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Numerical study of conductance fluctuations in disordered metals

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The results of a numerical study of conductance fluctuations in weakly disordered one- and two-dimensional metals are presented and compared with recent perturbative calculations. Two models are considered: the usual Anderson tight-binding model with a uniform distribution of site energies, and a second tight-binding model in which the site energies of a given concentration of "impurities" are $\pm\Delta E$. For the Anderson model we calculate the fluctuations of the conductance among members of an ensemble of statistically similar samples (i.e., samples with the same amount of disorder, etc.). We find that the magnitude of these fluctuations agrees fairly well with the "universal" value predicted by the perturbative calculations, even for systems as small as a few lattice spacings on a side. For the second model, we calculate the conductance fluctuations which occur when only a single impurity is moved, and our results are in reasonable agreement with the analytic results for this case.

I. INTRODUCTION

The problem of quantum transport in disordered metals has attracted a good deal of attention in recent years, and a number of surprising results have emerged from work in this area.[1] In particular, the occurrence of sizable conductance fluctuations among members of an ensemble of statistically similar systems has been studied in several contexts. For strictly one-dimensional systems (e.g., a linear chain) it has been shown that for such an ensemble, the zero-temperature conductance is a statistically "ill-behaved" quantity.[2–4] More precisely, it was found that the probability distribution, $P(G)$ where G is the conductance, cannot, as is usually the case, be characterized by the mean conductance, but has a long "tail" for large G. A similar type of fluctuation involves the phenomena of resonant tunneling[5] which is manifest as large fluctuations in the conductance, as a function of (Fermi) energy, and which has been studied in most detail in the strongly disordered limit. In addition, recent work by Lee and co-workers[6] has shown that variable range hopping in finite, one-dimensional systems can also lead to large fluctuations in the conductance.

In the last year or so a different type of fluctuation has been discovered in weakly disordered systems. It has been shown[7–10] that the zero temperature conductance of an ensemble of statistically similar systems exhibits fluctuations which have a "universal" magnitude $\overline{\delta G} \approx e^2/h$, independent of system size, or G, so long as the system is a "good" conductor, i.e., $G \gtrsim e^2/h$. More precisely, $\overline{\delta G}$ is predicted to have this universal value when the electron motion is diffusive, which is the case for length scales which are both greater than the elastic mean free path, l_e, and smaller than the spatial extent of the wave function, i.e., the localization length.[1] This phenomenon is predicted to be only weakly dependent on dimensionality, with $\overline{\delta G} = Be^2/h$, where B is a constant of order unity, which varies only about 30% as one goes from one to three dimensions.[8,9] It turns out that there is no diffusive regime for a strictly one-dimensional system,[11] so these universal fluctuations can only be studied in systems which are (at least) two or more lattice sites wide. However, systems can be one dimensional as far as the universal fluctuations are concerned, so long as their width is less than the localization length.

Universal conductance fluctuations are manifest experimentally in a number of different ways, and the theoretical predictions concerning these fluctuations seem to be in good agreement with various experiments.[12] These predictions stem largely from perturbation theory[8–10] performed to lowest order in $(k_F l_e)^{-1}$, where k_F is the Fermi wave vector, and l_e is the elastic mean free path. There has, in addition, been numerical work on this problem,[7,8,13] and indeed, the numerical work of Stone[7] helped lead to the discovery of the universality of the fluctuations. Some very interesting arguments, based on the theory of random matrices, have also been put forth by Imry[14] (see also Al'tshuler and Shklovskii[15]), and attractive physical arguments exist to explain why these systems should exhibit large fluctuations.[16,10] Nevertheless, while this problem has attracted a great deal of interest, many of the detailed, quantitative predictions of the perturbative calculations have not yet been tested, either by experiments or through other types of calculations, such as numerical simulations. In this paper we present the results of a numerical study of this phenomenon.

We have studied two models. The first is the usual Anderson tight-binding model[17] in which the site energies are uniformly (and randomly) distributed over a given range. For this model we have calculated the fluctuations of the conductance as a function of system size, energy, etc. For the case of weak disorder we find that the fluctuations have a "universal" magnitude, which is in reasonable agreement with the perturbative calculations in both one and two dimensions. We have also considered the effects of strong disorder, and find that when the conductance is less than about e^2/h, the size

of the fluctuations decreases significantly as G decreases. The second model we have studied is also a tight-binding model, but one which contains "impurity" sites, which have an energy $\pm\Delta E$. For this model we concentrate on the fluctuations which occur when a single impurity is moved, and compare with recent calculations of Feng *et al.*[13,18] In this case, comparison with the theory requires an independent estimate of parameters such as the mean free path. If simple nearly-free-electron theory is used to determine these parameters, we find reasonable, though not perfect, agreement with the theory.[13] It has been proposed that the fluctuations which occur when single impurities are moved could be a significant source of $1/f$ noise in metals. The relatively minor disagreement we find with the theory for this case may have significant implications for the magnitude of the $1/f$ noise generated by these fluctuations.

Our work has been performed along the same lines as the numerical work of Stone[7] (see also Refs. 8 and 13), and our calculations confirm many of his results. We also test several of the theoretical predictions not considered in the previous numerical work.

II. METHOD

The models we consider are both described by the Hamiltonian

$$H=\sum_i E_i+V\sum_{i,j}(\,|\,i\,\rangle\langle\,j\,|+|\,j\,\rangle\langle\,i\,|\,), \tag{1}$$

where the indices i and j refer to different sites. For the usual Anderson tight binding model[17] the site energies E_i are randomly and uniformly distributed in the range $-W/2$ to $+W/2$. For the second model we consider, which we will refer to as the $\pm\Delta E$ model, the site energy is zero except for a given concentration of impurity sites which have energies $\pm\Delta E$, with both the locations of the impurities, and the signs of their site energies, chosen at random. For both models we assume overlap only between nearest neighbors, with $V=1$ generally taken for convenience. We have studied systems in which the sites were situated on a square lattice, with various values of length and width.

The conductance was calculated using the method developed by Fisher and Lee.[19] In this approach, the ends of the system described by (1) are attached to perfect leads, which are semi-infinite regions which are also described by (1) but with the site energies all zero. Fisher and Lee have shown that, subject to some very reasonable assumptions, the conductance G for the total system, the leads plus the disordered region, is related to the transmission matrix, t, by

$$G=2\frac{e^2}{h}\mathrm{Tr}(t^\dagger t), \tag{2}$$

where $t^\dagger$ is the transpose of t, and Tr denotes the trace. It is convenient to work in terms of the dimensionless conductance, $g=G/(e^2/h)$, and we will do so in the rest of this paper.

The relation between g and t, (2), is a generalization of the well-known Landauer formula.[3,2] The original Landauer formula[3] applies to a strictly one-dimensional chain, while (2) is intended to be applicable in more general situations. However, it has been shown that the precise form of the relationship between g and t depends on how one chooses to attach leads to the system, etc., and this problem has been discussed extensively in the literature.[20] While no "best" generalization of the Landauer formula has yet emerged, it is believed that (2) is applicable when g is not too large; that is, when the transmission coefficients of the individual conducting channels (see below) are all small. This is always the case for the models and parameter values studied in this paper, so it is hoped that our results are not affected by any limitations inherent in (2). We note that a number of previous related studies[21,7,8,13] have also employed (2).

We have calculated the transmission matrix, t, using the method of Fisher and Lee[19] (see also Thouless and Kirkpatrick[22]). One starts in one of the perfect leads, at, for example, the right side of the disordered region. It is straightforward to calculate the eigenstates and eigenvalues in the leads.[23] Now, consider the (disordered) system to consist of L rows of sites, with row 1 at the left edge. From the eigenvalues in the adjacent (right) lead, one can calculate the right self-energy for row L. Since the eigenstates in the leads are characterized by the wave vectors, k_x and k_y, it is simplest to write this self-energy in terms of a "channel" basis, with the channels labeled by their values of k_y (here the x direction runs along the system, from one lead to the other, while y runs along each row). However, in the disordered region k_x and k_y are no longer good quantum numbers, and it is more convenient to work in terms of a "site" basis [i.e., (1)]. It is straightforward to transform the self-energy matrix from one basis to the other.[23] After one obtains the right self-energy of row $L+1$, S^R_{L+1}, one can then obtain the self-energy of row L using the relation[22]

$$S^R_i=\frac{V^2}{EI-H_{i+1}-S^R_{i+1}}, \tag{3}$$

where the i denotes the row, E is the (Fermi) energy, I is the unit matrix, and H_i is the Hamiltonian for row i. Note that S^R and H_i in (3) are matrices of rank M, where M is the number of sites in a row.

The next step is to iterate (3) to the left edge of the disordered region, where one then "connects" to the left lead, and obtains the Green function using the relation[22]

$$G(i,j)=(EI-H_i-S^R_i-S^L_i)^{-1}\prod_{k=i}^{j-1}\left[\frac{S^R_k}{V}\right], \tag{4}$$

where S^L is the left self-energy. The Green functions $G(0,L+1)$ and $G(0,0)$ can then be used to obtain the elements of the transmission and reflection matrices. For this, it is easiest to work in the channel basis, since the relations are[19,7]

$$|\,r_{\beta\gamma}\,|^2=|\,\sqrt{-1}\,(v_\beta v_\gamma)^{1/2}G(0,0)_{\beta\gamma}-\delta_{\beta\gamma}\,|^2, \tag{5}$$

and

$$|\,t_{\beta\gamma}\,|^2=v_\beta v_\gamma\,|\,G(0,L+1)_{\beta\gamma}\,|^2. \tag{6}$$

Here and below, β and γ are channel indices. The channel velocities, v_β, are given by

$$v_\beta = \frac{\partial E_\beta}{\partial k_\beta}, \tag{7}$$

where E_β and k_β are the energy and wave vector (in the x direction) for channel β. It useful to note that $E = 2V\cos(k_x a) + 2V\cos(k_y a)$ where a is the lattice constant.

Given (6) and (7), the conductance can be obtained from [see (2)][19,7]

$$g = 2\sum_{\beta\gamma} |t_{\beta\gamma}|^2, \tag{8}$$

or

$$g = 2\left[N - \sum_{\beta\gamma} |r_{\beta\gamma}|^2\right]. \tag{9}$$

The expressions (8) and (9) are equivalent, since the total reflection and transmission coefficients (which are just the sum of the square moduli of the coefficients for the individual channels) must sum to N, the number of channels. Note also that only channels with real velocities (i.e., propagating modes) are included in the sums (8) and (9).[19] Finally, we note that while the expressions above contain V and a explicitly, in the calculations below we have set them both to unity.

It is straightforward to implement the above scheme numerically. Nevertheless, it is desirable to test the computer program wherever possible, and we have performed the following tests. First, as can be seen from (8) and (9), for the case of no disorder ($|t_{\beta\gamma}| = 1$, $|r_{\beta\gamma}| = 0$) one finds that g is equal to $2N$, and this result was always found. In addition, we calculated g using both (8) and (9), and the results always agreed to at least six or seven decimal places. Since (8) and (9) employ quite different Green functions, this is a strong test. This also tests for the possibility of round-off errors, and the agreement indicates that this was not a problem.

III. RESULTS FOR THE ANDERSON MODEL

We have calculated the conductance for the Anderson model for systems with various sizes, and values of E and W. Typical results for systems which are size $L \times L$ lattice spacings, i.e., two dimensional, are shown in Fig. 1, while Fig. 2 shows results for systems which are $10L \times L$, and hence expected to be one dimensional. Here we plot var(g), where

$$\mathrm{var}(g) = \langle g^2 \rangle - \langle g \rangle^2. \tag{10}$$

These results were obtained by computing the conductance of a number of different independent samples, typically 100 to 1000. From Fig. 1 we see that for a wide range of parameter values, var(g) is equal to approximately 0.60, with a spread of about ± 0.08. This is in reasonable agreement with the value 0.74 predicted by Lee and Stone,[8] and by Al'tshuler.[9] The difference is slightly outside the uncertainty we would estimate from Fig. 1, but we are not certain if it is truly significant. The numerical work of Lee and Stone[8] gave agreement

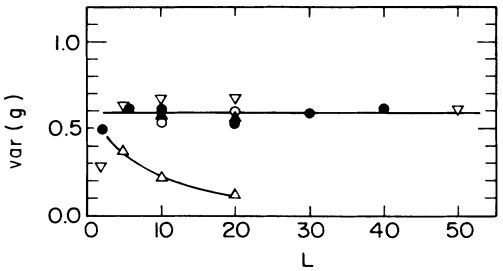

FIG. 1. var(g) as a function of L for an $L \times L$ Anderson model, for various values of E and W. (●) $E=0$, $W=5$; (▽) $E=0$, $W=3$; (▲) $E=0.4$, $W=5$; (△) $E=0$, $W=8$; (○) $E=0$, $W=1$. The solid lines are guides to the eye. All of the data are from the regime $g \gtrsim 1$, except for the results for $E=0$, $W=8$; in this case $\langle g \rangle$ varied from 0.7 for $L=5$ to 0.14 for $L=20$.

with the theoretical value (0.74), but given the combined uncertainties our findings are not in disagreement with those numerical results. We note from Fig. 1 that this "universal" value of var(g) is observed only for the weakly disordered case; that is, $\langle g \rangle \sim 1$ or larger. The △ symbols in Fig. 1 show data for the case of strong disorder, i.e., large W, for which $\langle g \rangle$ is reduced below unity. In this limit, var(g) is correspondingly reduced. For example, for the case $E=0$, $W=8$, in Fig. 1, the sample with $L=20$ had $\langle g \rangle = 0.14$, and var(g)$=0.13$, well below the value var(g)$=0.53$ found for $L=20$, $E=0$, $W=5$, which had $\langle g \rangle = 1.33$.

Figure 2 shows that var(g) in one dimension is also universal for large $\langle g \rangle$, with a value near 0.50. This is very close to the value 0.53 predicted by Lee and Stone,[8] and slightly, but definitely, higher than the value 0.38

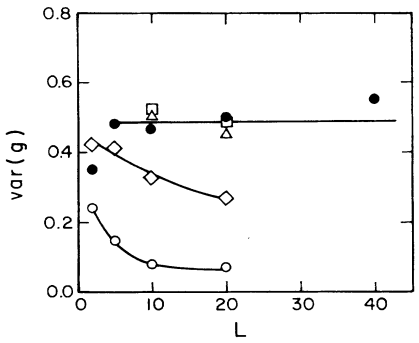

FIG. 2. var(g) as a function of L for a $(10L) \times L$ Anderson model, for various values of E and W. (●) $E=0$, $W=1$; (◇) $E=0.4$, $W=2$; (○) $E=0$, $W=3$; (□) $E=0$, $W=0.5$; (△) $E=0.2$, $W=1$. The solid lines are guides to the eye. All of the data are from the regime $g \gtrsim 1$, except for the results for $W=2$ and $W=3$; in these cases $\langle g \rangle$ varied from 0.9 for $L=5$ and $W=2$ to 0.12 for $L=20$ and $W=3$.

found by Al'tshuler.[9] Our findings also agree with the numerical results of Lee and Stone.[8] Our results thus also confirm that var(g) does depend on dimensionality. In addition, we find that, as in two dimensions, var(g) is depressed below the universal value when $\langle g \rangle < 1$. This can be seen from the $\diamond$ and $\circ$ data sets in Fig. 2.

A very interesting aspect of the results in Figs. 1 and 2 is that the fluctuations attain the "universal" values even for quite small system sizes. Systems as small as $L=5$ lattice spacings on a side exhibit values of var(g) which are in good agreement with the theory, and systems with $L=2$ exhibit values which are not much different. In this connection, one should recall that systems which are strictly one-dimensional (i.e., $L=1$) do not exhibit the universal fluctuations, since[11] in that case the localization length is of order l_e, and hence there is no length scale on which the electronic motion is diffusive.

The very large, and non-universal, fluctuations found in the strictly one-dimensional case, are also associated with probability distributions, $P(g)$, which are distinctly non-Gaussian.[2-4] We have therefore investigated $P(g)$ for the present case, and some typical results are shown in Figs. 3 and 4. We have not performed any statistical tests on the distributions, as meaningful tests would require far more samples than we have obtained at this time. Nevertheless, from the results for two dimensions shown in Fig. 3 we see that for $\langle g \rangle \gtrsim 1$ [Figs. 3(a)–3(d)], the regime in which var(g) has a universal magnitude, the distributions appear, at least by eye, to be well behaved. This is in agreement with the results of Al'tshuler *et al.*[24] We should note, however, that the distributions in Figs. 3(a), 3(c), and 3(d) do seem to show significant deviations from a Gaussian form, with "tails" for large g. In contrast to the nearly Gaussian behavior seen for large $\langle g \rangle$, systems with small $\langle g \rangle$, Fig. 3(e), exhibit very ill-behaved distributions. This is also in agreement with Al'tshuler *et al.*[24] Unfortunately they do not give any explicit form for $P(g)$, so it is not possible to make a quantitative comparison with any of the results in Fig. 3. Returning to Fig. 3(e), it seems as if the system "wants" to have a large value of var(g), but the small value of $\langle g \rangle$ prevents it from attaining the "universal" value. It is also interesting to note from Fig. 3(e) that $P(g)$ appears to follow a roughly $1/g^{\nu}$ form with $\nu \sim 0.5$ for small g in this case (although ν may well depend on E and W). Figure 4 shows corresponding results for $P(g)$ for one dimension. Again we see that $P(g)$ is well behaved for large $\langle g \rangle$, but [Fig. 4(b)] with a significant "tail" for large g.

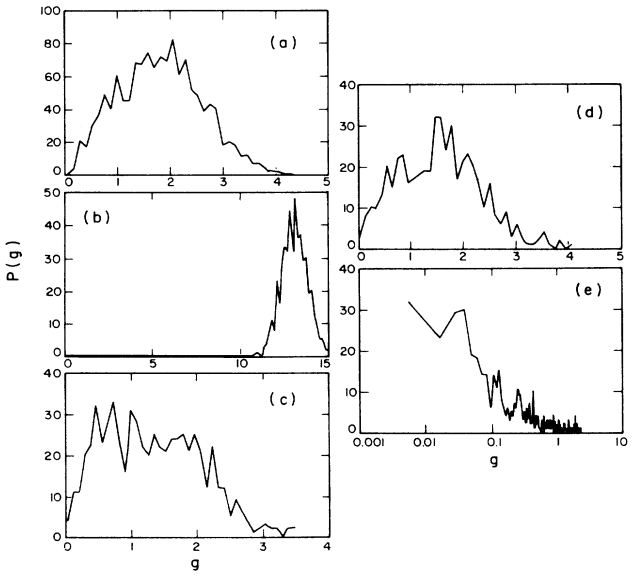

FIG. 3. Probability distributions $P(G)$ for $L \times L$ Anderson models for various values of L, E and W. (a) $L=5$, $E=0$, $W=5$; (b) $L=10$, $E=0$, $W=1$; (c) $L=20$, $E=0$, $W=5$; (d) $L=20$, $E=0.4$, $W=5$; (e) $L=10$, $E=0$, $W=8$. Note that for (e) the horizontal scale is logarithmic. The vertical scales are all in arbitrary units.

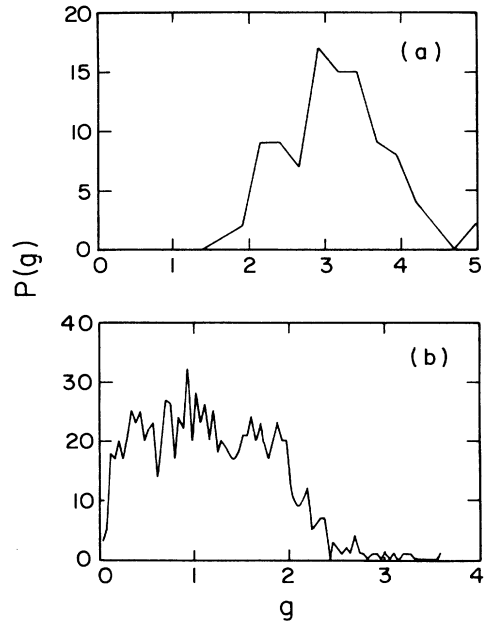

FIG. 4. Probability distributions $P(G)$ for $(10L) \times L$ Anderson models for different values of L, E and W. (a) $L=10$, $E=0$, $W=1$; (b) $L=2$, $E=0$, $W=2$.

IV. $\pm\Delta E$ MODEL

Our calculations for the $\pm\Delta E$ model have concentrated on determining $\langle (\delta g)^2 \rangle$, where δg is the conductance change produced when a single impurity is moved. Our intention was to test the predictions of Feng *et al.*,[13] who have considered this case in detail. Previous numerical work on this problem[13] has employed the usual Anderson model, and has been concerned with how $\langle (\delta g)^2 \rangle$ scales with system size. The perturbation-theory calculations of Feng *et al.* predict that if a single impurity moves, there will be a conductance change δg, where

$$\langle(\delta g)^2\rangle = \frac{B}{C l_e^d}\left[\frac{L}{l_e}\right]^{2-d} \alpha(k_F \delta r), \tag{11}$$

for a d-dimensional hypercube of length L. Here B is a numerical factor[25] equal to 0.226, C is the concentration of impurities, δr is the distance which the impurity moves, and the function $\alpha(x)=1-\{[\sin(x/2)]/(x/2)\}^2$. The overall size of $\langle(\delta g)^2\rangle$ is determined by the factor $(C/l_e^d)(L/l_e)^{2-d}$, which Feng *et al.* have shown to be proportional to $(k_F l_e)^{-1}$ for strong individual scatterers in the strictly two-dimensional geometry we have studied. The dependence on δr is controlled by $\alpha(k_F \delta r)$, which approaches zero as $\delta r \to 0$, as one would expect.

We have calculated $\langle(\delta g)^2\rangle$ as a function of δr for a wide range of parameter values. This was accomplished by generating an impurity distribution, calculating g, then moving one impurity "atom" a given amount $\delta \mathbf{r}$, recalculating g, and then calculating $(\delta g)^2$. This procedure was then repeated many times (generally 50 to 200) and the average value, $\langle(\delta g)^2\rangle$ was computed. The impurities were moved about according to the following rules. An impurity site was first picked at random. A site a distance δr (which was allowed to be in a direction along, perpendicular, or at some angle with respect to the x axis) from the impurity was then chosen, and if the site energy at this location was different from that of the original site, then the two site energies were interchanged. Hence, these rules guaranteed that the new configuration would be different from the original one. In the limit of very small impurity concentrations this would always be the case, even without this restriction, although this would not be true for the relatively large concentrations of impurities typical of a numerical simulation such as ours.

Some typical results for $\langle(\delta g)^2\rangle$ as a function of δr are shown as the solid symbols in Fig. 5, where we show results for a wide range of parameter values. We first note that $\langle(\delta g)^2\rangle$ is generally somewhat smaller than what one finds when the entire sample is changed. This can be seen from a comparison with Fig. 1, if one notes that $\langle(\delta g)^2\rangle$ in Fig. 5 is related to var(g) in Fig. 1 by $(\delta g)^2 = 4\,\mathrm{var}(g)$, when var($g$) is computed for the two configurations needed to calculate dg. In addition, values of var(g) obtained from complete changes of the impurity distribution for the $\pm\Delta E$ model, were in reasonable agreement with the results in Fig. 1.

Figure 5 also shows the theoretical predictions, (11). Here, two theoretical curves are shown, corresponding to two different ways of estimating the parameters which enter (11). Feng *et al.*[13] have shown that for the case of strong individual scatterers, one can use the standard relations[13] $l_e^{-1} = C\sigma$, and $\sigma = 4/k_F$, where σ is the scattering cross-section of an individual scatterer; note that we are now considering the case of two dimensions. Using these relations, (11) becomes (again for $d=2$)

$$\langle(\delta g)^2\rangle = \frac{16BC}{k_F^2}\alpha(k_F \delta r), \tag{12}$$

The solid curves in Fig. 5 were computed from (12) using $k_F = \sqrt{4-E}$, as determined from the dispersion relation.[26] Note that (11) and (12) should be applicable only when $\langle(\delta g)^2\rangle$ is less than unity, which is the value of (i.e., the upper bound on) $\langle(\delta g)^2\rangle$ for a complete change in the impurity configuration. This bound is seen to be important only for the parameters used in Fig. 5(b), and in that case we have therefore not plotted values of $\langle(\delta g)^2\rangle$ from (12) which are larger than unity. We see that, except for the case of Fig. 5(d), (12) yields values of

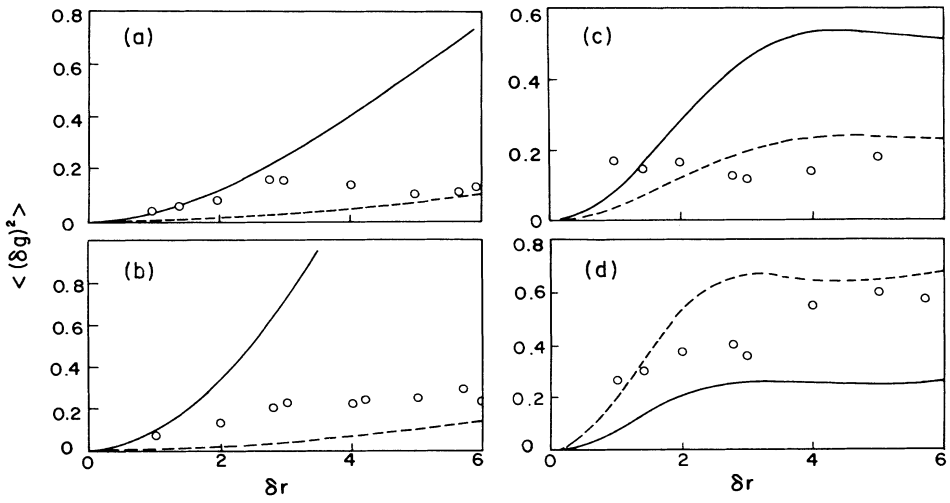

FIG. 5. $\langle(\delta g)^2\rangle$ as a function of δr for several $L \times L$ $\pm\Delta E$ models. (a) $L=30$, $E=3.7$, $\Delta E=1.5$, $C=0.1$, $\langle g\rangle=2.1$; (b) $L=10$, $E=3.7$, $\Delta E=1$, $C=0.3$, $\langle g\rangle=1.0$; (c) $L=20$, $E=2$, $\Delta E=2$, $C=0.3$, $\langle g\rangle=2.5$; (d) $L=10$, $E=0$, $\Delta E=5$, $C=0.3$, $\langle g\rangle=2.1$. The symbols are the calculated values, the solid lines are the prediction (12), and the dashed lines are the prediction (11) using nearly-free-electron theory to estimate l_e as described in the text.

$\langle(\delta g)^2\rangle$ which are significantly higher than found in our calculations, and which are not in agreement with our results.

It is also possible to estimate l_e using the standard relations of nearly-free-electron theory.[27] Doing this, and inserting the values obtained for l_e directly in (11) yields the results shown as the dashed lines in Fig. 5. These predictions are seen to be in fairly good agreement with our calculations. At large δr the agreement is to within better than $\pm 50\%$ in all cases. We conclude that, given the uncertainties is estimating all of the parameters involved (e.g., l_e), the theory (11) is quite consistent with our results. It is not clear why the value of l_e obtained from nearly-free-electron theory should be preferred over that from standard scattering theory. However, a change of l_e by only a factor of $\sim 3-5$ is all that would be required to bring (12) into agreement with our results. It does not seem entirely unreasonable that the "standard" relations between l_e, σ, and k_F, which apply in the limit of very strong individual scatters, could be in error by this amount in our case. It is also worth noting that for the purposes of comparing with experiment, (12) is far more convenient than the use of nearly-free-electron theory to evaluate (11), since the latter requires an estimate of C. In any case, our results suggest that for typical cases (12) may be in error by a factor of 4, and perhaps more.

Our comparisons with the theory, (11), have to this point all focussed on the overall magnitude of $\langle(\delta g)^2\rangle$. The predicted dependence on δr, which arises through the function $\alpha(x)$ in (11), is also important. This dependence is a function only of k_F, which can be derived directly from the dispersion relation. We see from Fig. 5 that our calculations do indicate that $\langle(\delta g)^2\rangle$ becomes smaller at small δr, as predicted by the theory (e.g., the dashed curves). However, it does not appear that the calculated values of $\langle(\delta g)^2\rangle$ approach zero as quickly as predicted for small δr. Thus, $\langle(\delta g)^2\rangle$ is somewhat larger than predicted by (11) for small δr, and impurity motions of only one or two lattice sites have *much larger* effects than predicted. Whether this discrepancy is significant or not is unclear. There are certainly many difficulties which enter the numerical calculations. For example, computer limitations restrict the size of the system which can be studied, and for large E [as in Figs. 5(a) and 5(b)] this in turn limits the number of propagating channels [see (8) and (9)]. The multichannel Landauer formula (2) is valid only for a large number of channels, so this presents a problem. However, we have obtained results like those in Fig. 5 for many values of E, ΔE, C, and system size, and the results shown in Fig. 5 are typical. We therefore do not believe that difficulties of this kind were a problem in our calculations.

V. SUMMARY AND CONCLUSIONS

We have reported the results of a numerical study of conductance fluctuations in weakly disordered metals. Our results for var(g) for complete changes of the impurity distribution for the Anderson model for large g ($g \gtrsim 1$) in both one and two dimensions agree well with theoretical predictions based on perturbation calculations. The universal behavior is surprisingly robust, in that systems as small as a few lattice spacings on a side are, to a good approximation, in the universal regime. For small g ($g \lesssim 1$), i.e., strong disorder, we find that var(g) is depressed below the universal value, as expected theoretically. Probability distributions are well behaved for weak disorder, but are ill-behaved, with long tails for large g, in the strongly disordered limit. We have also obtained results for a related tight binding model, with which we have studied the magnitude of the conductance change when a single impurity is moved by various amounts. Our results for this case are in reasonable agreement with the theory, if one uses standard nearly-free-electron theory to estimate l_e. However, we do find that for small displacements of the impurities, e.g., one lattice spacing, $\langle(\delta g)^2\rangle$ is somewhat larger than predicted by the theory. This result could be of importance with regards to $1/f$ noise caused by impurity motion[13] for the following reason. The theory (11) predicts that $\langle(\delta g)^2\rangle$ is small for small impurity displacement, δr. As a result, the magnitude of $1/f$ noise due to small impurity displacements is significantly smaller than that due to the much less probable large displacements. Our results suggest instead that the fluctuations in the two cases are nearly the same size. If so, then a given magnitude of $1/f$ noise could be caused by the motions of *far fewer* impurities than if (11) is accurate. This would appear to make this proposed explanation of $1/f$ noise in metals even more plausible.

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[26]Given the dispersion relation $E_i=2V\cos(k_x a)+2V\cos(k_i a)$, one finds near $E=4V$ that $k_F=(4-E_F)^{1/2}$. This should be accurate for E near 4, a reasonable approximation for $E=2$, and at least qualitatively correct for $E=0$, the cases considered in Fig. 5
[27]From the standard relation for the conductivity of an electron gas, $\sigma=ne^2\tau/m$, one can show that $l_e=gL/Mk_F$, where L and M are the length and width of the system, respectively.