

Mapping between random central-force networks and random resistor networks

W. Tang and M. F. Thorpe

*Department of Physics and Astronomy and Center for Fundamental Materials Research,
Michigan State University, East Lansing, Michigan 48824*

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We show that the random-resistor-network problem can be mapped on to a related network of Hooke springs of natural length zero stretched on a frame. The conductance of the network is equivalent to the pressure on the frame. The new viewpoint leads to a useful visualization of conductivity on random networks. The mapping can also be used on tight-binding Hamiltonians. We use this method to study the conductivity and superconductivity of random networks in two dimensions.

I. INTRODUCTION

The properties of random resistor networks have been well studied and are now understood.¹ Excellent numerical simulations can be performed in which Kirchoff's laws are solved. These provide as accurate numerical solutions as are desired. If the resistors on a network are progressively removed, the conductance vanishes at the percolation concentration p_c . The description of this phase transition in terms of critical exponents, scaling behavior, etc. are all well understood.² This problem is equivalent to a tight-binding Hamiltonian.³ It is also equivalent to the problem of spin waves in a Heisenberg ferromagnet at low temperatures.³

In this paper we explore yet another mapping. This one is slightly more subtle and leads to some new geometrical insights. Of course, no new results are found as the original (equivalent) problem is well understood. Our purpose in this paper is primarily to develop and explore the mapping.

In Sec. II we develop the mapping and formula. We also show the relationship to the low-energy density of states of a tight-binding Hamiltonian. In Sec. III we apply the mapping to dilute resistor networks on the square lattice as an illustration. In Sec. IV we give a similar discussion for random networks in two dimensions. A dual random network is constructed and it is shown how the superconducting-normal network on the dual lattice maps on to the dilute resistor network on the original lattice and vice versa. Finally, we mention the relationship of this work to rigidity percolation in the conclusion.

II. GENERAL FORMULATION

A. The mapping

To illustrate the mapping, let us consider a resistor network in which a voltage difference V_0 is applied across the two opposite sides as shown in Fig. 1 (see unrelaxed network). If the current flows in the x direction across the voltage difference V_0 , the conductivity G_{xx} is defined by $E = G_{xx} V_0^2$, where E is the electrical energy stored in the resistor network. Let us label the sites by

i, j , etc. so that a current I_{ij} flows in the ij bond which has a conductance σ_{ij} . The range of the σ_{ij} is arbitrary; and can be nearest neighbor, next nearest neighbor, etc. Sites that are not connected have $\sigma_{ij} = 0$. Current conservation at each site leads to

$$\sum_j I_{ij} = 0 \tag{1}$$

which using $I_{ij} = \sigma_{ij}(V_i - V_j)$, where V_i is the voltage at site i , leads to

$$\sum_j \sigma_{ij}(V_i - V_j) = 0 \tag{2a}$$

or

$$V_i = \frac{\sum_j \sigma_{ij} V_j}{\sum_j \sigma_{ij}} \tag{3a}$$

The total energy stored in the network is

$$E = \sum_{\langle i,j \rangle} \sigma_{ij} (V_i - V_j)^2 = G_{xx} V_0^2, \tag{4a}$$

where the angular brackets denote that each bond is only counted once in the sum. The conductivity is given by

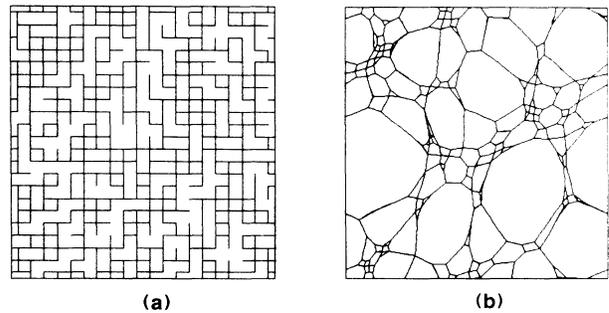


FIG. 1. A square net with 70% of the bonds present ($p = 0.70$) is shown (a) before and (b) after relaxation with the centroid condition. See also Fig. 2.

$$G_{xx} = \sum_{\langle i,j \rangle} \sigma_{ij} (V_i - V_j)^2 / V_0^2 . \quad (5a)$$

Now let us imagine a central-force (Hooke) spring network in which each spring has natural length *zero*. The whole system is held on a frame to prevent it collapsing to a point. For a large system, the shape of the frame is irrelevant, but it is convenient to imagine it to be a square. The equilibrium condition for each site i is that the total force acting on it vanishes. We have in the x direction

$$\sum_j K_{ij} (R_{ix} - R_{jx}) = 0 , \quad (2b)$$

where \mathbf{R}_i is the position of the i th atom and K_{ij} is the spring constant. From this we see that

$$R_{ix} = \frac{\sum_j K_{ij} R_{jx}}{\sum_j K_{ij}} . \quad (3b)$$

There are similar equations to (2b) and (3b) in the y direction. The total energy stored in the spring system is

$$\begin{aligned} E &= \frac{1}{2} \sum_{\langle i,j \rangle} K_{ij} [(R_{ix} - R_{jx})^2 + (R_{iy} - R_{jy})^2] \\ &= \frac{1}{2} (K_{xx} + K_{yy}) L^2 , \end{aligned} \quad (4b)$$

where L is the length of the sample and

$$K_{xx} = \sum_{\langle i,j \rangle} K_{ij} (R_{ix} - R_{jx})^2 / L^2 \quad (5b)$$

and a similar expression for K_{yy} .

We see that the (a) and (b) equations can be made to coincide through the mapping,

$$\begin{aligned} \sigma_{ij} &\rightarrow K_{ij} , \\ V_i &\rightarrow R_{ix} , \\ G_{xx} &\rightarrow K_{xx} . \end{aligned} \quad (6)$$

There is an important subtlety in this mapping associated with the boundary conditions. In the resistor problem, the net current flow is in a particular direction; x in the case considered here. This leads to the conductivity G_{xx} . In the spring problem, the frame acts equally in the x and y directions so that K_{xx} and K_{yy} cannot be obtained separately. This is not a concern for high symmetry networks, where the tensors $G_{\alpha\beta}$ and $K_{\alpha\beta}$ are proportional to the unit tensor. We shall refer to such networks as *electrically isotropic*. We shall only discuss such networks in this paper. These include square nets, random square nets, triangular nets, random triangular nets and the random networks considered in Sec. IV. We are thus led to

$$\begin{aligned} \text{Tr} \vec{G} &= G_{xx} + G_{yy} \rightarrow K_{xx} + K_{yy} \\ &= \sum_{\langle i,j \rangle} K_{ij} (\mathbf{R}_i - \mathbf{R}_j)^2 / L^2 . \end{aligned} \quad (7)$$

The conductance can be calculated from the spring net-

work via (7) and the equilibrium condition (2b) that may be written in vector form

$$\sum_j K_{ij} (\mathbf{R}_i - \mathbf{R}_j) = 0 . \quad (8)$$

In two dimensions, the conductance and conductivity of a square sample are the *same*. For a sample of hypercubic shape in d dimensions (volume L^d), the current flows between parallel hyperplanes (area L^{d-1}) so that the conductance σ is given by

$$\sigma = G / L^{d-2} , \quad (9)$$

where G comes from (7). Note that (9) holds for electrically isotropic networks. Of course (7) must be modified so that the trace is over all d dimensions and L^2 in (4b), (5b), and (7) must be replaced by L^d so that

$$\text{Tr} \vec{G} \rightarrow \sum_{\langle i,j \rangle} K_{ij} (\mathbf{R}_i - \mathbf{R}_j)^2 / L^d \quad (7')$$

and Eq. (4b) can be extended to read

$$\begin{aligned} E &= \frac{1}{2} \text{Tr} \vec{K} L^2 \\ &\rightarrow \frac{1}{2} \text{Tr} \vec{G} L^2 = \frac{d}{2} G L^2 = \frac{d}{2} \sigma L^d . \end{aligned} \quad (4c)$$

The tension T (i.e., inward pressure) can be obtained from

$$T = \frac{\partial E}{\partial L^d} \rightarrow \frac{G}{L^{d-2}} = \sigma . \quad (10)$$

Thus the *conductivity* is obtained from (7') while the *conductance* is equivalent to the outward tension needed to prevent the system collapsing to a point. For $2d$ systems, this pressure is independent of the size of the system L . For $1d$ systems, the tension increases with size linearly, whereas for $d > 2$, the tension decreases as the size gets larger. The total *force* required to prevent the system collapsing, *always* increases with size. It may be convenient in some cases to have the spring network stretched on a circular frame or tennis racket, so that the boundary condition is isotropic.

B. Tight-binding Hamiltonian

A useful alternative viewpoint is given by the tight-binding Hamiltonian⁴

$$H = \frac{1}{2} \sum_{i,j} K_{ij} (|i\rangle\langle i| - |i\rangle\langle j|) , \quad (11)$$

where the $|i\rangle$ is a localized s state on the i th site. The overlap integral K_{ij} is the same as the spring constant in Eq. (2b). It is convenient to take a (large) unit *supercell* containing n atoms. This cell is repeated periodically. Each atom is designated by a labeling (l, n) , where l designates the cell and n is the atom within the cell. This label pair is conveniently denoted by $i \leftrightarrow (l, n)$. We can use Bloch's theorem on the supercell translations \mathbf{R}_l to transform to a new basis

$$|\mathbf{q}, n\rangle = \frac{1}{(N_c)^{1/2}} \sum_l e^{i\mathbf{q}\cdot\mathbf{R}_{l+n}} |l+n\rangle . \quad (12)$$

The total number of atoms $N = nN_c$ where N_c is the

number of supercells. The Hamiltonian (10) is block diagonal in this basis and within a \mathbf{q} block the matrix elements are given by

$$\langle \mathbf{q}, n | H | \mathbf{q}, n' \rangle = \frac{1}{N_c} \sum_{i, i'} K_{ii'} (1 - e^{i\mathbf{q} \cdot (\mathbf{R}_i - \mathbf{R}_{i'})}) , \quad (13)$$

where atoms i and i' are connected by the hopping matrix element $K_{ii'}$. These two atoms may either be in the same unit cell or in adjacent unit cells.

The Hamiltonian (11) deliberately has related diagonal and off diagonal terms so that the band edge is at $E = 0$. The eigenvector corresponding to zero energy is denoted to $|0\rangle$

$$|0\rangle = \frac{1}{(n)^{1/2}} \sum_n |\mathbf{q} = \mathbf{0}, n\rangle . \quad (14)$$

Labeling the other $n - 1$ eigenvalues at $\mathbf{q} = \mathbf{0}$ by E_e , we use the matrix analogue of $\mathbf{k} \cdot \mathbf{p}$ perturbation theory.⁵ From expanding Eq. (13),

$$\begin{aligned} \langle \mathbf{q}, n | H | \mathbf{q}, n' \rangle = & \frac{1}{N_c} \sum_{i, i'} K_{ii'} \{ -i\mathbf{q} \cdot (\mathbf{R}_i - \mathbf{R}_{i'}) \\ & + \frac{1}{2} [\mathbf{q} \cdot (\mathbf{R}_i - \mathbf{R}_{i'})]^2 \} \end{aligned} \quad (15)$$

then to second order,

$$\begin{aligned} E_q = & \frac{1}{2N} \sum_{i, i'} K_{ii'} [\mathbf{q} \cdot (\mathbf{R}_i - \mathbf{R}_{i'})]^2 \\ & - \sum_e \left| \sum_{i, i'} K_{ii'} \frac{\mathbf{q} \cdot (\mathbf{R}_i - \mathbf{R}_{i'})}{N_c \sqrt{n}} \langle \mathbf{q} = \mathbf{0}, n | E_e \rangle \right|^2 / E_e . \end{aligned} \quad (16)$$

There is no term linear in \mathbf{q} in (16) as this vanishes when the sum over all sites is done. If we are interested only in the density of states $\rho(E)$, then *distances* are irrelevant. We imagine forming an *auxiliary lattice* in which the second term in (16) vanishes. This can be achieved if

$$\sum_j K_{ij} (\mathbf{R}_i - \mathbf{R}_j) = 0 \quad (17)$$

this is the same condition as (8) and defines the same set of new atomic positions \mathbf{R}_i . Thus the sites in the auxiliary lattice are coincident with the positions of the sites in the central-force network discussed in Sec. II A. In the auxiliary lattice

$$E_q = \frac{1}{2N} \sum_{i, j} K_{ij} [\mathbf{q} \cdot (\mathbf{R}_i - \mathbf{R}_j)]^2 \quad (18)$$

when i, j go over *all* sites. The unit supercell is not explicitly appearing in the summations in Eqs. (17) and (18). Note that (17) can be obtained by minimizing the energy in Eq. (18) with respect to the \mathbf{R}_i if the \mathbf{R}_i are initially regarded as arbitrary.

From (17) we may define principal axes where

$$E_q = D_{xx} q_x^2 + D_{yy} q_y^2 . \quad (19)$$

The area of the ellipse containing states up to an energy E is $\pi E (D_{xx} D_{yy})^{1/2}$ so that the low-energy density of states is

$$\rho(E) = \left(\frac{L}{2\pi} \right)^2 \frac{\pi}{(D_{xx} D_{yy})^{1/2}} , \quad (20)$$

where L^2 is the area of the sample. This can be written as

$$\rho(E) = \left(\frac{L}{2\pi} \right)^2 \frac{\pi}{(\text{Det} \vec{D})^{1/2}} , \quad (21)$$

where \vec{D} is the diffusion matrix which from (18) and (19) has matrix elements

$$D_{\alpha\beta} = \frac{1}{2N} \sum_{i, j} K_{ij} (R_{i\alpha} - R_{j\alpha})(R_{i\beta} - R_{j\beta}) . \quad (22)$$

Note that

$$\text{Tr} \vec{D} = \frac{L^2}{N} \text{Tr} \vec{G} \quad (23)$$

where $\text{Tr} \vec{G}$ is defined by Eq. (7). For electrically isotropic $2d$ systems we also have

$$(\text{Det} \vec{D})^{1/2} = \frac{1}{2} \text{Tr} \vec{D} = \frac{L^2}{2N} \text{Tr} \vec{G} = \frac{L^2 \sigma}{N} , \quad (24)$$

where σ is the conductance. Hence

$$\rho(E) = \frac{N}{4\pi\sigma} . \quad (25)$$

This density of states is normalized over all energies to the number of sites N , and the low-energy part depends only on the conductance σ . Note that this result is very general and applies to any network that behaves as a continuum in the long-wavelength limit. This *excludes* fractals. The geometry of the network enters only through σ .

These equations are easily generalized to higher dimensions. For Eqs. (11)–(18) there are no changes. We have to modify Eq. (19) to

$$E_q = \sum_{\alpha} D_{\alpha\alpha} q_{\alpha}^2 , \quad (19')$$

where α goes over the d principal directions. The density of states given in Eqs. (20) and (21) becomes

$$\rho(E) = \frac{L^2}{4\pi\Gamma(d/2)(\text{Det} \vec{D})^{1/2}} \left(\frac{L^2 E}{4\pi} \right)^{d/2-1} , \quad (21')$$

where $\Gamma(d/2)$ is the gamma function.⁶ Equations (22)–(23) are unchanged but (24) and (25) become

$$(\text{Det} \vec{D})^{1/2} = \frac{1}{d} \text{Tr} \vec{D} = \frac{L^2}{dN} \text{Tr} \vec{G} = \frac{L^d \sigma}{N} \quad (24')$$

and, hence,

$$\rho(E) = \frac{N}{4\pi\sigma} \frac{1}{\Gamma(d/2)} \left(\frac{E}{4\pi} \right)^{d/2-1} . \quad (25')$$

Thus as in two dimensions, the low-energy density of states is determined by the conductance σ and the total number of atoms N in the network and $\rho(E)$ is an extensive quantity. An atom is counted as being in the network if it is coupled to the backbone or conducting path.

This includes regions that are connected to the backbone but carry no current. They nevertheless contribute to the inertia and so must be counted. Isolated regions that are not coupled to the backbone are ignored.

This method of using the tight-binding Hamiltonian (11) to derive the centroid condition (17) is less satisfactory than mapping onto central springs of natural length zero. This is because we had to introduce a “supercell” in order to have a \mathbf{q} vector to work with. The final answer did not explicitly contain reference to this supercell and so we regard the result (25') as being quite general and true for any conducting network that is homogeneous on distances greater than some correlation length ξ . The derivation that lead to (25') is only valid for $q \ll \xi^{-1}$ and hence for vanishingly small energies as $\xi \rightarrow \infty$. We believe the result is actually much more general as the relevant length ξ is probably *not* the supercell size but rather the distance that characterizes the structural correlations in the random network. A similar quantity ξ is commonly used in percolation theory.¹ There is no easy way around this problem in a dynamic approach as given here.

In the static approach in Sec. II A, everything is rigorous. In order to get the dynamic result (25') one has to assume an Einstein relation between the conductance and the diffusivity.¹ It is here that the problem is glossed over. This area needs further study.

III. ILLUSTRATION OF THE MAPPING

In order to illustrate the mapping we consider the square net shown in Fig. 1. All the nearest-neighbor bonds have resistance σ_0^{-1} and a fraction $1-p$ have been randomly removed. The solution to this problem is well known and so it will serve as an illustration.^{1,2} The condition (8) becomes just the condition that every site is at the *centroid* of its nearest neighbors that are present.

In these kinds of percolation problems one really wants to study the system in the thermodynamic limit ($N \rightarrow \infty$). Periodic boundary conditions are better than any other kind as every atom is properly coordinated. We therefore periodically repeat the “supercell” in Fig. 1. Bonds are removed randomly in the reference supercell and also in all others. Rather than put the network on a frame when we go to the central spring model, we hold the supercell repeat vectors *constant*. This is equivalent to an external tension and more convenient in practice. The spring constants become $K=1/R$. The network is relaxed iteratively, site by site, until the energy stored in the springs is minimized. We see from (7) that the conductivity G is proportional to the mean-square nearest-neighbor distance

$$G \sim \sum_{\langle i,j \rangle} \delta_{ij}^2 / L^2, \quad (26)$$

where $\delta_{ij} = \mathbf{R}_i - \mathbf{R}_j$ is a nearest-neighbor distance in the *relaxed* network. Bonds that are absent are counted as zero in performing the average in (26). In Fig. 2 we show some typical results for various values of p . Isolated islands do not contribute to the conductivity and so relax to points. Side groups on the backbone that do not

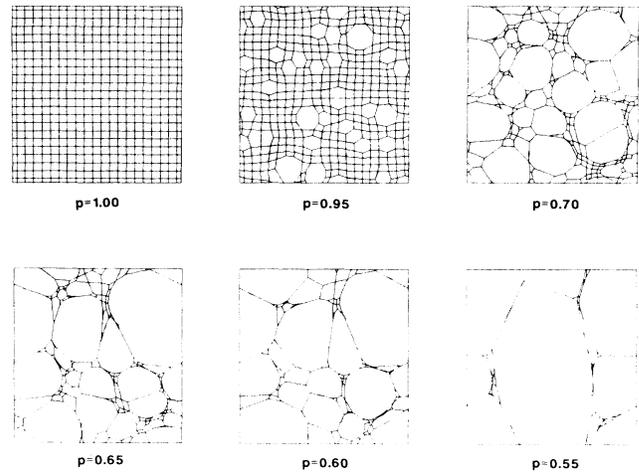


FIG. 2. Square nets with a fraction p of bonds present have been relaxed with the centroid condition. The supercell shown is periodically repeated. Note that $p_c=0.5$ for bond percolation on the square net.

carry current also relax to points. Thus the conducting backbone is clearly and simply exposed by this algorithm. Those bonds that carry most current are stretched the most and so make the largest contribution to G . Note that many of the long straight connections in Fig. 2 are actually made up of many bonds and collapsed side groups. The networks for $p=0.70$ in Fig. 2 and Fig. 1(a) are examples of different random configurations.

In Fig. 3 we show results for the conductance for site percolation on the square net.^{1,7,8} As our method is

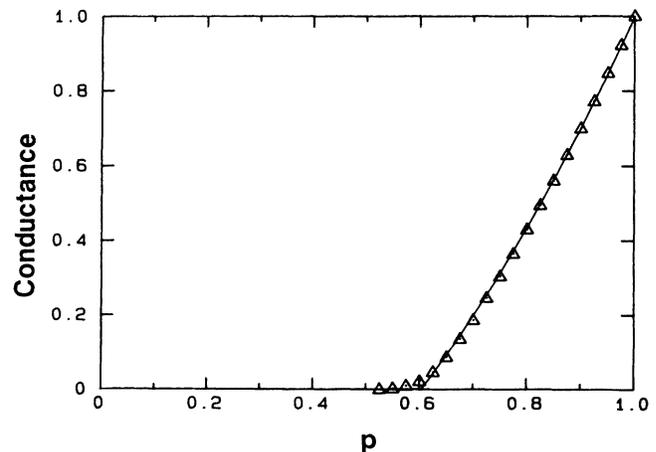


FIG. 3. The conductance of the square net with a fraction p of the sites present. The solid line is the effective-medium result of Watson and Leath (Ref. 7) which agrees closely with their direct numerical solution of Kirchoff's laws except very close to p_c . Percolation occurs at $p_c=0.59$ and the conductance of the undepleted system is set equal to 1 for convenience. The tail just below p_c is due to the finite supercell size. Results are obtained by averaging over 15 samples of 1600 sites each.

quite general, it works equally well for bond or site depleted networks. It can be seen that our results lie close to the effective-medium theory which agrees well with a direct solution of Kirchoff's laws except very close to p_c . The tail that extends slightly below $p_c=0.59$ in our simulations is due to finite-size effects.

Before ending this section, we consider a general two-dimensional network with nearest-neighbor conductors of magnitude σ_0 . The relaxed network and tight-binding model [using Eqs. (17) and (18)] with the nearest-neighbor $K_{ij} \rightarrow \sigma_0$, leads to

$$E_q = \frac{q^2}{4N} \sigma_0 \sum_{i,j} \delta_{ij}^2. \quad (27)$$

From this the density of states is

$$\rho(E) = NL^2 / \left[\pi \sigma_0 \sum_{i,j} \delta_{ij}^2 \right] \quad (28)$$

and hence from (25), the conductance σ is given by

$$\sigma = \sigma_0 \frac{\sum_{i,j} \delta_{ij}^2}{4L^2}. \quad (29)$$

This is a convenient form. For the square net with N atoms and nearest-neighbor distance a , we have $L^2 = Na^2$ so that

$$\sigma = \sigma_0. \quad (30)$$

Similarly, for the triangular net

$$\sigma = \sqrt{3} \sigma_0, \quad (31)$$

and for the honeycomb lattice

$$\sigma = \sigma_0 / \sqrt{3}. \quad (32)$$

IV. RANDOM NETWORKS

We consider a random network where every site is threefold coordinated as shown in Fig. 4(a). This 800-site network was constructed by successively disordering a honeycomb lattice while maintaining the coordination. The lattice was then relaxed to make lengths as equal as possible. There is a small ($\sim 5\%$) variation in the bond lengths. We make all the bonds conductors of equal magnitudes of σ_0 . We could have made the resistance proportional to the bond length but this seems unnecessarily complicated.

In order to calculate the conductivity, every atom is moved to the centroid of its three neighbors. The relaxed network is shown in Fig. 5(a). The changes are surprisingly large. The large polygons have grown and the small polygons have shrunk. There is also much more variability in the bond lengths than in Fig. 4(a).

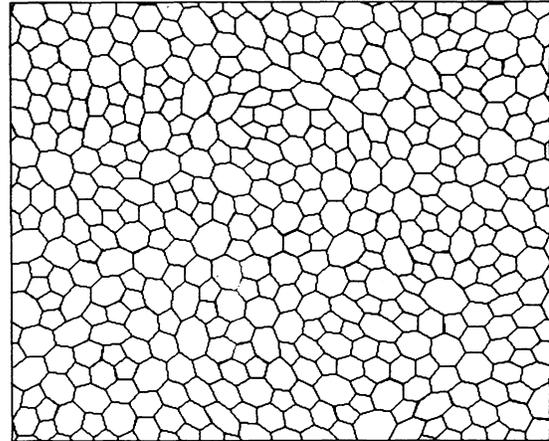
Because the supercell of the sample shown in Figs. 4(a) and 5(a) is not a square some additional care must be taken. The supercell shown has repeat distances L_x and L_y , where $L_x/L_y = 2/\sqrt{3} = 1.16$. These are held fixed during the relaxation. Minor changes in the previous formalism can easily be made for such a network. It can be shown that Eq. (29) holds for any electrically iso-

tropic network, independent of the shape, if L^2 is replaced by the area A :

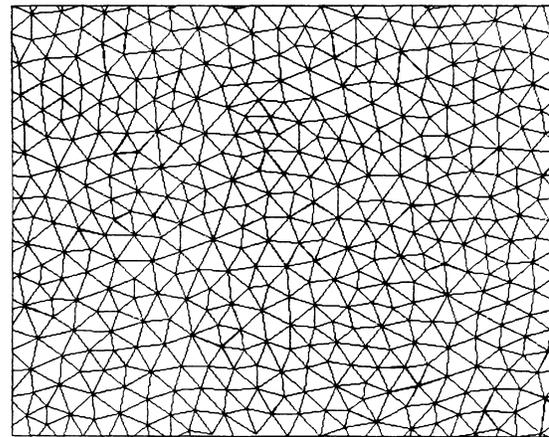
$$\sigma = \sigma_0 \frac{\sum_{i,j} \delta_{ij}^2}{4A}. \quad (29')$$

We use the relaxed network of Fig. 5(a), to compute $\sigma = 0.555\sigma_0$. This should be compared to $\sigma = 0.577\sigma_0$ for the honeycomb lattice Eq. (32).

The dual lattice of Fig. 4(b) is relaxed using the centroid condition to obtain the relaxed lattice of Fig. 5(b). Using Eq. (29') we find that $\sigma = 1.81\sigma_0$. This should be compared to $\sigma = 1.73\sigma_0$ for the triangular net in Eq. (31). An elementary application of Euler's theorem⁹ shows that the mean size of a polygon in the network of Fig. 4(a) is 6. Therefore in the dual lattice, the *mean coordination* is 6, although it is not constant but varies from site to site; the minimum coordination is 5 and the maximum is 9.



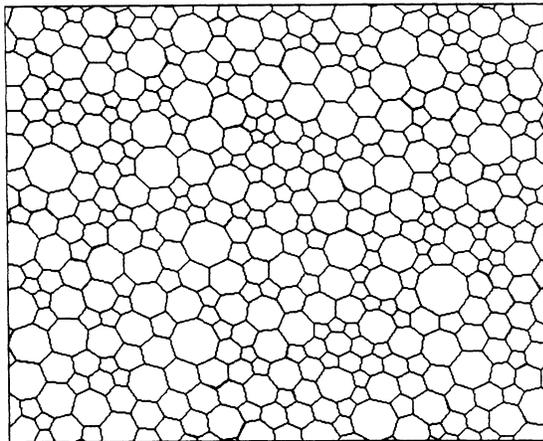
(a)



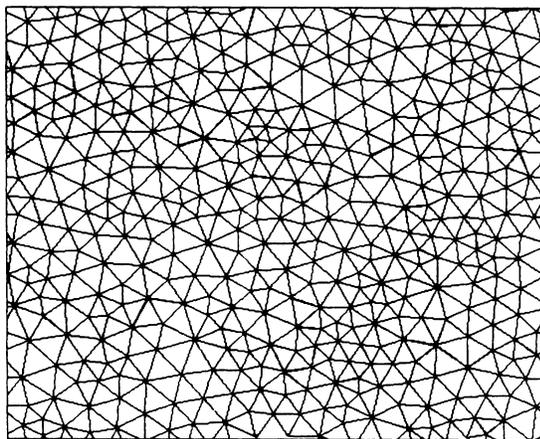
(b)

FIG. 4. (a) Showing a threefold coordinated random network (Ref. 11). A single supercell is shown that repeated periodically. (b) The dual lattice of (a) in which a site is placed at the centroid of each polygon and then these sites are connected.

Finally we examine the effect of randomly diluting these networks. Quantities referring to the lattice of Fig. 5(a) are denoted with subscripts 1 and quantities referring to the dual network in Fig. 5(b) are denoted by subscripts 2. A fraction p of the bonds are present in each case. The depleted networks are relaxed again and the conductance computed by the methods that we have discussed in this paper. The results for σ_1 , against p_1 , and σ_2 against p_2 are shown in Figs. 6 and 7. These results have been obtained by averaging over 15 random depletions and σ_1 and σ_2 are set equal to 1 for the undepleted systems for convenience. As expected for such dual lattices, the critical points are related by $p_{1c} = 1 - p_{2c}$. Note that for the honeycomb lattice (also coordination 3), the bond percolation concentration⁸ $p_c = 0.65$. From our simulations this $p_c \approx p_{1c}$ as might be expected.¹ In Figs. 6 and 7, the solid lines are the effective-medium theory results¹



(a)



(b)

FIG. 5. (a) Showing the same threefold coordinated random network as in Fig. 4(a) except every atom has been moved to the centroid of its three neighbors. The topology is the same as Fig. 4(a). (b) Also shown is the dual lattice [Fig. 4(b)] which has been relaxed so that every atom is at the centroid of its neighbors. The topology is the same as Fig. 4(b).

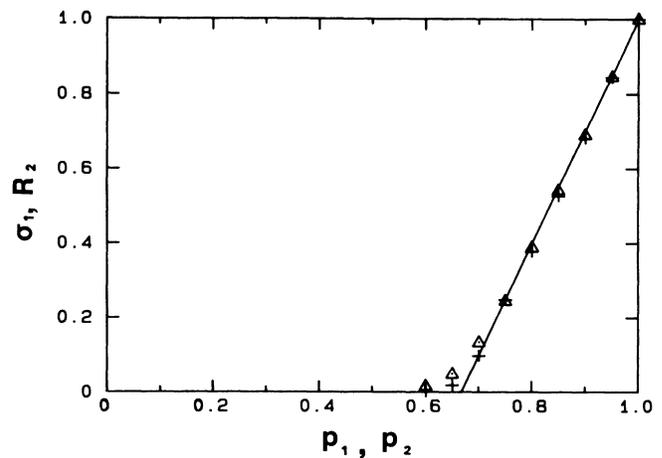


FIG. 6. Showing the conductance σ_1 of the threefold coordinated lattice of Fig. 4(a) as a function of the fraction of the bonds present p_1 . Also shown is the resistance R_2 of the dual lattice of Fig. 4(b) as a function of p_2 , the fraction of normal bonds present where $1 - p_2$ is the fraction of superconducting bonds. Results are obtained by averaging over 15 samples. The crosses are for random resistor networks and the triangles are for the superconducting-normal networks. The solid line is the effective-medium theory result [Eq. (33)]. The conductance and resistance of the undepleted systems are set equal to 1 for convenience.

$$\sigma_1 = 3p_1 - 2 \quad (33)$$

$$\sigma_2 = (3p_2 - 1)/2, \quad (34)$$

which are there as useful guides to the eye.

We have also computed the resistance R of these networks where a fraction $1 - p$ of bonds are superconducting and a fraction p are normal. The resistance of the normal systems is normalized to 1. The results for these networks, averaged over 15 samples, are also shown in

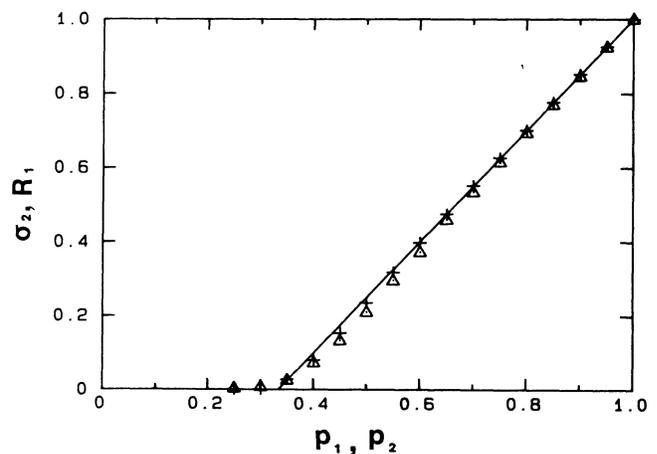


FIG. 7. Same as Fig. 6 except σ_2 and R_1 are substituted for σ_1 and R_2 . The solid line is the effective-medium result [Eq. (34)].

Figs. 6 and 7 and are seen to have the behavior expected of dual networks.² The correspondence between the data is maximized by using the same random-number generator to tag bonds in both the original lattice and their dual equivalents. It can be seen that the dilute resistor problem on a network maps onto the superconducting-normal network on the dual network as expected.

In order to facilitate numerical simulations we have used a ratio of conductivities of 1000:1 between superconducting and normal bonds. The error introduced by this is mainly near p_c when there is a small incremental enhancement in the finite-size tail. It would not be hard to take a larger conductivity ratio. Note that in the relaxed network, the superconducting bonds are *fused together* to a point so that, consequently, the normal bonds are elongated as the sample must remain fully connected and is not allowed to collapse.

V. CONCLUSIONS

We have introduced a new mapping for the random resistor problem. Its utility is that it reduces the electrical problem to a geometry problem and hence provides a useful visualization of the conduction process. It can be seen that effective-medium theories do a better job than might have been expected (see Figs. 3, 6, and 7). They are designed to get the initial slope correct near $p = 1$,

but are uncontrolled elsewhere. In the unstretched central-force network problem, effective-medium theories are even better than here.¹⁰

We have applied these results to a few well-studied problems. We have also studied depleted random networks. The results for the depleted three-coordinated random network are very close to those for the depleted honeycomb lattice. This is to be expected since the coordination is the single most important parameter characterizing a percolating network.

The depletion of central-force networks has recently been studied by many authors. These networks are such that every spring has its natural length in the absence of external stresses. This is often referred to as *rigidity percolation*. The elasticity vanishes at $p^* > p_c$ because connected paths are inelastically ineffective. This is *not* the case in the work here as all springs in the conducting backbone are stretched by the frame. Hence the conductivity and elasticity are intimately tied together as we have shown.

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