizes the DKM exponent and applies for all ground states of the system for spin-flip dynamics.

We now examine the spin modes associated with the lower branch, which we expect to be given by the order parameter in the critical region. From Eq. (3.10), it is seen that the correspondence $A_q^- \to \psi_q$ is controlled by the phase factor δ which is temperature dependent through Eq. (3.7b). Note that when $\delta = 0$, this correspondence holds manifestly since then $U\phi_q^+ = 0$, $U\phi_q^- = 1$, $A_q^- = \psi_q$, and thus $\chi_q^- = \chi_q$. When $\delta = \pi$ the relaxation is entirely in terms of the upper branch λ_q^+, since then $U\phi_q^- = 0$, $U\phi_q^+ = 1$, and $A_q^+ = \psi_q$. It can be shown that for $q = 0$, $\delta = 0$, for $J_1 > 0$, while for $J_1 < 0$, $\delta = \pi$. For $q = 2\pi/a$ we find the opposite, $\delta = \pi$, for $J_1 > 0$, while for $J_1 < 0$, $\delta = 0$. These results hold irrespective of J_2. Hence, for *precisely* these wave vectors, which are appropriate for the critical ferro- and antiferromagnetic systems, respectively, the dynamics is a simple decay of the order parameter with the relaxation rate, λ_q^-. The mixed regimes, $(J_1 J_2) < 0$, however, are more complicated in that the critical correspondence $\chi_q^- \to \chi_q$ holds only asymptotically. To see this, consider the "remainder" quantity χ_q^+, since $\chi_q^- = \chi_q - \chi_q^+$. We find

$$\chi_q^+ = \tfrac{1}{2}\chi_q \lambda_q^- \frac{1}{\Lambda_q} \begin{bmatrix} \Lambda_q - (\gamma_1+\gamma_2)\cos(qa/2) \\ 1 + (\gamma_1+\gamma_2)\cos(qa/2) \end{bmatrix} \ . \tag{3.15}$$

Hence, for $q = \pi/a$, χ_q^+ vanishes like $\xi^{1-z} = \xi^{-\rho}$ in the critical region and it is only in the limit as $T \to 0$ that $\chi_q^- \to \chi_q$. We see also from Eq. (3.15) that χ_q^+ is in general nonzero away from the special wave vectors $q = 0$, $2\pi/a$. The alternating bond environment therefore leads to a dynamic critical region narrower than that of the isotropic system, since one must be relatively closer to criticality for the order parameter to have become the dominant slow mode.

C. Dynamic nonuniversality at $T_c=0$

We now examine the nonuniversality in detail. As we have just shown, the *dominant* contribution to the time-displaced correlations of critical order parameter fluctuations is in terms of a relaxation of the zero-time correlations with the single, critical relaxation rate. It is therefore convenient to define an effective critical frequency via

$$\omega_c(q)=(\chi_q)^{-1}\Gamma_q \ , \tag{3.16}$$

where the kinetic coefficient is taken from

$$\Gamma_q=-\left.\frac{\partial\langle\psi_{-q}\psi_q(t)\rangle}{\partial t}\right|_{t=0} . \tag{3.17}$$

We can then obtain a dynamic exponent from $\omega_c(q)$, the advantage of which is that there is a simple physical interpretation for Γ_q. However, this identifies z from the initial response of the system, whereas dynamic critical phenomena is associated with the long time response. We defer momentarily the justification of such a procedure. We will see that nonuniversality arises from the critical behavior of Γ_q. We find

$$\Gamma_q=\alpha\Gamma_R \ , \tag{3.18}$$

where

$$\Gamma_R=\langle W_{2n}(\sigma)\rangle=\langle W_{2n+1}(\sigma)\rangle=\frac{(1-u_1^2)(1-u_2^2)}{(1-u_1^2u_2^2)} . \tag{3.19}$$

We see that Γ_q is simply the average flip-rate per spin in the presence of interactions, which here is identical for the two sublattices. Conventional theory presumes this quantity to be finite at T_c, the argument being that it should not exhibit critical behavior since it depends on local interactions. In two dimensions, Γ_R is indeed a smooth function of temperature at T_c.[8] The conventional dynamic exponent is thus given by $z_c=\gamma/\nu=2-\eta$, in terms of the (universal) *static* critical exponents. However, for the one-dimensional system $T_c=0$, the significance of which is that now Γ_R is forced to critically *vanish*. As $T\to 0$, large domains develop, with possible excitations occurring only at domain boundaries. Detailed balance then requires the *average* flip rate to vanish in such a case. This "anomalous" kinetic coefficient results in a "nonconventional" dynamic exponent. For the alternating bond system, it is clear that the vanishing kinetic coefficient is the mechanism for nonuniversality, since Γ_R depends on short-range correlations, and hence its vanishing occurs *nonuniversally*. Indeed, from Eq. (3.19) we find $\Gamma_R\sim\xi^{-\rho}$. Combined with the divergence of χ_q, Eq. (3.16) yields the exponent $z=1+\rho$, in agreement with our exact calculation. Generally, if the kinetic coefficient vanishes in the critical region as ξ^{-x}, then the conventional exponent for model A is given by[11] $z_c=2-\eta+x$. Γ_R is the canonically averaged flip rate for *all* spins of the system. A more microscopic interpretation of the low-temperature dynamics in one dimension focuses on the spins most likely to flip, the domain boundary spins. As shown by Cordery, Sarker, and Tobochnik,[16] the physics of the one-dimensional dynamic critical exponent can be understood for the isotropic, ferromagnetic chain in terms of the random-walk motions of domain walls.[17] We have not formulated arguments of this type which encompass the ground state structures for which our result $z=1+\rho$ applies.

For kinetic Ising systems, the initial response rate is always larger than the subsequent response rate,[10] implying that the conventional exponent is in principle the lower bound,[8] $z\geq z_c$. As in the above example, however, it may happen that the exponent computed from the initial response equals the (true) exponent characterizing the long-time response. The long time response is naturally discussed in the frequency domain in terms of the associated memory function,[18] $\phi(q,z)$, defined via

$$C(q,z)=\chi_q(z+\phi(q,z))^{-1} \ , \tag{3.20}$$

where

$$C(q,z)=\int_0^\infty dt e^{-zt}\langle\psi_{-q}\psi_q(t)\rangle \tag{3.21}$$

is the dynamic structure factor. The above inequality can then be formulated as,[8]

$$\phi_s(q)\geq\phi(q,z) \ , \tag{3.22}$$

where

$$\phi_s(q)=\lim_{z\to\infty}\phi(q,z) \tag{3.23}$$

is the static part of the memory function. For the spin-flip dynamics of the alternating-bond system, an exact solution could be obtained. For the Kawasaki dynamics this is not the case and we must resort to approximate methods. By focusing on the memory function for this system, we obtain the rigorous lower bound to the dynamic exponent in all cases. It will also be possible to ascertain when the conventional exponent is exact. We now turn to the spin-exchange dynamics of the alternating chain.

IV. SPIN-EXCHANGE DYNAMICS

In this section we set up the EOM for spin-exchange dynamics. We obtain $\phi_s(q)$, from which we extract the conventional critical exponents. We then analyze the remaining frequency dependence of $\phi(q,z)$. We present evidence that the conventional exponent is exact for the case of the conserved order parameter.

A. Equations of motion

The EOM for nearest-neighbor spin-exchange dynamics is of the form[9,19]

$$\tilde{D}_\sigma\sigma_i=-\beta\sum_{a=\pm1}(\sigma_i-\sigma_{i+a})W_{i,i+a}(\sigma) \ , \tag{4.1}$$

where β is the exchange rate in the absence of interactions, the sum is over nearest neighbors and $W_{i,i+a}$ is the probability of near-neighbor exchange (see below). The total spin is manifestly conserved as can be seen by summing Eq. (4.1) over all sites. Spin-exchange dynamics is a two-site process, and thus for the alternating bond envi-

ronment anisotropic transition rates must be established. The generalization of Eq. (4.1) is therefore

$$\tilde{D}_\sigma \sigma_{2n} = -\beta[(\sigma_{2n}-\sigma_{2n+1})W_{2n,2n+1}(\sigma) + (\sigma_{2n}-\sigma_{2n-1})W_{2n-1,2n}(\sigma)] , \quad (4.2a)$$

$$\tilde{D}_\sigma \sigma_{2n+1} = -\beta[(\sigma_{2n+1}-\sigma_{2n+2})W_{2n+1,2n+2}(\sigma) + (\sigma_{2n+1}-\sigma_{2n})W_{2n,2n+1}(\sigma)] . \quad (4.2b)$$

The simplest transition rate in one-dimension is that due to Zwerger[4] which we adapt to this system. For even-odd exchange (taken from left to right, respectively) we find

$$W_{2n,2n+1}(\sigma) = 1 - \tfrac{1}{2}\eta_1(\sigma_{2n-1}\sigma_{2n} + \sigma_{2n+1}\sigma_{2n+2}) , \quad (4.3a)$$

while for odd-even exchange (taken left to right)

$$W_{2n-1,2n}(\sigma) = 1 - \tfrac{1}{2}\eta_2(\sigma_{2n-2}\sigma_{2n-1} + \sigma_{2n}\sigma_{2n+1}) , \quad (4.3b)$$

where

$$\eta_{1,2} \equiv \tanh(2K_{1,2}) . \quad (4.4)$$

Note that the transition rate for even-odd (odd-even) exchange, i.e., between spins coupled by J_2 (J_1), is actually set by the *other* coupling, J_1 (J_2). This is because the product of the exchanged spins is an invariant of the dynamics. Detailed balance is satisfied irrespective of the bond coupling the exchanging spins, since the pair contribution to the Hamiltonian from these spins is invariant. The transition rate, however, *is* sensitive to the change in energetic environment into which exchange occurs. This is controlled by the "outside" bonds, which, because of the bond-periodicity are the same on adjacent sides of the exchanging pair.

The single-spin EOM generates nonlinear products of three *consecutive* spins, as can be seen from Eqs. (4.2) and (4.3). The occurrence of this type of nonlinearity is specific to the Zwerger model and is the feature which allows it to be tractable.[9] Hence, in addition to the single spin transforms, Eq. (2.5), we require the Fourier transforms of consecutive three-spin terms, of which there are two types, centered on alternating sites,

$$T_q = \frac{1}{\sqrt{N}} \sum_n \exp(iqna)\sigma_{2n-1}\sigma_{2n}\sigma_{2n+1} , \quad (4.5a)$$

$$V_q = \frac{1}{\sqrt{N}} \sum_n \exp[iq(2n+1)a/2]\sigma_{2n}\sigma_{2n+1}\sigma_{2n+2} . \quad (4.5b)$$

The order parameter EOM is then given by

$$\frac{\sqrt{2}}{\beta}\tilde{D}_\sigma \psi_q = a_q R_q + a_{-q} S_q + b_q T_q + b_{-q} V_q , \quad (4.6)$$

where

$$a_q = -4\sin^2(qa/4) + \tfrac{1}{2}(\eta_1+\eta_2)[\cos(qa/2)-\cos(qa)] + \frac{i}{2}(\eta_1-\eta_2)[\sin(qa)-\sin(qa/2)] \quad (4.7a)$$

and

$$b_q = -(\eta_1+\eta_2)\sin^2(qa/4) + \frac{i}{2}(\eta_1-\eta_2)\sin(qa/2) . \quad (4.7b)$$

The total magnetization, which is the $q=0$ component, is readily seen to be conserved. Note that the terms *odd* in q are associated with anisotropy of the coupling strengths, i.e., they vanish in the isotropic limit. These terms can be traced to the anisotropy of exchange rates which breaks the invariance of the dynamical equations under $q \to -q$. An attempt to find an explicit solution for ψ_q would entail confronting an infinite hierarchy of coupled dynamical equations. EOM for T_q and V_q couple to five-spin terms, etc. As discussed in the previous section, information concerning the critical response can be directly obtained from the associated memory function.

The EOM for ψ_q is complicated in that it couples *separately* to the modes R_q, S_q, T_q, and V_q. This proliferation of modes is characteristic of the alternating bond system, with its extra periodicity. To achieve simplicity, then, we explicitly project the order parameter out from the EOM,

$$\tilde{D}_\sigma \psi_q \equiv -\phi_s(q)\psi_q + \beta I_q , \quad (4.8)$$

where Eq. (4.8) combined with Eq. (4.6) *defines* I_q such that

$$\langle \psi_{-q} I_q \rangle = 0 , \quad (4.9)$$

and where

$$\phi_s(q) = -(\chi_q)^{-1}\langle \psi_{-q}\tilde{D}_\sigma \psi_q \rangle . \quad (4.10)$$

Equation (4.10) is equivalent to Eqs. (3.16) and (3.17). Note that the two terms in Eq. (4.8) are orthogonal at zero time only. Whether the degrees of freedom which comprise I_q are relevant to the long-time critical response is addressed below. It is natural to separate from $\phi(q,z)$ its frequency-dependent, "dynamic part," $\phi_d(q,z)$ via

$$\phi(q,z) \equiv \phi_s(q) + \phi_d(q,z) . \quad (4.11)$$

$\phi_d(q,z)$ arises *solely* from the contributions to the equation of motion which initially have no component along the order parameter,[8]

$$\phi_d(q,z) = -(\chi_q)^{-1}\beta^2 P_{q,z}(I_{-q}, I_q) , \quad (4.12)$$

where the projectorlike quantity

$$P_{q,z}(A_{-q}, B_q) \equiv \langle A_{-q} R(z) B_q \rangle - \langle A_{-q} R(z)\psi_q \rangle C^{-1}(q,z)\langle \psi_{-q} R(z) B_q \rangle \quad (4.13)$$

will be useful in what follows. $P_{q,z}(A,B)$ effectively projects any component of the order parameter out from the dynamical correlation function between variables A and B. $R(z)$ is the resolvent operator

$$R(z) \equiv [z - \tilde{D}_\sigma]^{-1} . \quad (4.14)$$

$\phi_d(q,z)$ is a complicated object which reflects the nonlinear nature of the problem. On the other hand, $\phi_s(q)$ is

exactly calculable. We first extract the conventional results for this system from $\phi_s(q)$. We then return to the effects of the remaining dynamic piece upon the critical response.

B. Conventional critical dynamics

We find from Eq. (4.10)

$$\phi_s(q)=2\beta \sin^2(qa/4)\Gamma_D(\chi_q)^{-1} , \tag{4.15}$$

where

$$\Gamma_D=\Gamma_1(1-u_2)+\Gamma_2(1-u_1) , \tag{4.16}$$

and where Γ_1 and Γ_2 are the two average exchange rates

$$\Gamma_1=\langle W_{2n,2n+1}(\sigma)\rangle=\frac{1-u_1^2}{1+u_1^2} , \tag{4.17a}$$

$$\Gamma_2=\langle W_{2n+1,2n+2}(\sigma)\rangle=\frac{1-u_2^2}{1+u_2^2} . \tag{4.17b}$$

Equation (4.15) prescribes the lifetimes of fluctuations under Kawasaki dynamics for the alternating chain in the conventional approximation, which, as we will see, *can* be the limiting form of the $\phi(q,z)$ in the critical region. We note that Eq. (4.15) becomes exact in the high-temperature limit. Its nontrivial q dependence in this limit reflects the *dynamical* coupling needed to maintain spin diffusion even in the absence of interactions. As we remarked previously, the divergence of χ_q is universal, i.e., independent of ground state. Thus, the nonuniversality again occurs via the vanishing of Γ_D as $T\to 0$. We now examine the various cases.

1. Dominant interaction ferromagnetic

When both interactions are ferromagnetic, critical fluctuations occur at long wavelength and the dynamics is dominated by the conserved order parameter. The spin diffusion coefficient associated with this hydrodynamic mode is then obtained via[18]

$$D=\lim_{q\to 0}\lim_{z\to 0}\frac{1}{q^2}\phi(q,z) . \tag{4.18}$$

$\phi(q,z)$ has an overall, leading q^2 dependence for small q [see Eq. (4.28)] and the limit can be taken directly. We thus obtain the conventional diffusion coefficient

$$D_s=\frac{\beta a^2}{8}\frac{1}{\chi_0}\Gamma_D . \tag{4.19}$$

In the isotropic limit D_s reduces to the *exact* Zwerger diffusion coefficient.[4] As we will show, D_s is the dominant contribution to the full diffusion coefficient in the critical region [see Eq. (4.33)]. In the critical region,

$$\Gamma_D\simeq(1-u_1)(1-u_2)\sim\xi^{-(1+\rho)} . \tag{4.20}$$

Combined with the divergence of the susceptibility, the diffusion coefficient then vanishes *asymptotically* as

$$D\simeq\xi^{-(2+\rho)} . \tag{4.21}$$

We obtain from Eq. (4.15),

$$\phi_s(q)\simeq\xi^{-(4+\rho)}(q\xi)^2[1+\tfrac{1}{4}(q\xi)^2+\cdots] \quad (q\xi\ll 1) . \tag{4.22}$$

Thus, the conventional exponent is $z=4+\rho$. Note that since $\rho\geq 1$, our result is smaller than that of DKM, who obtain $3+2\rho$ from an analysis of domain wall motions. As noted in Ref. (11), domain wall random walk arguments seek to identify the *fastest* dynamical mechanism by which a domain can traverse a distance ξ and therefore arguments of this type should yield a *lower* bound to the dynamic exponent in agreement with conventional theory. The discrepancy here is not understood. Our result would imply a still faster mechanism for domain diffusion than that considered by DKM.

For $J_2<0$, the ground state is a dimerized structure consisting of alternately oriented spin pairs. The vanishing of Γ_D is

$$\Gamma_D\simeq 1-u_1\sim\xi^{-\rho} . \tag{4.23}$$

Thus, the conventional exponent in this case is $z=1+\rho$. The smaller exponent (faster dynamics) for this system as compared with the ferromagnetic chain is due to two effects. The order parameter is not conserved which lowers the exponent by two. Secondly, however, spin exchange proceeds more readily in the presence of the "interfaces" of the dimerized structure. Correlations decay more rapidly as the domains can "dissolve" rather than diffuse. This is reflected in the less severe vanishing of Γ_D, i.e., $\Gamma_D\sim\xi^{-\rho}$ as opposed to $\xi^{-(1+\rho)}$ for $J_2>0$.

2. Dominant interaction antiferromagnetic

For this system we obtain $z=2$ irrespective of the weaker bond, J_2. For $J_2>0$, the vanishing of Γ_D is characterized by

$$\Gamma_D\simeq 1-u_2\sim\xi^{-1} , \tag{4.24}$$

whereas for $J_2<0$,

$$\Gamma_D\simeq 1-|u_1u_2|\sim\xi^{-1} . \tag{4.25}$$

Thus, we obtain $z=2$ for both cases. Note that we also obtain $z=2$ for the isotropic antiferromagnetic exchange dynamics. This case has not received prior investigation to the best of our knowledge. The exponents from Eqs. (4.23)–(4.25), i.e., $z=1+\rho$ and $z=2$, resemble spin-flip exponents. This is to be expected, since while the total magnetization is conserved by the dynamics, the order parameter for these cases is not.

C. Dynamic memory function

As we have stated, the conventional exponent is in principle the lower bound to the full exponent. This can be readily seen from the Kawasaki inequality, Eq. (3.22), as both $\phi_s(q)$ and $\phi(q,z)$ are positive. One can also infer that $\phi_d(q,z)\leq 0$. If the long-time critical response is to be *slower* than the conventional result,[8] we see that in the critical region the form of $\phi_d(q_0,0)$ must be such that it is composed of $-\phi_s(q_0)$ plus a piece $A\xi^{-z}$ with $z>z_c$ ($A>0$). Thus, if it can be shown in the critical region

that $\phi_s(q_0) \gg |\phi_d(q_0,0)|$, then conventional theory yields the exact exponent. While we are unable to evaluate $\phi_d(q,z)$ explicitly, we can gain insight from its overall structure. Combining I_q from Eqs. (4.6) and (4.8) with Eqs. (4.12) and (4.13), it is readily shown that $\phi_d(q,z)$ consists of higher-order correlations among the split linear (L) single-spin modes R_q, S_q and the nonlinear (N) three-spin modes, T_q, V_q,

$$-\chi_q \phi_d(q,z) \equiv \beta^2 [G_{LL}(q,z) + G_{LN}(q,z) + G_{NN}(q,z)] \tag{4.26}$$

where

$$G_{LL}(q,z) = \tfrac{1}{4}(\eta_1 - \eta_2)^2 \sin^2(qa/2)[2\cos(qa/2) - 1]^2 P_{q,z}(R-S, R-S) , \tag{4.27a}$$

$$G_{LN}(q,z) = \tfrac{1}{4}(\eta_1 - \eta_2)^2 \sin^2(qa/2)[2\cos(qa/2) - 1][P_{q,z}(R-S, T-V) + P_{q,z}(T-V, R-S)] + \frac{i}{2}(\eta_1^2 - \eta_2^2)\sin^2(qa/4)\sin(qa/2)[2\cos(qa/2) - 1][P_{q,z}(R-S, T-V) - P_{q,z}(T-V, R-S)] , \tag{4.27b}$$

and

$$G_{NN}(q,z) = \tfrac{1}{4}(\eta_1 - \eta_2)^2 \sin^2(qa/2) P_{q,z}(T-V, T-V) + i(\eta_1^2 - \eta_2^2)\sin(qa/2)\sin^2(qa/4)[P_{q,z}(T,V) - P_{q,z}(V,T)] + (\eta_1 + \eta_2)^2 \sin^4(qa/4) P_{q,z}(T+V, T+V) , \tag{4.27c}$$

where the function $P_{q,z}(A,B)$ is defined by Eq. (4.13). Note that $\phi_d(q,z)$ is a real-valued function. $\phi_d(q,z)$ reflects the full many-body nature of the problem, which is complicated by the splittings induced by the extra bond periodicity. The following, however, is readily observed. If the sublattice degrees of freedom were identically equivalent dynamical variables, note that only the third contribution to $G_{NN}(q,z)$ would remain in $\phi_d(q,z)$. These same cancellations also occur in the isotropic limit, $J_1 = J_2$. Yet even with anisotropy, these terms eventually vanish in the critical region for much the same reason: the components of ψ_q order together. Once again, we see a narrowing of the dynamic critical region. We notice, however, that the terms associated with anisotropy involve correlations among relatively "fast" modes, $R-S$, $T-V$. Hence, we expect that, asymptotically, correlations among the "split" linear and nonlinear modes are irrelevant to the long-time critical response of ψ_q, and furthermore, that only the "intrinsically" nonlinear correlations, $P_{q,z}(T+V, T+V)$, could lead to a nonconventional exponent. However, note that the "slow" nonlinear correlation begins only at $O(q^4)$ for small q, as seen from Eq. (4.27c). Thus, for the same reasons that the exponent is exactly known for the isotropic ferromagnetic exchange dynamics, these considerations suggest that the conventional exponent could also be exact for this system in the case of the conserved order parameter. We must show that the contributions to $\phi_d(q,z)$ at $O(q^2)$ do not vanish any slower than the conventional response.

In the long-wavelength limit, the separate contributions in Eq. (4.26) combine into the following form

$$\phi_d(q,z) = -\frac{\beta^2 (qa)^2}{16\chi_0}(\eta_1 - \eta_2)^2 \times P_{0,z}(R-S+T-V, R-S+T-V) + O(q^4) . \tag{4.28}$$

The exact expression for the diffusion coefficient is therefore

$$D = D_s - \frac{\beta^2 a^2 (\eta_1 - \eta_2)^2}{16\chi_0} \times P_{0,0}(R-S+T-V, R-S+T-V) , \tag{4.29}$$

where D_s is given by Eq. (4.19). Note that $(\eta_1 - \eta_2)^2 \sim \xi^{-4}$ and thus the prefactor in the "dynamical" term vanishes as ξ^{-5}. The above correlation function is among the nonconserved set of variables $R_0 - S_0 + T_0 - V_0$, which relaxes rapidly when compared with the (conserved) order parameter. For convenience we define the combination of variables $W_q \equiv R_q - S_q + T_q - V_q$. It is shown in the Appendix that

$$P_{0,0}(W,W) = \frac{\langle W_0^2 \rangle}{K_{W,W}(0,0)} , \tag{4.30}$$

where $K_{W,W}$ is the memory function associated with the variable W_q. The remaining "projection" terms [see Eq. (4.13)] start at $O(q^4)$ and thus do not contribute at long wavelength, i.e., dynamically there is no component of W_0 along the uniform order parameter. We show in the Appendix that in the critical region [Eqs. (A4) and (A8)],

$$\langle W_0^2 \rangle \sim \xi^{-\rho} , \tag{4.31}$$

$$K_{W,W}(0,0) \sim \xi^{-2} , \tag{4.32}$$

which, when combined with Eq. (4.29) yields as $T \to 0$,

$$D = D_s - O(\xi^{-(3+\rho)}) . \tag{4.33}$$

Since $D_s \sim \xi^{-(2+\rho)}$ [Eq. (4.21)], the *conventional result is the leading contribution to the diffusion coefficient in the critical region.* This implies, then, that the conventional exponent $z = 4 + \rho$ is exact.

V. DISCUSSION

We have examined the critical dynamics of the alternating bond Ising chain for two types of kinetic processes and for arbitrary bond strengths. We find nonuniversal

dynamic critical exponents which are governed by the ratio of the couplings. For the general Glauber dynamics and for the ferromagnetic Kawasaki dynamics we obtain exact dynamic exponents. For the remaining cases of spin-exchange dynamics we obtain rigorous lower bounds corresponding to conventional theory, which may also be asymptotically exact. Dynamic nonuniversality was shown to arise from the nonuniversal vanishing of the kinetic coefficients, which depend on local correlations. The nonuniversal critical dynamics of this system is therefore a consequence of $T_c=0$. However, not every system with $T_c=0$ may in fact show nonuniversal dynamic critical phenomena due to anisotropy. A counter-example is the fractal Sierpinski gasket, which due to its finite ramification has a zero temperature critical point. In this case the self-similar geometry greatly extends the *static* critical region[20] via an anomalously divergent correlation length. As for the conventional critical dynamics of this system,[21] the strongly diverging correlation length forces the kinetic coefficient, with its dependence on local correlations, to vanish not as a power of ξ but instead only *logarithmically*. Thus, we would not expect nonuniversal critical dynamics for a Sierpinski gasket with anisotropic couplings, unless the *static* critical phenomena were also nonuniversal. The dynamic exponents for the alternating chain are *larger* than those for the isotropic system, which result from the energetic "bottlenecks" produced by the stronger bonds. One can speculate whether the larger exponents of this system can be seen as a remote precursor to the breakdown of dynamic scaling seen in less regular geometries.[22] Formulated differently, the extra periodicity of the alternating bond system forces the dynamical critical region to be narrower than that for the isotropic system. This is due to the extra dynamical freedom of the "split" order parameter components, where only asymptotically in the critical region does the order parameter become the dominant slow mode. One can then envisage a route to the breakdown of dynamic scaling as a progressive narrowing of the dynamic critical region as further periodicities are introduced.

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APPENDIX

We calculate the higher-order correlation function which contributes to the diffusion coefficient, Eq. (4.29). From Eq. (4.13),

$$P_{q,0}(W,W)=\frac{\langle W_{-q}W_q\rangle}{K_{W,W}(q,0)}-\frac{\phi(q,0)}{\chi_q}\frac{|\langle W_{-q}\psi_q\rangle|^2}{|K_{W,\psi}(q,0)|^2}\ , \tag{A1}$$

where $K_{W,W}$ and $K_{W,\psi}$ are the respective memory functions. We find that

$$|\langle W_{-q}\psi_q\rangle|^2\sim O(q^2)\ , \tag{A2}$$

and the equilibrium correlation between W_q,ψ_q vanishes at long wavelength. Furthermore, we find that the memory function $K_{W,\psi}$ is independent of q as $q\to 0$. The "projection" piece in Eq. (A1) therefore starts at $O(q^4)$. This is to be expected since W_0 is nonconserved and thus is a faster mode than ψ_q. Therefore, the $q=0$ limit can be taken in Eq. (A1). It can be shown that

$$\langle W_0^2\rangle=2u_1u_2(2-u_1-u_2)+\frac{2}{1-u_1u_2}[(1-u_1)^3+(1-u_2)^3-(u_1-u_2)^2]\ , \tag{A3}$$

which vanishes in the critical region as

$$\langle W_0^2\rangle\sim\xi^{-\rho}\ . \tag{A4}$$

We calculate the memory function associated with W_0 utilizing its initial relaxation rate. As the fluctuations of W_0 have no critical weight [Eq. (A4)], we expect this to be a good approximation. Thus, we have

$$\langle W_0^2\rangle K_{W,W}(0,0)=-\langle W_0\tilde{D}_\sigma W_0\rangle\ . \tag{A5}$$

It can be shown that

$$\langle W_0\tilde{D}_\sigma W_0\rangle=-2\beta[\Gamma_1(1-u_2)^2(3-u_2)+\Gamma_2(1-u_1)^2(3-u_1)]\ , \tag{A6}$$

which vanishes in the critical region as

$$-\langle W_0\tilde{D}_\sigma W_0\rangle\sim\xi^{-(2+\rho)}\ . \tag{A7}$$

From Eqs. (A4), (A5), and (A7),

$$K_{W,W}(0,0)\sim\xi^{-2}\ . \tag{A8}$$

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