RAPID COMMUNICATIONS

PHYSICAL REVIEW B VOLUME 36, NUMBER 7 1 SEPTEMBER 1987

Dielectric breakdown in the presence of random conductors

S. S. Manna and B. K. Chakrabarti
Saha Institute of Nuclear Physics, 92 Acharya Prafulla Chandra Road, Calcutta 700 009, India
(Received 10 June 1987)

The potential distribution in a random conductor-insulator mixture is obtained here by numerically solving Laplace's equation on a lattice with arbitrary potential boundaries. A new algorithm is introduced for estimating the minimum insulation gap. The average breakdown voltage and the minimum gap for such lattices are estimated for different concentrations of conductors below percolation threshold. Detailed computer-simulation studies in two dimensions indicate that both the average breakdown voltage and the minimum gap variations near the percolation threshold are characterized by the same exponent, equal to the percolation correlation-length exponent.

Properties of a random mixture of conductors and insulators have been studied extensively in the past. The response of such systems can be both linear as well as nonlinear. Linear response, like the electrical conductivity, decreases with the addition of insulating impurities (dilution). Near the percolation threshold, this decrease with dilution becomes nonanalytic, characterized by critical exponents. Such behavior for the conductivity[1] is now fairly well established.

Recently, the problem of the nonlinear response of such networks or their (irreversible) breakdown properties has received considerable attention. This has been studied theoretically in two kinds of electrical networks. In the random-fuse-network model,[2] each conducting bond is a fuse, occupied randomly with a concentration p greater than the percolation threshold p_c, so that there is a spanning cluster of fuses in the lattice. A fuse becomes an insulator if the potential difference across it exceeds a fixed value. For $p > p_c$, such a lattice is conducting for small applied voltage across the lattice. However, if the external voltage is gradually increased, breakdown starts when the potential difference across any fuse exceeds the threshold value and the fuse burns into the insulator. The breakdown voltage (at which the lattice becomes just nonconducting) of the entire network then diverges near p_c, essentially because of the tortuosity of the percolating paths for p near p_c.

In the dielectric-breakdown model[3] of a conductor-insulator random mixture, the random occupation concentration of conductors is just below p_c, so that the total system is nonconducting. Each insulating bond can withstand a fixed potential difference across it and becomes a conductor if that potential difference is exceeded. Therefore when the voltage applied to the lattice is raised to some critical voltage (called the breakdown voltage, which depends on the configuration), some dielectric bonds break down into conductors and the lattice becomes conducting. Such systems have interesting properties near $p \lesssim p_c$. For $p=0$, the breakdown voltage (V_B) of the pure insulating system scales as the linear size L of the system: $V_B/L = 1$. For $p \geq p_c$, $V_B/L = 0$ and as $p \to p_c$ from below, V_B/L vanishes as $(p_c - p)^{t'}$, where t' is the breakdown exponent.

In the first attempt to solve this problem, the breakdown voltage was approximated by the minimum insulation gap in the random lattice.[4,5] The minimum gap of a nonpercolating lattice configuration is the minimum number of conducting bonds which are to be added to get a connection between the opposite sides of the lattice across which the external voltage is applied. This minimum gap (g) averaged over many configurations is supposed to vary as $(p_c - p)^{t_g}$ and this gap exponent t_g was identified with the breakdown exponent t'. Duxbury, Shukla, Stinchcombe, and Yeomans[4] indicated $t_g = \nu$, where ν is the percolation correlation-length exponent. In another context, Chayes, Chayes, and Durrett[6] studied the critical behavior of the two-dimensional first-passage time in a model where bonds have zero and unit passage times with the probabilities p and $1-p$, respectively, and showed exactly that the exponent associated with the first-passage time is equal to the exponent ν, indicating that the gap exponent t_g should be exactly equal to ν. However, the real breakdown voltage V_B is not identical with the minimum gap g; they are quite different quantities and V_B is approximately evaluated in terms of the minimum gap g in this model.[4,5]

The direct solution of this problem requires a knowledge of the potential distribution on such lattices. As the shapes of potential boundaries are completely arbitrary, the only way to get the potential distribution is to solve Laplace's equation numerically in this geometry. When the potential distribution is known, the macroscopic breakdown-initiation voltage is obtained by knowing the maximum of the potential differences across the dielectric bonds. In fact such a straightforward solution of Laplace's equation for a dielectric with a random concentration p of conducting sites or bonds is still lacking in the literature (see, however, Bowman and Stroud[3]). Here we give the results of a detailed solution of Laplace's equation for a square lattice containing a random fraction (p) of conducting sites. For each concentration p, a large number of configurations are considered and for each configuration the breakdown voltage V_B as well as the minimum gap g are evaluated. The configurationally averaged breakdown voltage $V_B(p)$ is then compared with the average minimum gap $g(p)$; the values of the two quantities are seen to be quite different except at $p=0$ and $p=p_c$, although the exponents characterizing the varia-

tions of both the quantities near p_c seem to be the same (difference being in the prefactors), indicating $t' = t_g = \nu$ for the real breakdown-voltage exponent.

In the simulation process, we took a square lattice of size L; all sites on it are randomly occupied (conducting sites) with probability p and unoccupied (dielectric sites) with probability $1-p$. For p below p_c we check for each configuration whether or not the lattice is nonpercolating via conducting sites (for percolating lattices $V_B = 0$). We apply a unit potential difference between the left and right sides of the lattice and assume periodic boundary conditions along the up-down direction. All the conducting sites connected to the left sides are at the left potential and those conducting sites connected to the right side will be at the right potential. In between there will be other isolated conducting clusters surrounded by dielectric sites. All sites on any of these clusters will be at the same potential. Different isolated clusters will be at different potentials. To get the potential distribution we proceed through the following steps.

(a) First, we ignore the conducting isolated clusters and treat the corresponding sites as dielectric sites. We assign potential values unity to all conducting sites connected to the left side, zero to all conducting sites connected to the right side, and any arbitrarily fixed initial value to all intermediate sites. Then, in the first iteration, the potentials at all intermediate sites are replaced by the solution of the discretized version of Laplace's equation:

$$V_{i,j} = (V_{i+1,j} + V_{i,j+1} + V_{i-1,j} + V_{i,j-1})/4 . \qquad (1)$$

This iteration process over dielectric sites is continued until the maximum of the difference between the potentials at any site in two successive iterations of the lattice converges to a value less than a small number δ ($=10^{-4}$).

(b) Now we consider intermediate clusters one by one (as if they are now being added to the lattice). We take the algebraic average of the potentials for all the sites of a cluster and assign this value to all sites of this cluster. Other clusters are treated in the same way.

(c) Potentials at all conducting sites are held fixed, whereas those at the dielectric sites are obtained iterating Eq. (1) with the accuracy δ for these new boundary conditions. After that, the potentials at all the conducting sites are found using the same iteration method, keeping the potentials at the dielectric sites fixed (at the values obtained in the operations just preceding).

The steps (b) and (c) are repeated until the maximum of the difference between the potentials at a site in two successive scannings of the whole lattice becomes less than δ. Obtaining the potential distribution in this way, we find out the potential differences across all the bonds of the lattice. We then search for the maximum of these values. The reciprocal of this maximum value gives us the initiation voltage V_I. We assume that the potential difference needed to break a dielectric bond is unity. Therefore, when the applied voltage is raised up to the initiation voltage, some dielectric bonds of the lattice have a potential difference unity [because of linear superposition solutions of Eq. (1) for that configuration]. At this voltage, these bonds will become conductors and the potential will be redistributed. It is seen that a larger number of

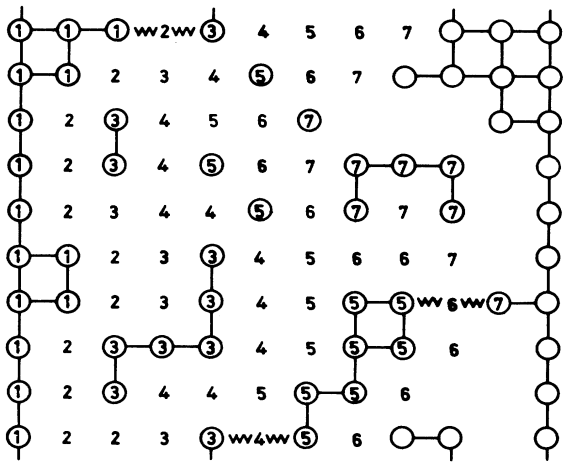

FIG. 1. This explains the algorithm for estimating minimum gaps. Liquid is injected from left connected cluster. The position of the liquid front at different time instants are shown by different numbers. The time it takes to reach the right side gives the minimum gap. The minimum gap is denoted by the wavy lines.

bonds now face a potential difference greater than unity. This leads to a cascade of bond breaking, resulting in the breakdown of the lattice. Therefore the breakdown-initiation voltage practically gives us the final breakdown voltage (checked over many configurations).

To estimate the minimum gap, we generalized the algorithm by Havlin,[7] which was used to find the minimum connecting path length between two points on percolating

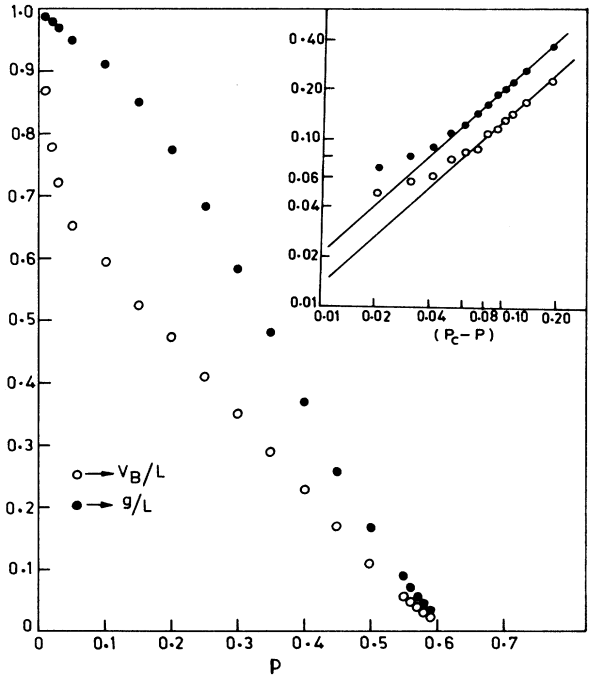

FIG. 2. V_B/L and g/L against p for $L=25$. The inset shows the log-log plot of $V_B(p)$ and $g(p)$ against $(p-p_c)$.

clusters. In our case, the minimum gap is the minimum path between the left connected clusters and the right connected clusters through dielectric bonds. Essentially, this algorithm deals with the profile of a moving liquid front through the dielectric sites when it is injected from the left connected conducting sites and reaches the right connected conducting sites, assuming that the liquid takes no time to fill up an intermediate isolated conducting cluster. For site-diluted square lattices we assume a bond between every two neighboring conducting sites and estimate the minimum gap from this bond picture. We thus proceed through the following steps (see Fig. 1).

(a) Assign a value 1 to all conducting sites connected to the left side.

(b) Assign a value 2 to all dielectric sites which have at least one 1 in the neighboring sites.

(c) Get all sites which have at least one 2 in the neighboring sites. If they are dielectric sites, assign 3 to each of them, and if any of them is a conducting site, assign 3 to all the sites of the cluster containing that site.

(d) This process of assigning new numbers is repeated until a conducting site connected to the right side is reached. The minimum gap of the lattice is equal to one less than the number which is to be assigned to this (last) site.

For each p below p_c, we simulated 1000 configurations of the site-diluted lattice of linear size $L=25$. Each such simulation for a particular configuration took about half a minute in a Norsk Data 500 computer. The average

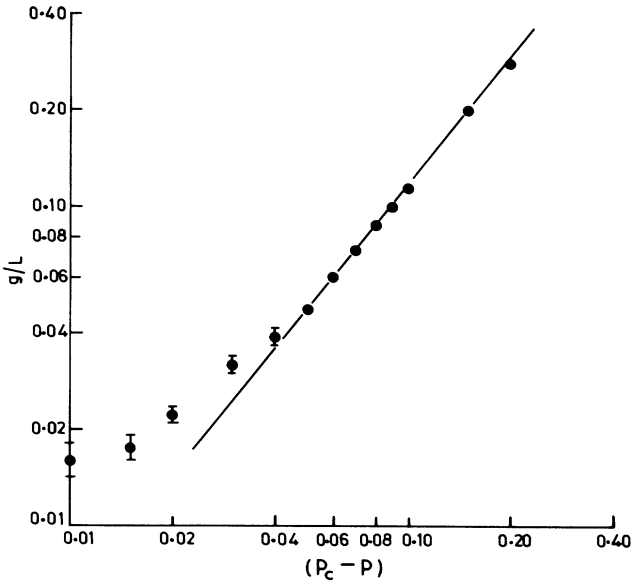

FIG. 3. Log-log plot of $g(p)$ against $(p-p_c)$ for $L=100$.

breakdown voltage $V_B(p)$ and the average gap $g(p)$ are plotted in Fig. 2 for various values of p. It may be noted from the figure, that in general $V_B(p)$ is less than $g(p)$ except at $p=0$ and $p \geq p_c$. This happens essentially because the gaps are measured along the lattice edges while the

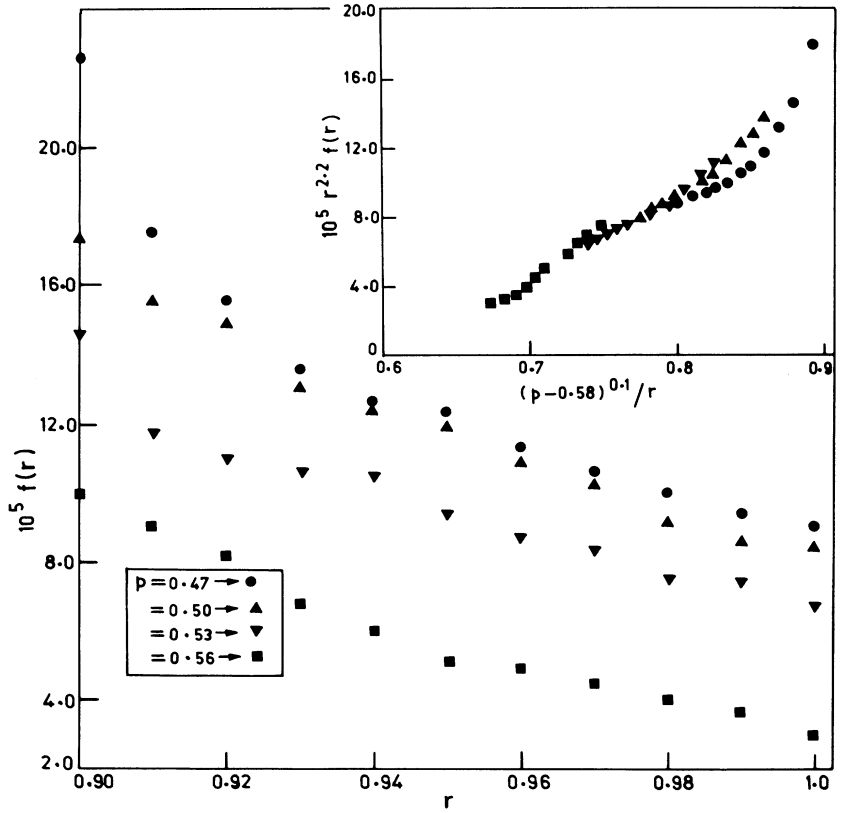

FIG. 4. Potential distribution function $f(r)$ against $r(=V/V_m)$. The inset shows a possible fitting with scaling relation $f(r) \sim r^{-2.2} Y(r/(p-p_c)^{0.1})$.

equipotential lines (coming from solutions of Laplace's equation for arbitrary conductor distribution) often overlap along the lattice diagonals. For example, the potential gradient along the lattice edge is more in the case when two nearest conducting sites are at diagonally opposite positions than when the conducting sites are on the same lattice edge but have one insulating site in between; in both cases the minimum gap (along the lattice edges) is two. We take the value of the percolation threshold p_c for this finite lattice size ($L=25$) to be 0.58.[8] The inset in Fig. 2 shows the log-log plot of $V_B(p)$ and $g(p)$ vs (p_c-p). It gives $t'\simeq t_g=1.00\pm0.05$, indicating that both the exponents for the breakdown voltage and the minimum gap are the same; the differences in V_B and g values near p_c being essentially because of the prefactors. It may be noted that a similar value for t_g was obtained[5] for an $L=20$ bond-diluted square and site-diluted triangular lattices (with of course a large error due to averaging over a very small number of configurations). In view of the exact value[6] of $t_g=\nu=\frac{4}{3}$ in two dimensions, we attribute this smaller value of $t_g\simeq1.0$ to be due to smallness of our lattice size. In fact, Roussenq *et al.*[8] also obtained $\nu\simeq1.1$ for such lattice sizes. These simulations thus clearly indicate $t'=t_g=\nu$. This is also indicated ($t'\simeq t_g\simeq1.1$) in our simulations for $L=50$. However, the configurational averagings could not be made very accurate here as each configuration took about 25 min to get V_B and g with the same value as δ. In fact, for a comparatively bigger size ($L=100$) bond-diluted square lattice, averaging over 100 configurations at each p, we find (see Fig. 3) from the log-log plot of $g(p)$ against (p_c-p) (with $p_c=\frac{1}{2}$), $t_g=1.31\pm0.05$, which is close to the exact value $\frac{4}{3}$.

For unit voltage across the lattice we found, solving the Laplace's equation for each configuration at each p, the distribution of potential differences across the bonds. From the results for $L=25$ site-diluted lattices, we fitted the density distribution $f(r)$ of bonds, having potential difference $V=rV_m$, near its maximum V_m, to a scaling form[5] $f(r)\sim r^{-x}Y(r/(p-p_c)^y)$ (see Fig. 4) and obtained $x\simeq2.2$, $y\simeq0.1$ for two dimension with $Y(z)\to0$ as $z\to\infty$.

Note added in proof

After the acceptance of this paper, agreement with our result $t'\simeq1.1$ in the breakdown experiment near $p=p_c$ in a 20×20 diode-resistor network has been brought to our notice (Professor L. Benguigui, private communication). However, the difference in slopes of the curves for $g(p)$ and $V_B(p)$ near $p=0$, as observed in our paper, needs some clarification.

In our site-impurity model, neither g nor V_B will be affected by the presence of isolated single-site conductors in a dielectric environment, but will be affected by fluctuating pairs of conducting sites (occurring in p^2 order). However, the linear variation in p for g and V_B observed in our model is because, in the layers adjacent to the left and right boundary, where voltage difference is applied, a single conductor anywhere reduces the gap by unity (see Fig. 1). Because of the sharp branch-tip geometry of the conductor there, the electric field becomes concentrated (similar to the stress concentration in elastic medium with sharp cracks) and greatly affects V_B, which is responsible for the large slope for $V_B(p)$ near $p=0$. The slope would be strictly infinity in the continuum limit because of infinite field concentration (from solution of Laplace's equation) at sharp notches of needlelike conductors.

We are grateful to J. M. Yeomans for useful communications and comments and to Variable energy Cyclotron Centre, Computer Centre, Calcutta, for their generous help in getting the long CPU times.

[1]R. B. Stinchcombe and B. P. Watson, J. Phys. C **9**, 3221 (1976); B. Derrida, J. G. Zabolitzsky, J. Vannimenus, and D. Stauffer, J. Stat. Phys. **36**, 31 (1984).

[2]L de Arcangelis, S. Redner, and H. J. Herrmann, J. Phys. (Paris) Lett. **46**, L585 (1985).

[3]D. R. Bowman and D. Stroud, Bull. Am. Phys. Soc. **30**, 563 (1985); P. M. Duxbury, P. D. Beale, and P. L. Leath, Phys. Rev. Lett. **57**, 1052 (1986).

[4]R. B. Stinchcombe, P. M. Duxbury, and P. Shukla, J. Phys. A **19**, 3903 (1986); P. M. Duxbury, P. Shukla, R. B. Stinchcombe, and J. M. Yeomans (unpublished).

[5]B. K. Chakrabarti, K. K. Bardhan, and P. Ray, J. Phys. C **20**, L57 (1987).

[6]J. T. Chayes, L. Chayes, and R. Durrett (unpublished).

[7]S. Havlin, in *Kinetics of Aggregation and Gelation,* edited by F. Family and D. P. Landau (North-Holland, Amsterdam, 1984), p. 145.

[8]J. Roussenq, J. Clerc, G. Giraud, E. Guyon, and H. Ottavi, J. Phys. (Paris) Lett. **37**, L99 (1976); R. M. Ziff and B. Sapoval, J. Phys. A **19**, L1169 (1986).