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Spin-spin correlations in finite systems with O(n) symmetry: Scaling hypothesis and corrections to bulk behavior

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A scaling hypothesis is set up for the correlation function $G(\mathbf{R},T;L)$ of a finite-sized system, with O(n) symmetry ($n \geq 2$), confined to geometry $L^{d-d'} \times \infty^{d'}$ ($2<d<4$, $d' \leq 2$) and subjected to periodic boundary conditions. Finite-size effects in the correlation length $\xi(T;L)$ are predicted in the region of first-order phase transition ($T<T_c$) as well as the region of second-order phase transition ($T \simeq T_c$); for the correlation function, the predictions relate mostly to the regime $T<T_c$. To test these predictions, a detailed analytical study is carried out in the case of the spherical model and all predictions are seen to be fully borne out. In the process we find that, though a *single* correlation length ξ governs the scaling of the variable $\mathbf{R}$ in *all* directions, the functional dependence of the correlation function on the scaled parameter $\mathbf{R}/\xi$ is highly anisotropic, in that it depends very differently on the components, $\mathbf{R}_\perp$ and $\mathbf{R}_\parallel$, of $\mathbf{R}$ pertaining to the finite and the infinite dimensions, respectively. Most importantly, while long-range order at temperatures below T_c prevails all the way in the direction of $\mathbf{R}_\perp$ it is severely limited in the direction of $\mathbf{R}_\parallel$. In fact, for $R_\parallel \gg L$, the qualitative features of the correlation function become characteristic of a d'-dimensional bulk system, though ξ still pertains to the actual, finite-sized system; the net result is that long-range order in the direction of $\mathbf{R}_\parallel$ persists only to distances small in comparison with ξ which, for $d'<2$, is found to be $O(L(L/a)^{(d-2)/(2-d')})$, a being the lattice constant.

I. INTRODUCTION

In a recent paper,[1] hereafter referred to as I, we carried out a detailed analysis of the thermodynamic properties of the spherical model of ferromagnetism confined to geometry $L^{d-d'} \times \infty^{d'} (2<d<4, d' \leq 2)$ and subjected to periodic boundary conditions. We showed that, in the region of the second-order phase transition ($T \simeq T_c$), our analytical results for the "singular" part of the free energy density $f^{(s)}$, the specific heat per unit volume $c^{(s)}$ and the magnetic susceptibility χ of the system were in full conformity with the Privman-Fisher hypothesis[2] on the hyperuniversality of finite systems. In a subsequent investigation,[3] we extended our analysis to the region of the first-order phase transition ($T<T_c$) and found a similar conformity with a *generalized* version of the aforementioned hypothesis in which the scaling parameters of the system were not only model dependent but temperature dependent as well and covered the temperature range from $T \simeq T_c$ down to $T=0$. A little reflection showed[4,5] that, although the motivation for this generalization had come originally from our study of the spherical model ($n=\infty$), the predictions of the generalized hypothesis were, in fact, applicable to *all* O(n) models with $n \geq 2$. At that stage we decided to turn our attention to the problem of spin-spin correlations in these systems, with a view to studying finite-size effects in the correlation function $G(\mathbf{R},T;L)$ as well as the correlation length $\xi(T;L)$. This study has been carried out in detail, in the region of both first- and second-order phase transitions, and some of the relevant results were reported in an earlier communication.[6]

According to the generalized version of the Privman-Fisher hypothesis, the "singular" part of the free-energy density of a system confined to geometry $L^{d-d'} \times \infty^{d'}$ and subjected to periodic boundary conditions may be written as

$$f^{(s)}(T,H;L) \approx TL^{-d}Y(x_1,x_2) , \tag{1}$$

where x_1 and x_2 are the *scaled variables* of the system,

$$x_1=\tilde{C}_1 L^{1/\nu}\tilde{t}, \quad x_2=\tilde{C}_2 L^{\Delta/\nu}H/T . \tag{2}$$

$\tilde{C}_1$ and $\tilde{C}_2$ are certain *nonuniversal*, model-dependent (and, in general, temperature-dependent) scale factors while $\tilde{t}$ is a generalization of the conventional temperature variable $t[=(T-T_c)/T_c]$; here, T_c denotes the bulk critical temperature of the system while other symbols have their usual meanings. The function $Y(x_1,x_2)$ is then a *universal* function—common to all systems in the same universality class. Encouraged by the success of hypothesis (1), we introduce, in Sec. II, a corresponding hypothesis for the correlation function $G(\mathbf{R},T,H;L)$ of the finite-sized system which embodies a subsidiary hypothesis for the correlation length $\xi(T,H;L)$ as well. A comparison of the proposed form of the *finite-system* correlation function with known expression for the bulk correlation function at temperatures below T_c enables us to determine the scaling parameters $\tilde{C}_1\tilde{t}$ and $\tilde{C}_2$ for all temperatures from $T \simeq T_c$ down to $T=0$. Using our knowledge of the mathematical nature of the singularity encountered by the system as $T \to 0$, we then make specific predictions about the size dependence as well as the temperature dependence of the correlation length ξ

in the region of both first- and second-order phase transitions; this is done in Sec. III, which also includes a comparative discussion of the magnetic susceptibility of the system.

In Sec. IV we carry out an exact analysis of the zero-field correlation function $G(\mathbf{R},T;L)$ and the zero-field correlation length $\xi(T;L)$ of a finite-sized spherical model under periodic boundary conditions; in Secs. V and VI we compare our analytical results with the scaling predictions made in Sec. III. We find that all our predictions are fully borne out and comparison with earlier results, wherever possible, is perfect. In the process we isolate the finite-size effect, $G^*(\mathbf{R},T;L)$, in the correlation function of the system and analyze it in detail for various values of R $(=|\mathbf{R}|)$ in relation to the parameters L and ξ. Not surprisingly, the effect associated with the component $\mathbf{R}_\parallel$ of $\mathbf{R}$ (which pertains to directions in which the system is infinite) turns out to be very different in nature from the one associated with the component $\mathbf{R}_\perp$ (which pertains to directions in which the system is finite). While for short distances the two effects are comparable in magnitude but opposite in sign, for long distances they are radically different. In fact, the contrast between the two becomes dramatic when one considers the propagation of long-range order in the system at temperatures below T_c over distances much greater than L in the direction of $\mathbf{R}_\parallel$. One is thereby led to a "splitting theorem," for the correlation function $G(\mathbf{R},T;L)$, which constitutes Sec. VII of the paper and highlights the fact that, for $R_\parallel \gg L$, the mathematical form of the correlation function bears no resemblance to the one for a d-dimensional bulk system; it becomes characteristic of a d'-dimensional bulk system instead. Clearly, no such problem arises in the case of a fully finite system $(d'=0)$.

II. SCALING HYPOTHESIS FOR THE CORRELATION FUNCTION

We propose that the correlation function $G(\mathbf{R},T,H;L)$ of a finite-sized system subject to periodic boundary conditions may be written as

$$G(\mathbf{R},T,H;L)\approx \frac{\tilde{D}}{R^{d-2+\eta}}X\left[\frac{\mathbf{R}}{\xi},\frac{L}{\xi},x_2\right] \quad (\mathbf{R}=\mathbf{R}_i-\mathbf{R}_j,\quad R=|\mathbf{R}|)\ , \tag{3}$$

where ξ is the *finite-system* correlation length which obeys the subsidiary hypothesis

$$\xi(T,H;L)\approx LS(x_1,x_2)\ . \tag{4}$$

It seems important to note that, as in the case of expression (1), no *nonuniversal* scale factor appears in front of the scaling function $S(x_1,x_2)$ in expression (4); the same is not true of expression (3) where a *nonuniversal* factor $\tilde{D}$ does indeed appear in front of the scaling function $X(\mathbf{r},l,x_2)$. To determine $\tilde{D}$, we observe that the magnetic susceptibility $\chi(T,H;L)$ of the system may be derived from Eq. (3) as well as from Eq. (1) and require that the two expressions thus derived be identical. In the former case,

$$\chi(T,H;L)\equiv\frac{1}{T}\sum_{\mathbf{R}_i,\mathbf{R}_j}G(\mathbf{R}_i-\mathbf{R}_j,T,H;L) \approx\frac{N\tilde{D}\xi^{2-\eta}}{Ta^d}Z_1\left[\frac{L}{\xi},x_2\right]\ , \tag{5}$$

where N is the total number of spins in the system, a is a microscopic length (such as the lattice constant) while Z_1 is some universal function of the variables L/ξ and x_2. Using (4), and remembering that the volume of the system is $\approx Na^d$, we may write *per unit volume* of the system

$$\chi(T,H;L)\approx\frac{\tilde{D}L^{2-\eta}}{Ta^{2d}}Z_2(x_1,x_2)\ , \tag{6}$$

where Z_2 is another universal function—this time of the variables x_1 and x_2. Starting from (1), on the other hand, we obtain (again *per unit volume* of the system)

$$\chi(T,H;L)\equiv-\left[\frac{\partial^2 f^{(s)}}{\partial H^2}\right]_{T,L}\approx\frac{\tilde{C}_2^2L^{\gamma/\nu}}{T}Z(x_1,x_2)\ , \tag{7}$$

where Z is also a universal function of the variables x_1 and x_2. Comparing (6) and (7), we conclude that the ratio $\tilde{D}/a^{2d}\tilde{C}_2^2$ must be a universal number. Since this number can be absorbed into the function $X(\mathbf{r},l,x_2)$ of Eq. (3), we may write

$$\tilde{D}=a^{2d}\tilde{C}_2^2\ . \tag{8}$$

A comparison of the present formulation for the finite-system correlation function $G(\mathbf{R},T,H;L)$ with that of Privman and Fisher[2] for the bulk correlation function $G(\mathbf{R},T,H;\infty)$ shows that, up to universal factors, our parameters $\tilde{D}$ and $\tilde{C}_2$ are equivalent to their parameters D_1 and $D_2c_0^{-\Delta/\nu}$, respectively. Accordingly, our Eq. (8) amounts, in their notation, to the statement

$$D_1\sim a^{2d}D_2^2c_0^{-2\Delta/\nu}\sim a^{2d}A_1^{\Psi}A_2^2\ ,$$

which, however, differs from the last result stated in Eq. (A8) of Privman and Fisher, in that a factor such as a^{2d} is not there. The reason for this (minor) discrepancy lies in the fact that Eq. (A2) of Privman and Fisher lacks a factor of v^{-2}, where v denotes the "volume of the unit cell" in the lattice. While this factor is necessary for mutual consistency of their Eqs. (A2) and (A7), it *does* indeed arise if the passage from Eq. (A1) to Eq. (A2) is made with caution. With this modification introduced into Eqs. (A2) and (A8) of Privman and Fisher, the result obtained here would be in complete agreement with theirs.

To determine $\tilde{C}_1\tilde{t}$ and $\tilde{C}_2$ appearing in Eqs. (2), we observe that, under appropriate limiting conditions, hypothesis (3) must reproduce known expression for the field-free, *bulk* correlation function, namely[7-9]

$$G(\mathbf{R},T,0;\infty)=M_0^2(T)+\frac{A(T)}{R^{d-2}}\quad (T<T_c) \tag{9}$$

where $M_0(T)$ is the spontaneous magnetization and $A(T)$ a system-dependent coefficient. Assuming that, for $T<T_c$ and $L\to\infty$, the scaling function S in (4) takes the limiting form

$$S(x_1,0)\approx S_-\,|x_1|^{\sigma}\quad (x_1\to-\infty)\,, \tag{10}$$

whence

$$\xi(T;L)\approx S_-(\tilde{C}_1\,|\tilde{t}\,|)^{\sigma}L^{(\nu+\sigma)/\nu}\quad (T<T_c,\ \ L\to\infty)\,, \tag{11}$$

we conclude that the function X in (3) must possess the asymptotic behavior

$$[X(\mathbf{r},l,0)]_{(r,l)\to 0}\approx X_1\left[\frac{r}{l^{(\nu+\sigma)/\sigma}}\right]^{d-2+\eta}+X_2\left[\frac{r}{l^{(\nu+\sigma)/\sigma}}\right]^{\eta}+X^*(\mathbf{r},l)\,, \tag{12}$$

with *universal* amplitudes

$$X_1=M_0^2(T)/a^{2d}\tilde{C}_2^2(S_-^{1/\sigma}\tilde{C}_1\,|\tilde{t}\,|)^{\nu(d-2+\eta)} \tag{13}$$

and

$$X_2=A(T)/a^{2d}\tilde{C}_2^2(S_-^{1/\sigma}\tilde{C}_1\,|\tilde{t}\,|)^{\nu\eta}\,, \tag{14}$$

while $X^*(\mathbf{r},l)$ represents the finite-size effect in X. It follows that

$$\tilde{C}_1\,|\tilde{t}\,|=S_-^{-1/\sigma}[X_2M_0^2(T)/X_1A(T)]^{1/\nu(d-2)} \tag{15}$$

and

$$\tilde{C}_2=a^{-d}[A(T)/X_2]^{1/2}[X_1A(T)/X_2M_0^2(T)]^{\eta/2(d-2)}\,. \tag{16}$$

Now, in view of the fact that the quantity $A(T)\Upsilon(T)/TM_0^2(T)$, where $\Upsilon(T)$ is the "helicity modulus" of the (bulk) system, is *universal*,[9]

$$A(T)\Upsilon(T)/TM_0^2(T)=a_0\,, \tag{17}$$

relations (15) and (16) may be written in the alternative form

$$\tilde{C}_1\,|\tilde{t}\,|=S_-^{-1/\sigma}[X_2\Upsilon(T)/a_0X_1T]^{1/\nu(d-2)} \tag{18}$$

and

$$\tilde{C}_2=a^{-d}[a_0TM_0^2(T)/X_2\Upsilon(T)]^{1/2}\times[a_0X_1T/X_2\Upsilon(T)]^{\eta/2(d-2)}\,. \tag{19}$$

The nonuniversal parameters $\tilde{C}_1\tilde{t}$ and $\tilde{C}_2$ are thus determined in terms of the bulk properties of the system.

We note that, as $T\to T_{c-}$, $M_0(T)\sim|t|^{\beta}$, while $A(T)\sim|t|^{\nu\eta}$, t being the conventional temperature variable $(T-T_c)/T_c$; it follows that $\Upsilon(T)$ in that case would vary as $|t|^{\nu(d-2)}$, with the result that

$$\tilde{C}_1\,|\tilde{t}\,|\to C_1\,|t\,|\,,\quad \tilde{C}_2\to C_2\,, \tag{20}$$

where C_1 and C_2 are the *temperature-independent* scale factors pertaining to the original hypothesis of Privman and Fisher for $T\simeq T_c$; our generalized hypothesis, therefore, reduces to the Privman-Fisher form as the region of second-order phase transition is approached. On the other hand, as $T\to 0$, the quantities $M_0(T)$ and $\Upsilon(T)$ tend to become constant, so that

$$\tilde{C}_1\,|\tilde{t}\,|\sim T^{-1/\nu(d-2)},\quad \tilde{C}_2\sim T^{\beta/\nu(d-2)}\,, \tag{21}$$

in perfect agreement with the recent proposal of Shapiro;[4] see also Ref. 5. Equations (21) are a clear signal of the singularity lurking at $T=0$ K.

The results stated so far are supposed to be general, in the sense that they may hold for *all* O(n) models with $n\geq 2$. In the special case of the spherical model ($n=\infty$), the relevant bulk results are known to be[7–9]

$$M_0(T)=\left[1-\frac{T}{T_c}\right]^{1/2},\quad A(T)=\Gamma\left[\frac{d-2}{2}\right]\frac{a^{d-2}T}{8\pi^{d/2}J}\,,$$
$$\Upsilon(T)=\frac{2J}{a^{d-2}}\left[1-\frac{T}{T_c}\right]\,, \tag{22}$$

so that

$$a_0=\Gamma\left[\frac{d-2}{2}\right]\Big/4\pi^{d/2}\,; \tag{23}$$

here, J denotes the interaction parameter of the model while a stands for the lattice constant. It follows that for the spherical model [for which $\eta=0$ and $\nu=1/(d-2)$], the desired scale factors may be written as

$$\tilde{C}_1\tilde{t}=\frac{J}{a^{d-2}}\left[\frac{1}{T_c}-\frac{1}{T}\right],\quad \tilde{C}_2=\left[\frac{T}{a^{d+2}J}\right]^{1/2}\,. \tag{24}$$

Referring to Eqs. (13) and (14), we observe that, with this choice of $\tilde{C}_1\tilde{t}$ and $\tilde{C}_2$, the universal numbers X_1 and X_2 would be given by

$$X_1=S_-^{-1/\sigma},\quad X_2=\tfrac{1}{2}a_0\,. \tag{25}$$

It may also be noted that, in the limit $T\to T_c$, our scale factors (24) reduce precisely to the ones adopted in I, i.e.,

$$\tilde{C}_1\tilde{t}\to\frac{J}{a^{d-2}T_c}t=C_1t,\quad \tilde{C}_2\to\left[\frac{T_c}{a^{d+2}J}\right]^{1/2}=C_2\,; \tag{26}$$

see Eqs. (20) of I. On the other hand, as $T\to 0$,

$$\tilde{C}_1\,|\tilde{t}\,|\sim T^{-1},\quad \tilde{C}_2\sim T^{1/2}\,, \tag{27}$$

in perfect agreement with (21). We shall now make predictions on the basis of the formulation of this section.

III. PREDICTIONS BASED ON SCALING HYPOTHESIS

For this we go back to the general O(n) model with $n\geq 2$ and examine various regimes of T and L one by one. Regarding the correlation length, we note that, for $T<T_c$ and $L\to\infty$ (which makes $x_1\to-\infty$), the desired result is given by Eqs. (11) and (18), so that

$$\xi(T;L)\approx[X_2\Upsilon(T)/a_0X_1T]^{\sigma/\nu(d-2)}L^{(\nu+\sigma)/\nu}\quad (T<T_c,\ L\to\infty)\,. \tag{28}$$

As $T\to 0$, $\Upsilon(T)$ tends to be a constant, with the result that $\xi\sim T^{-\sigma/\nu(d-2)}$. However, as argued earlier,[3] the mathematical nature of the singularity encountered by the system in geometry $L^{d-d'}\times\infty^{d'}$ at $T=0$ should be characteristic of the d'-*dimensional* bulk system. We, therefore, conclude that $\sigma=\dot{\nu}\nu(d-2)$, where $\dot{\nu}$ is the critical exponent governing the behavior of the correlation length $\xi(T)$ of a d'-dimensional bulk system as $T\to T_c(d')=0$. Making use of the actual value of $\dot{\nu}$, see Eq. (10) of the appendix, we find that, for $d'<2$,

$$\sigma=\nu(d-2)/(2-d') \tag{29}$$

and hence for the system under study

$$\xi(T;L)\approx[X_2\Upsilon(T)/a_0X_1T]^{1/(2-d')}L^{(d-d')/(2-d')} \tag{30}$$

$$\sim L[L^{d-2}\Upsilon(T)/T]^{1/(2-d')}\quad(T<T_c,\ L\to\infty)\ . \tag{31}$$

It will be noted that the *approach exponent*, $(d-d')/(2-d')$, appearing in (30), though dependent on d, is totally independent of the critical exponents pertaining to the d-dimensional bulk system. Equation (30) gives us complete dependence of ξ on *both* T and L throughout the region of the first-order phase transition. In passing, we note that Fisher and Privman[10] have also derived an asymptotic expression for ξ which holds for all $n\geq2$ but pertains to the "cylindrical" geometry ($d'=1$). A comparison of their result with ours suggests that the universal quantity

$$(X_2/a_0X_1)_{d'=1}=2n/(n-1)\ . \tag{32}$$

Likewise, the zero-field susceptibility of the system in the regime $T<T_c$ would take the form, see Eq. (7),

$$\chi(T;L)\approx Z_-(\tilde{C}_2^2/T)(\tilde{C}_1|\tilde{t}|)^{\theta}L^{(\gamma+\theta)/\nu}\ , \tag{33}$$

with Z_- universal and θ as yet unknown. Arguing as before, we conclude, with the help of Eqs. (21), that

$$\theta=(\dot{\gamma}-1)\nu(d-2)+2\beta=\nu[\dot{\gamma}(d-2)+\eta] \tag{34}$$

where $\dot{\gamma}$ is the corresponding d'-dimensional critical exponent. It follows that

$$\chi(T;L)\approx\frac{Z_-a_0}{X_2S_-^{[\dot{\gamma}(d-2)+\eta]/\dot{\nu}(d-2)}}\frac{M_0^2(T)}{a^{2d}\Upsilon(T)}\left[\frac{X_2\Upsilon(T)}{a_0X_1T}\right]^{\dot{\gamma}}L^{2+\dot{\gamma}(d-2)}\ . \tag{35}$$

Making use of the actual values of $\dot{\nu}$ and $\dot{\gamma}$, see again Eq. (A10) of the Appendix, we find that, for $d'<2$,

$$\chi(T;L)\approx\frac{Z_-a_0}{X_2S_-^{(4\beta-d'\nu\eta)/\nu(d-2)}}\frac{M_0^2(T)}{a^{2d}\Upsilon(T)}\left[\frac{X_2\Upsilon(T)}{a_0X_1T}\right]^{2/(2-d')}L^{2(d-d')/(2-d')} \tag{36}$$

$$\sim TL^{-d}[L^dM_0(T)/T]^2[L^{d-2}\Upsilon(T)/T]^{d'/(2-d')}\quad(T<T_c,\ L\to\infty)\ . \tag{37}$$

Equation (36) gives us complete dependence of χ on *both* T and L throughout the region of the first-order phase transition. Once again, we note that the *approach exponent*, $2(d-d')/(2-d')$, appearing in (36), though dependent on d, is totally independent of the critical exponents pertaining to the d-dimensional bulk system.

At this point it seems worthwhile to note that, in the regime under study, the quantity χ/ξ^2 is *independent* of L and depends only on T:

$$\frac{\chi}{\xi^2}\approx\frac{Z_-a_0}{X_2S_-^{(4\beta-d'\nu\eta)/\nu(d-2)}}\frac{M_0^2(T)}{a^{2d}\Upsilon(T)}\sim\frac{A(T)}{T}\ ; \tag{38}$$

in the case of the spherical model, even T dependence is absent and we obtain an especially simple result, viz.

$$\left[\frac{\chi}{\xi^2}\right]_{n=\infty}\approx\frac{Z_-}{S_-^2Ja^{d+2}}\ . \tag{38a}$$

In passing, we observe that, as $T\to T_{c-}$, Eqs. (31) and (37) assume the form

$$\xi\sim|t|^{\nu(d-2)/(2-d')}L^{(d-d')/(2-d')}$$

$$\sim|t|^{-\nu}(L|t|^{\nu})^{(d-d')/(2-d')}\quad(T\lesssim T_c,\ L\to\infty) \tag{39}$$

and

$$\chi\sim|t|^{2\beta+\nu(d-2)d'/(2-d')}L^{2(d-d')/(2-d')}$$

$$\sim|t|^{-\gamma}(L|t|^{\nu})^{2(d-d')/(2-d')}\quad(T\lesssim T_c,\ L\to\infty)\ , \tag{40}$$

a clear indication of the fact that the system is preparing to enter the region of the second-order phase transition where $\chi/\xi^2\sim|t|^{\nu\eta}$.

At this stage we would like to point out that the foregoing results were obtained for $d'<2$. If $d'=2$, then power laws such as (30), (31), (36), (37), (39), and (40) would get replaced by exponential ones; see Eqs. (A13) of the Appendix.

In the "core" region, where $|x_1|=O(1)$ and hence $|t|=O(L^{-1/\nu})$, the functions ξ and χ, for a fixed value of x_1, are proportional to L and $L^{\gamma/\nu}$, respectively; see Eqs. (4) and (7). Accordingly, the quantities

$$\xi(T_c;L)L^{-1}\text{ and }\chi(T_c;L)(T_c/C_2^2)L^{-\gamma/\nu}\ , \tag{41}$$

evaluated at the erstwhile critical point $T=T_c$, would be universal.

Finally, for $T\gtrsim T_c$ and $L\to\infty$, we expect to recover standard *bulk* results, with corrections arising from the finiteness of the system. Under periodic boundary conditions, these corrections are expected to be exponentially small. We shall now verify these predictions in the case of the spherical model of ferromagnetism.

IV. ANALYTICAL RESULTS FOR THE SPHERICAL MODEL

For analytical study we consider a field-free spherical model, of size $L_1\times\cdots\times L_d$, for which the spin-spin correlation function under periodic boundary conditions is given by[8,11]

$$G(\mathbf{R},T;L_j)=\frac{T}{2N}\sum_{\{n_j\}}\frac{\cos(\mathbf{k}\cdot\mathbf{R})}{\lambda-2J\sum_j\cos(k_ja)}\ , \tag{42}$$

where $n_j=0,1,\ldots,N_j-1$, $N_j=L_j/a$, $k_j=2\pi n_j/L_j$ $(j=1,\ldots,d)$ while λ denotes the "spherical field" pertaining to the model. Following the procedure developed in I, we obtain (for $R_j,L_j\gg a$)

$$G(\mathbf{R},T;L_j)=\frac{T}{4\pi J}\left[\frac{\phi}{2\pi}\right]^{(d-2)/2}\sum_{\{q_j\}=-\infty}^{\infty}\frac{K_{(d-2)/2}(\Lambda(q_j))}{[\Lambda(q_j)]^{(d-2)/2}}\ , \tag{43}$$

where $K_\nu(z)$ are modified Bessel functions,

$$\Lambda(q_j)=\frac{\sqrt{\phi}}{a}\left[\sum_j(q_jL_j+R_j)^2\right]^{1/2}\ ,\qquad \phi=\left[\frac{\lambda}{J}-2d\right]\ll 1\ ; \tag{44}$$

the parameter $\phi(T;L_j)$ is determined by the constraint equation of the system, see Eq. (65) of **I**,

$$8\pi J\left[\frac{1}{T_c}-\frac{1}{T}\right]=\left[\frac{\phi}{4\pi}\right]^{(d-2)/2}\left[\left|\Gamma\left[\frac{2-d}{2}\right]\right|-2^{d/2}\sum_{\mathbf{q}\,(\neq 0)}\frac{K_{(d-2)/2}(\Lambda_0(q_j))}{[\Lambda_0(q_j)]^{(d-2)/2}}\right] \tag{45}$$

where Λ_0 denotes $(\Lambda)_{\mathbf{R}=0}$.

The structure of the quantity $\Lambda(q_j)$ suggests that we define a *finite-system* correlation length $\xi(T;L_j)$, through the relationship

$$\xi(T;L_j)=a/[\phi(T;L_j)]^{1/2}\ , \tag{46}$$

and write $\Lambda(q_j)$ in the *scaled* form

$$\Lambda(q_j)=\left[\sum_j(q_jl_j+r_j)^2\right]^{1/2}\ , \tag{47}$$

where

$$l_j=L_j/\xi,\quad r_j=R_j/\xi\ . \tag{48}$$

For geometry $L^{d^*}\times\infty^{d'}$, where $d^*+d'=d$, only those $\Lambda(q_j)$ contribute to the sum in (43) which are of the form

$$\Lambda(q_j)=\left[\sum_{j=1}^{d^*}(q_jl+r_j)^2+r_\parallel^2\right]^{1/2}\ , \tag{49}$$

where

$$l=L/\xi,\quad r_\parallel^2=r^2-r_\perp^2,\quad r_\perp^2=\sum_{j=1}^{d^*}r_j^2\ . \tag{50}$$

It is now straightforward to see that, with nonuniversal parameter

$$\tilde{D}=a^{2d}\tilde{C}_2^2=a^{d-2}T/J\ , \tag{51}$$

see Eqs. (8) and (24), our analytical result for $G(\mathbf{R},T;L)$ is in full conformity with the scaling hypothesis (3), with scaling function

$$X(\mathbf{r},l,0)=\frac{r^{d-2}}{2(2\pi)^{d/2}}\sum_{\mathbf{q}(d^*)}\frac{K_{(d-2)/2}(\Lambda(\mathbf{q}))}{[\Lambda(\mathbf{q})]^{(d-2)/2}}\ . \tag{52}$$

As for the correlation length, we invoke the *thermogeometric parameter y* through the standard relationship[1]

$$y=\frac{1}{2}\frac{L}{a}\sqrt{\phi}=\frac{L}{2\xi}\ , \tag{53}$$

which shows quite readily that ξ indeed satisfies the subsidiary hypothesis (4), with scaling function

$$S(x_1,0)=\frac{1}{2y(x_1)}\ ; \tag{54}$$

the parameter $y(x_1)$, in turn, is determined by the constraint equation (45) which may now be written in the *scaled* form, see Eq. (24) for the nonuniversal parameter $\tilde{C}_1\tilde{t}$,

$$x_1=\frac{y^{d-2}}{8\pi^{d/2}}\left[\left|\Gamma\left[\frac{2-d}{2}\right]\right|-2\sum_{\mathbf{q}(d^*)}{}'\frac{K_{(d-2)/2}(2yq)}{(yq)^{(d-2)/2}}\right]$$
$$[q=(q_1^2+\cdots+q_{d^*}^2)^{1/2}>0]\ . \tag{55}$$

We shall now examine our results for $\xi(T;L)$ and $G(\mathbf{R},T;L)$ in different regimes of T and L, and for various values of R (in relation to the parameters L and ξ).

V. FINITE-SIZE EFFECTS IN THE CORRELATION LENGTH

A. Case (a): $T<T_c$, $L\to\infty$

In this case, $x_1\to-\infty$ and Eq. (55) gives

$$y\approx\begin{cases}\left[\left[\frac{1}{8\pi^{d'/2}}\Gamma\left[\frac{2-d'}{2}\right]\right]\Big/|x_1|\right]^{1/(2-d')} & (d'<2)\ ,\\ \text{const}\times\exp(-4\pi|x_1|) & (d'=2)\ .\end{cases} \tag{56}$$

The scaling function $S(x_1,0)$ then assumes the asymptotic form (10), with

$$S_-=\frac{1}{2}\left[8\pi^{d'/2}\Big/\Gamma\left[\frac{2-d'}{2}\right]\right]^{1/(2-d')},\qquad \sigma=1/(2-d')\qquad (d'<2)\ . \tag{57}$$

The value of σ obtained here is in perfect agreement with the scaling prediction (29), with $\nu=1/(d-2)$. It may also be noted that, in the case of the "cylindrical" geometry $(d'=1)$, S_- turns out to be 4 which, by Eq. (25), gives $X_1=\frac{1}{4}$; it then follows that the ratio (X_2/a_0X_1) in this case equals 2—in perfect agreement with Eq. (32), with $n\to\infty$.

The correlation length itself is given by Eqs. (53) and (56), with the result that, for $d'<2$,

$$\xi \approx \frac{1}{2}\left[8\pi^{d'/2}\Big/\Gamma\left[\frac{2-d'}{2}\right]\right]^{1/(2-d')}\left[\frac{J}{a^{d-2}}\left[\frac{1}{T}-\frac{1}{T_c}\right]\right]^{1/(2-d')} L^{(d-d')/(2-d')} \tag{58}$$

$$=\frac{1}{2}\left[4\pi^{d'/2}\Big/\Gamma\left[\frac{2-d'}{2}\right]\right]^{1/(2-d')} L\left[\frac{L^{d-2}\Upsilon(T)}{T}\right]^{1/(2-d')} ; \tag{59}$$

here, use has also been made of Eqs. (22) and (24). We find that Eqs. (58) and (59) do indeed agree with predictions (30) and (31), with

$$X_1=\frac{1}{2(4\pi)^{d'/2}}\Gamma\left[\frac{2-d'}{2}\right], \quad X_2=\tfrac{1}{2}a_0 ; \tag{60}$$

see Eqs. (22)–(25). For $d'=2$, we obtain instead

$$\xi \approx \text{const}\times L \exp\left[4\pi J\left[\frac{L}{a}\right]^{d-2}\left[\frac{1}{T}-\frac{1}{T_c}\right]\right] \quad (d'=2) ; \tag{61}$$

for $d=3$, the constant in front of this expression turns out to be unity (see also Ref. 11).

For the model under study, the foregoing results may as well be expressed in terms of the spontaneous magnetization, $M_0(T)$, of the bulk system; thus,

$$\frac{\xi}{L}\approx\begin{cases}\frac{1}{2}\left[8\pi^{d'/2}\Big/\Gamma\left[\frac{2-d'}{2}\right]\right]^{1/(2-d')}\left[\frac{J}{T}\left[\frac{L}{a}\right]^{d-2}M_0^2(T)\right]^{1/(2-d')} & (d'<2) ,\\ \text{const}\times\exp\left[4\pi\frac{J}{T}\left[\frac{L}{a}\right]^{d-2}M_0^2(T)\right] & (d'=2) .\end{cases} \tag{62}$$

It is gratifying to note that, in the special case $d'=1$, this result is in complete agreement with the one derived recently by Fisher and Privman.[12]

At this point it seems instructive to recall that the magnetic susceptibility per unit volume of the system is given by the expression[1,11]

$$\chi=\frac{1}{2Ja^d\phi}=\frac{L^2}{8Ja^{d+2}y^2} . \tag{63}$$

It is readily seen that expression (63) conforms to the scaling relation (7), with scaling function

$$Z(x_1,0)=\frac{1}{8y^2}=\frac{1}{2}S^2(x_1,0) ; \tag{64}$$

cf. (54). Equations (46) and (63) show that for the spherical model of ferromagnetism the ratio χ/ξ^2 is, by definition, a constant—independent of both T and L:

$$\chi/\xi^2=1/2Ja^{d+2} \tag{65}$$

in perfect agreement with the scaling prediction (38a), with

$$Z_-=\tfrac{1}{2}S_-^2 , \tag{66}$$

which follows from Eq. (64). It should be emphasized, however, that, for general $n\geq 2$, the ratio χ/ξ^2 is independent of L *only* for $T<T_c$, but is still a function of T; the case of the spherical model ($n=\infty$) is special, in that this ratio is independent of *both* T and L in *all* regimes of T.

B. Case (b): The "Core" Region

In the "core" region, where $|x_1|=O(1)$ and hence $|t|=O(L^{-(d-2)})$, the thermogeometric parameter y in general is also $O(1)$; accordingly, $\xi(L)=O(L)$ and its precise value can be determined only numerically. In the exceptional cases $d\gtrsim 2$ and $d\lesssim 4$, however, y does turn out to be much less than unity and we obtain the following analytical results:

$$\left[\frac{\xi(L)}{L}\right]_{T=T_c}\approx\frac{1}{2\pi^{1/2}}\times\begin{cases}[\frac{1}{2}\Gamma((2-d')/2)\epsilon]^{-1/(2-d')} & [\epsilon=(d-2)\ll 1] ,\\ [\frac{1}{2}\Gamma((2-d')/2)\epsilon]^{-1/(4-d')} & [\epsilon=(4-d)\ll 1] ,\end{cases} \tag{67}$$

valid for $d'<2$. Equations (67) generalize some recent results of Luck[13] (valid for $d'=1$) and some of Brézin[14] (valid for $d'=0$ and 1). Of course, for any value of d', the ratio

$$\left[\frac{\xi(L)}{L}\right]_{T=T_c}=\frac{1}{2y(x_1=0)} , \tag{68}$$

clearly, a universal number.

C. Case (c): $T \gtrsim T_c$, $L \to \infty$

In this case, $x_1 \to +\infty$ and Eq. (55) gives

$$y \approx \left[\frac{8\pi^{d/2} x_1}{|\Gamma((2-d)/2)|} \right]^{1/(d-2)} \left[1 + \frac{d^* \pi^{1/2}}{\Gamma\{(4-d)/2\}} \left[\frac{2\xi(\infty)}{L} \right]^{(d-1)/2} e^{-L/\xi(\infty)} \right] ; \tag{69}$$

accordingly,

$$\xi(L) \approx \xi(\infty) \left[1 - \frac{d^* \pi^{1/2}}{\Gamma\{(4-d)/2\}} \left[\frac{2\xi(\infty)}{L} \right]^{(d-1)/2} e^{-L/\xi(\infty)} \right] , \tag{70}$$

where $\xi(\infty)$ is the corresponding bulk correlation length, viz.

$$\xi(\infty) = \frac{1}{2} \left[\frac{|\Gamma\{(2-d)/2\}| T_c}{8\pi^{d/2} J} \right]^{1/(d-2)} a t^{-1/(d-2)} \quad (0 < t \ll 1) . \tag{71}$$

As expected under periodic boundary conditions, finite-size correction in this regime is indeed exponentially small. In passing, we note that Eq. (70) generalizes another result of Luck[13] which pertained to the special case $d'=1$, i.e., $d^* = d-1$.

VI. FINITE-SIZE EFFECTS IN THE CORRELATION FUNCTION

The correlation function of the system under study is given by Eq. (43) which, for geometry $L^{d^*} \times \infty^{d'}$ $(d^* + d' = d)$, may be written in the form

$$G(\mathbf{R},T;L) = \frac{T}{2(2\pi)^{d/2} J} \left[\frac{a^2}{\xi R} \right]^{(d-2)/2} K_{(d-2)/2}(R/\xi) + \frac{T}{2(2\pi)^{d/2} J} \left[\frac{a}{\xi} \right]^{d-2} \sum_{\mathbf{q}(d^*)}{}' \frac{K_{(d-2)/2}(\Lambda(\mathbf{q}))}{[\Lambda(\mathbf{q})]^{(d-2)/2}} , \tag{72}$$

where the $\mathbf{q}=0$ term has been taken out of the summation and the parameter ϕ has been replaced by ξ through the defining relationship (46); of course, it will be remembered that the quantity $\Lambda(\mathbf{q})$ appearing here is given by Eqs. (49) and (50). Now, combining Eq. (72) with (45), we obtain a more useful form of the function $G(\mathbf{R},T;L)$, namely

$$G(\mathbf{R},T;L) = \left[1 - \frac{T}{T_c} \right] + \frac{T}{2(4\pi)^{d/2} J} \left[\frac{a}{\xi} \right]^{d-2} \left[2^{d/2} \left[\frac{\xi}{R} \right]^{(d-2)/2} K_{(d-2)/2}(R/\xi) + \left| \Gamma \left[\frac{2-d}{2} \right] \right| \right]$$
$$+ \frac{T}{2(2\pi)^{d/2} J} \left[\frac{a}{\xi} \right]^{d-2} \sum_{\mathbf{q}(d^*)}{}' \left[\frac{K_{(d-2)/2}(\Lambda(\mathbf{q}))}{[\Lambda(\mathbf{q})]^{(d-2)/2}} - \frac{K_{(d-2)/2}(\Lambda_0(\mathbf{q}))}{[\Lambda_0(\mathbf{q})]^{(d-2)/2}} \right] , \tag{73}$$

where Λ_0, as before, denotes $(\Lambda)_{\mathbf{R}=0}$. For $R \ll \xi$, we make use of the limiting form[15]

$$K_\nu(z) = \frac{\pi}{2\sin(\nu\pi)} [I_{-\nu}(z) - I_\nu(z)] = \frac{1}{2}\Gamma(\nu) \left[\frac{1}{2} z \right]^{-\nu} - \frac{1}{2\nu}\Gamma(1-\nu) \left[\frac{1}{2} z \right]^{\nu} + O(z^{2-\nu}) \quad (0 < \nu < 1,\ z \ll 1) , \tag{74}$$

with the result that

$$G(\mathbf{R},T;L) = \left[1 - \frac{T}{T_c} \right] + \left[\Gamma \left[\frac{d-2}{2} \right] \frac{T}{8\pi^{d/2} J} \left[\frac{a}{R} \right]^{d-2} + O \left[\frac{a^{d-2} R^{4-d}}{\xi^2} \right] \right] + G^*(\mathbf{R},T;L) , \tag{75}$$

where

$$G^*(\mathbf{R},T;L) = \frac{T}{2(2\pi)^{d/2} J} \left[\frac{a}{\xi} \right]^{d-2} \sum_{\mathbf{q}(d^*)}{}' \left[\frac{K_{(d-2)/2}(\Lambda(\mathbf{q}))}{[\Lambda(\mathbf{q})]^{(d-2)/2}} - \frac{K_{(d-2)/2}(\Lambda_0(\mathbf{q}))}{[\Lambda_0(\mathbf{q})]^{(d-2)/2}} \right] . \tag{76}$$

It is now readily seen that, for $T < T_c$ and $L \to \infty$ (which makes $\xi \to \infty$), Eq. (75) correctly reproduces the *bulk* correlation function (9), with $M_0(T)$ and $A(T)$ given by Eqs. (22). Thus, our result for the finite-system correlation function is consistent with the expectation[7] that

$$\lim_{R\to\infty} [\lim_{L\to\infty} G(\mathbf{R},T;L)] = M_0^2(T) . \tag{77}$$

The above demonstration entails the fact that, in the limit $(r,l) \to 0$, the scaling function $X(\mathbf{r},l,0)$ of Eq. (52) would also reduce to the form anticipated in Eq. (12), with appropriate values of the universal numbers X_1 and X_2, with $\eta=0$ and with $\nu/\sigma=(2-d')/(d-2)$. To see this explicitly, we observe that the first term in (12) would essentially arise from an integration (instead of summation) over $\mathbf{q}(d^*)$ in (52) which, in the desired approximation, would give[15]

$$\frac{r^{d-2}}{2(2\pi)^{d/2}}\int_0^\infty \frac{K_{(d-2)/2}(ql)}{(ql)^{(d-2)/2}}\frac{2\pi^{d^*/2}}{\Gamma(d^*/2)}q^{d^*-1}dq = \frac{1}{2(4\pi)^{(d-d^*)/2}}\Gamma\left[\frac{2-d+d^*}{2}\right]\frac{r^{d-2}}{l^{d^*}}\ , \quad (78a)$$

in perfect agreement with the expected result $X_1 r^{d-2}/l^{d-d'}$, with X_1 given by (60). The second term, on the other hand, would arise exclusively from the $(\mathbf{q}=0)$-term of the sum in (52) which, in turn, would give

$$\frac{r^{(d-2)/2}}{2(2\pi)^{d/2}}K_{(d-2)/2}(r)\simeq\Gamma\left[\frac{d-2}{2}\right]\frac{1}{8\pi^{d/2}}\ , \quad (78b)$$

in perfect agreement with the expected result X_2, as given by Eqs. (23) and (25). The difference between the actual expression in (52) on one hand and the sum of the approximants (78a) and (78b) on the other would constitute the remainder function, $X^*(\mathbf{r},l)$, of Eq. (12).

We shall now examine finite-size effects in the correlation function $G(\mathbf{R},T;L)$, as given by Eqs. (75) and (76). Remembering that the expression in question is valid for $a \ll R \ll \xi$, we shall focus our attention on the region of the first-order phase transition $(T<T_c)$ and examine different regimes of the variable R in relation to the parameter L.

A. Case 1: $R \ll L$

Equation (76) may, in this case, be approximated as

$$G^*(\mathbf{R},T;L)\simeq\frac{T}{4(2\pi)^{d/2}J}\left[\frac{a}{\xi}\right]^{d-2}\sum_{\mathbf{q}(d^*)}{}'\left[-\frac{K_{d/2}(ql)}{(ql)^{d/2}}(r_\perp^2+r_\parallel^2)+\frac{K_{(d+2)/2}(ql)}{d^*(ql)^{(d-2)/2}}r_\perp^2\right]\ ; \quad (79)$$

it will be remembered that the parameters l and $\mathbf{r}$ here stand for the *scaled variables* L/ξ and $\mathbf{R}/\xi$, respectively. For the "block" geometry $(d^*=d)$, $\mathbf{r}_\parallel$ does not exist, so $\mathbf{r}_\perp$ represents the full vector $\mathbf{r}$; Eq. (79) then simplifies to

$$G^*(\mathbf{R},T;L)\simeq\frac{T}{4(2\pi)^{d/2}J}\left[\frac{a}{\xi}\right]^{d-2}\sum_{\mathbf{q}(d)}{}'\frac{K_{(d-2)/2}(ql)}{d\,(ql)^{(d-2)/2}}r^2\quad (d^*=d)\ . \quad (80)$$

Now, for $T<T_c$ and $L\to\infty$ (which makes $x_1\to-\infty$), the parameter $l\,(\equiv 2y)\to 0$; see Eq. (56). The sums appearing in expressions (79) and (80) may then be replaced by their asymptotic forms, as given by Eqs. (79) of I; we thus obtain

$$G^*(\mathbf{R},T;L)\simeq\frac{Ta^{d-2}}{4JL^d}\times\begin{cases}\dfrac{\Gamma(d/2)}{2\pi^{d/2}d^*}\left[\sum_{\mathbf{q}(d^*)}{}' q^{-d}\right](d'R_\perp^2-d^*R_\parallel^2)\quad (0<d'\le 2)\ ,\\[2mm] \dfrac{1}{d}R^2\quad (d'=0)\ ,\end{cases} \quad (81)$$

where d', as usual, is the number of dimensions in which the system is infinite, i.e., $d'=d-d^*$. Quite expectedly, the finite-size effect for $R\ll L$ is isotropic in the case of the "block" geometry but not so in general; in fact, as will be seen repeatedly, the effect is positive for the component $\mathbf{R}_\perp$ and negative for $\mathbf{R}_\parallel$. For the most practical cases, viz. $d=3$ and $d^*=1$, 2, or 3, Eqs. (81) simplify to

$$G^*(\mathbf{R},T;L)\simeq\frac{Ta}{8\pi JL^3}\times\begin{cases}\zeta(3)(2R_\perp^2-R_\parallel^2)\quad (d^*=1)\ , & (82a)\\ \zeta(\tfrac{3}{2})\beta(\tfrac{3}{2})(R_\perp^2-2R_\parallel^2)\quad (d^*=2)\ , & (82b)\\ (2\pi/3)R^2\quad (d^*=3)\ ; & (82c)\end{cases}$$

here, use has been made of the Hardy sum[16]

$$\sum_{q_{1,2}=-\infty}^{\infty}{}'(q_1^2+q_2^2)^{-s}=4\zeta(s)\beta(s)\quad (s>1)\ , \quad (83)$$

where

$$\zeta(s)=\sum_{l=0}^{\infty}(l+1)^{-s},\quad \beta(s)=\sum_{l=0}^{\infty}(-1)^l(2l+1)^{-s}\ . \quad (84)$$

Before proceeding further, it seems important to point out that the error term displayed in Eq. (75), which had its origin in the $(\mathbf{q}=0)$-term of the original sum in (72), is in the present regime negligible in comparison with the effect calculated here. For this, we note that the ratio of the term in question to the effect shown in Eqs. (81) is $O(L^d/R^{d-2}\xi^2)$, i.e.,

$$\sim\left[\frac{a}{R}\right]^{d-2}\left[\frac{a}{L}\right]^{d'(d-2)/(2-d')} \quad (85a)$$

for $d'<2$, and

$$\sim \left[\frac{a}{R}\right]^{d-2}\left\{\left[\frac{L}{a}\right]^{d-2}\exp\left[-8\pi J\left[\frac{1}{T}-\frac{1}{T_c}\right]\left[\frac{L}{a}\right]^{d-2}\right]\right\} \tag{85b}$$

for $d'=2$; see Eqs. (58) and (61). We thus find that, for all $d'\leq 2$, the ratio (85) is negligibly small. Equations (81), therefore, determine the most dominant finite-size effect for the case under study.

B. Case 2: $R_\perp = O(L)$, $R_\parallel = 0$

For obvious reasons this case is not so readily tractable as the one with $R \ll L$. For simplicity, therefore, we confine our discussion to a system in *three* dimensions, for which Eqs. (75) and (76) reduce to the form

$$G(\mathbf{R},T;L)\approx\left[1-\frac{T}{T_c}\right]+\frac{Ta}{8\pi JR}+G^*(\mathbf{R},T;L)\ , \tag{86}$$

where

$$G^*(\mathbf{R},T;L)=\frac{Ta}{8\pi JL}g(\boldsymbol{\epsilon})\quad(\boldsymbol{\epsilon}=\mathbf{R}/L)\ , \tag{87}$$

with

$$g(\boldsymbol{\epsilon})=\sum_{\mathbf{q}(d^*)}{}'\left[\frac{e^{-2y[(\mathbf{q}+\boldsymbol{\epsilon}_\perp)^2+\boldsymbol{\epsilon}_\parallel^2]^{1/2}}}{[(\mathbf{q}+\boldsymbol{\epsilon}_\perp)^2+\boldsymbol{\epsilon}_\parallel^2]^{1/2}}-\frac{e^{-2yq}}{q}\right]$$
$$\left[\boldsymbol{\epsilon}_\perp=\frac{\mathbf{R}_\perp}{L},\boldsymbol{\epsilon}_\parallel=\frac{\mathbf{R}_\parallel}{L}\right]. \tag{88}$$

The function $g(\boldsymbol{\epsilon})$ can be simplified considerably by making use of certain identities established in a paper by Chaba and Pathria,[17] hereafter referred to as II. For a detailed study we consider the geometries of the cube, the cylinder and the film and, for clarity of understanding, set $\mathbf{R}_\parallel=0$ (which, in the case of the cube, is true anyway); accordingly, $\boldsymbol{\epsilon}_\parallel=0$ and $\boldsymbol{\epsilon}_\perp$ may be designated as $\boldsymbol{\epsilon}$. The function $g(\boldsymbol{\epsilon})$ then assumes the form

$$g(\boldsymbol{\epsilon})=\sum_{\mathbf{q}(d^*)}{}'\left[\frac{e^{-2y|\mathbf{q}+\boldsymbol{\epsilon}|}}{|\mathbf{q}+\boldsymbol{\epsilon}|}-\frac{e^{-2yq}}{q}\right]. \tag{89}$$

(i) *The cubic geometry ($d^*=3$).* For this case we recall identities (62) and (63) of II, viz.

$$\sum_{\mathbf{q}(3)}{}'\frac{e^{-2y|\mathbf{q}+\boldsymbol{\epsilon}|}}{|\mathbf{q}+\boldsymbol{\epsilon}|}=\left[\frac{\pi}{y^2}+\frac{A(\boldsymbol{\epsilon})}{\pi}-\frac{y^2}{\pi}\sum_{\mathbf{n}(3)}{}'\frac{\cos[2\pi(\mathbf{n}\cdot\boldsymbol{\epsilon})]}{n^2(n^2\pi^2+y^2)}\right]-\frac{e^{-2y\epsilon}}{\epsilon} \tag{90}$$

and

$$\sum_{\mathbf{q}(3)}{}'\frac{e^{-2yq}}{q}=\frac{\pi}{y^2}+\frac{C_3}{\pi}+2y-\frac{y^2}{\pi}\sum_{\mathbf{n}(3)}{}'\frac{1}{n^2(n^2\pi^2+y^2)}\ , \tag{91}$$

where

$$A(\boldsymbol{\epsilon})=\sum_{\mathbf{n}(3)}{}'\frac{\cos[2\pi(\mathbf{n}\cdot\boldsymbol{\epsilon})]}{n^2}\quad(\epsilon>0)\ , \tag{92}$$

while C_3 is a universal number of value $-8.913\,633\ldots$; it will be noted that, for reasons of symmetry, it is sufficient to consider $0\leq\epsilon_j\leq\frac{1}{2}$. Now, combining (90) and (91), and remembering that in the temperature regime under study $y=O(a/L)^{1/2}\ll 1$, we obtain

$$g(\boldsymbol{\epsilon})\approx\frac{A(\boldsymbol{\epsilon})}{\pi}-\frac{1}{\epsilon}-\frac{C_3}{\pi}\quad(0\leq\epsilon_j\leq\tfrac{1}{2})\ . \tag{93}$$

For comparison, we recall that the R-dependent part of the *bulk* correlation function is given by

$$G_0(\mathbf{R},T)\equiv G(\mathbf{R},T;\infty)-M_0^2(T)=\frac{Ta}{8\pi JR}\ , \tag{94}$$

whereby the ratio

$$G^*(\mathbf{R},T;L)/G_0(\mathbf{R},T)=\epsilon g(\boldsymbol{\epsilon})=O(1)\ . \tag{95}$$

Some special values of this ratio may be noted:

$$\frac{\sqrt{3}}{2}g(\tfrac{1}{2},\tfrac{1}{2},\tfrac{1}{2})=0.762\,68\ ,$$
$$\frac{1}{\sqrt{2}}g(0,\tfrac{1}{2},\tfrac{1}{2})=0.594\,37\ , \tag{96}$$
$$\tfrac{1}{2}g(0,0,\tfrac{1}{2})=0.370\,68\ .$$

As $\epsilon\to 0$, this ratio tends to the limiting value $(2\pi/3)\epsilon^3$; see Eq. (82c).

(ii) *The cylindrical geometry ($d^*=2$).* For this case we recall identities (42) and (44) of II, viz.

$$\sum_{\mathbf{q}(2)}\frac{e^{-2y|\mathbf{q}+\boldsymbol{\epsilon}|}}{|\mathbf{q}+\boldsymbol{\epsilon}|}=\pi\sum_{\mathbf{n}(2)}\frac{\cos[2\pi(\mathbf{n}\cdot\boldsymbol{\epsilon})]}{(n^2\pi^2+y^2)^{1/2}}\quad(\epsilon>0) \tag{97}$$

and

$$\sum_{\mathbf{q}(2)}{}'\frac{e^{-2yq}}{q}=\frac{\pi}{y}+D+2y+\sum_{\mathbf{n}(2)}{}'\left[\frac{\pi}{(n^2\pi^2+y^2)^{1/2}}-\frac{1}{n}\right], \tag{98}$$

where D is a universal number of value $-3.900\,265\ldots$; note that the sums appearing in (97) are full, in the sense that the terms with $\mathbf{q}=0$ and $\mathbf{n}=0$ are included. With $y\ll 1$, the function $g(\boldsymbol{\epsilon})$ now turns out to be

$$g(\boldsymbol{\epsilon})\approx P(\boldsymbol{\epsilon})-\frac{1}{\epsilon}-D\quad(0\leq\epsilon_j\leq\tfrac{1}{2})\ , \tag{99}$$

where

$$P(\boldsymbol{\epsilon})=\sum_{\mathbf{n}(2)}{}'\frac{\cos[2\pi(\mathbf{n}\cdot\boldsymbol{\epsilon})]}{n}\quad(\epsilon>0)\ . \tag{100}$$

The ratio of the finite-size effect, $G^*(\mathbf{R},T;L)$, to the R-dependent part of the bulk correlation function, $G_0(\mathbf{R},T)$, is again $\epsilon g(\boldsymbol{\epsilon})$ which, as expected, is of order unity; some special values of this ratio are

$$\frac{1}{\sqrt{2}}g(\tfrac{1}{2},\tfrac{1}{2})=0.615\,54,\quad \tfrac{1}{2}g(0,\tfrac{1}{2})=0.378\,95\ . \tag{101}$$

Finally, as $\epsilon\to 0$, the ratio $\epsilon g(\epsilon)$ tends to the limiting value $\zeta(3/2)\beta(3/2)\epsilon^3$; see Eq. (82b).

(iii) *The film geometry* ($d^*=1$). In this case the limit $y\to 0$ may be taken right away, with the result that

$$g(\epsilon)\approx \sum_{q=-\infty}^{\infty}{}' \left[\frac{1}{|q+\epsilon|}-\frac{1}{|q|}\right]$$

$$=2\psi(1)-\psi(1+\epsilon)-\psi(1-\epsilon)\quad (0\le\epsilon\le\tfrac{1}{2})\ , \tag{102}$$

where $\psi(z)$ $[=d\ln\Gamma(z)/dz]$ is the digamma function.[15] The ratio $\epsilon g(\epsilon)$ in this case admits of an *exact* expansion in ascending powers of ϵ, viz.

$$\epsilon g(\epsilon)=2\sum_{s=1}^{\infty}\zeta(2s+1)\epsilon^{2s+1}\ . \tag{103}$$

Some special values of this ratio may be noted:

$$\begin{aligned}\tfrac{1}{2}g(\tfrac{1}{2})&=2\ln 2-1=0.386\,29\ ,\\ \tfrac{1}{3}g(\tfrac{1}{3})&=\ln 3-1=0.098\,61\ ,\\ \tfrac{1}{4}g(\tfrac{1}{4})&=\tfrac{3}{2}\ln 2-1=0.039\,72\ ,\\ \tfrac{1}{6}g(\tfrac{1}{6})&=\tfrac{2}{3}\ln 2+\tfrac{1}{2}\ln 3-1=0.011\,40\ ,\end{aligned} \tag{104}$$

tending to the limiting value $2\zeta(3)\epsilon^3$ as $\epsilon\to 0$; see Eqs. (82a) and (103).

C. Case 3: $R_\perp=O(L)$, $R_\parallel\neq 0$

For a study of this case we go back to Eq. (88) and examine the influence of the component $\mathbf{R}_\parallel$ on the function $g(\epsilon)$. In this connection we note that, since $\mathbf{R}_\parallel$ is irrelevant in the case of a cube, we have to consider the geometries of the cylinder and the film only.

(i) *The cylindrical geometry* ($d^*=2$). To analyze this case we need a generalization of identity (97), which turns out to be

$$\sum_{\mathbf{q}(2)}\frac{e^{-2y[(\mathbf{q}+\epsilon_\perp)^2+\epsilon_\parallel^2]^{1/2}}}{[(\mathbf{q}+\epsilon_\perp)^2+\epsilon_\parallel^2]^{1/2}}=\pi\sum_{\mathbf{n}(2)}\frac{\cos[2\pi(\mathbf{n}\cdot\epsilon_\perp)]e^{-2\epsilon_\parallel[n^2\pi^2+y^2]^{1/2}}}{(n^2\pi^2+y^2)^{1/2}}\quad(\epsilon>0)\ , \tag{105}$$

where $\epsilon_\perp=\mathbf{R}_\perp/L$ and $\epsilon_\parallel=\mathbf{R}_\parallel/L$. Substituting (98) and (105) into (88) and remembering that $y\ll 1$, we now obtain

$$g(\epsilon)\approx-\frac{\pi}{y}(1-e^{-2\epsilon_\parallel y})+\left[\sum_{\mathbf{n}(2)}{}'\frac{\cos[2\pi(\mathbf{n}\cdot\epsilon_\perp)]e^{-2\pi n\epsilon_\parallel}}{n}-\frac{1}{\epsilon}-D\right]\quad(0\le\epsilon_j\le\tfrac{1}{2})\ . \tag{106}$$

For $\epsilon_\parallel=0$, the first part of this expression vanishes while the second part reduces to (99), as it indeed should. As $\epsilon_\parallel$ increases to values of order unity but is still much less than $1/y$, i.e., $R_\parallel=O(L)\ll\xi$, the first part approximates $-2\pi\epsilon_\parallel$, which is comparable in value to the second part. As $\epsilon_\parallel$ becomes much greater than unity and $O(1/y)$, i.e., $L\ll R_\parallel=O(\xi)$, the first part completely dominates and we obtain, for the *full* correlation function (86),

$$G(\mathbf{R},T;L)\approx\left[1-\frac{T}{T_c}\right]-\frac{Ta}{8JLy}(1-e^{-2\epsilon_\parallel y})\ . \tag{107}$$

It will be noted that the finite-size effect due to $R_\parallel$ is, quite generally, negative—as also seen in case 1. Now, making use of the fact that $y=L/2\xi$ while ξ is given by Eq. (58) with $d=3$ and $d'=1$, i.e., by

$$\xi\approx\frac{4J}{a}\left[\frac{1}{T}-\frac{1}{T_c}\right]L^2\ , \tag{108}$$

we find a remarkable cancellation of the R-independent terms in (107) and are left with the result

$$G(\mathbf{R},T;L)\approx\frac{Ta\xi}{4JL^2}e^{-R_\parallel/\xi}\quad[L\ll R_\parallel=O(\xi)]\ . \tag{109}$$

Using (62), we may as well write

$$G(\mathbf{R},T;L)\approx M_0^2(T)e^{-R_\parallel/\xi}\quad[L\ll R_\parallel=O(\xi)]\ . \tag{110}$$

The foregoing results are highly instructive for the following reasons. First of all, they demonstrate mathematically what is indeed expected on physical grounds,[18] namely that in a system which is partially finite and partially infinite the propagation of long-range order at $T<T_c$ is severely limited; while in the finite directions it pervades all the way [for $R_\perp$ is at most $O(L)$ and $L\ll\xi$], in the infinite directions it is restricted to distances governed by the length scale ξ. As $R_\parallel$ becomes much greater than L and assumes values of order ξ, long-range order in the conventional sense is not seen; cf. Eqs. (9) and (110). At the same time, the finite-size effect in the correlation function is no longer a "correction" to the standard bulk result; it becomes, in fact, the most dominant feature of the correlation function. Furthermore, the manner in which the correlation function under these circumstances varies with $\mathbf{R}$ has little to do with the three-dimensional bulk system; it becomes instead characteristic of a *one-dimensional* bulk system; cf. Eq. (109) with Eq. (A18) of the Appendix. This comparison suggests that Eq. (109) may as well be written as

$$G(\mathbf{R},T;L^2\times\infty^1)\approx\left(\frac{a}{L}\right)^2 G(\mathbf{R}_\parallel,T;\infty^1)$$

$$[L \ll R_\parallel = O(\xi)] \ . \qquad (111)$$

Thus, under specified conditions, the correlation function of the finite-sized system "splits" into two factors—a constant one pertaining to the finite dimensions of the system and a variable one pertaining to the infinite dimensions; of course, the correlation length entering into the latter still pertains to the actual system.

(ii) *The film geometry ($d^*=1$).* For the analysis of this case we employ the one-dimensional identity[19]

$$\sum_{q=-\infty}^{\infty}\frac{e^{-2y[(q+\epsilon_\perp)^2+\epsilon_\parallel^2]^{1/2}}}{[(q+\epsilon_\perp)^2+\epsilon_\parallel^2]^{1/2}}=2\sum_{n=-\infty}^{\infty}\cos(2\pi n\epsilon_\perp)K_0[2\epsilon_\parallel(n^2\pi^2+y^2)^{1/2}] \quad (y>0,\ \epsilon_\parallel>0) \ , \qquad (112)$$

along with the standard sum

$$\sum_{q=-\infty}^{\infty}{}'\ \frac{e^{-2y|q|}}{|q|}=-2\ln(1-e^{-2y}) \quad (y>0) \ , \qquad (113)$$

to obtain (for $y\ll 1$)

$$g(\boldsymbol{\epsilon})\approx 2[\ln(2y)+K_0(2\epsilon_\parallel y)] + \left[4\sum_{n=1}^{\infty}\cos(2\pi n\epsilon_\perp)K_0(2\pi n\epsilon_\parallel)-\frac{1}{\epsilon}\right] \ . \qquad (114)$$

Once again, we find that, as $\epsilon_\parallel$ becomes much greater than unity and assumes values of order $(1/y)$, the first part of the above expression for $g(\boldsymbol{\epsilon})$ completely dominates—with the result that the *full* correlation function of the system is now given by

$$G(\mathbf{R},T;L)\approx\left(1-\frac{T}{T_c}\right)+\frac{Ta}{4\pi JL}[\ln(L/\xi)+K_0(R_\parallel/\xi)]$$

$$[L \ll R_\parallel = O(\xi)] \ . \qquad (115)$$

Recalling the relevant expression for ξ [see Eq. (61) and remember that the unknown constant in the present case is unity], we again find an exact cancellation of the **R**-independent terms, leaving us with the result

$$G(\mathbf{R},T;L)\approx\frac{Ta}{4\pi JL}K_0(R_\parallel/\xi) \quad [L \ll R_\parallel = O(\xi)] \ . \qquad (116)$$

Not surprisingly, the **R** dependence of the correlation function is now characteristic of a *two-dimensional* bulk system, see Eq. (A19) of the Appendix, such that

$$G(\mathbf{R},T;L^1\times\infty^2)\approx\left(\frac{a}{L}\right)^1 G(\mathbf{R}_\parallel,T;\infty^2)$$

$$[L \ll R_\parallel = O(\xi)] \ . \qquad (117)$$

Equations (111) and (117) call for an obvious generalization.

VII. "SPLITTING THEOREM" IN GENERAL GEOMETRY

For a generalization of the foregoing results, we go back to Eq. (43) for the full correlation function and, recognizing the fact that the geometry under consideration is $L^{d^*}\times\infty^{d'}$ with $d^*+d'=d$, render it into the form

$$G(\mathbf{R},T;L)=\frac{T}{4\pi^{d'/2}J}\left(\frac{a}{L}\right)^{d-2}\left(\frac{R_\parallel}{L}\right)^{(2-d')/2}\sum_{\mathbf{n}(d^*)}\frac{\cos[2\pi(\mathbf{n}\cdot\mathbf{R}_\perp)/L]K_{(2-d')/2}[2(n^2\pi^2+y^2)^{1/2}R_\parallel/L]}{(n^2\pi^2+y^2)^{(2-d')/4}} \quad (0<d'\le 2) \ . \qquad (118)$$

Now, separating the $\mathbf{n}(d^*)=0$ term from the rest, we may write (for $y\ll 1$)

$$G(\mathbf{R},T;L)\approx\frac{T}{4\pi^{d'/2}J}\left(\frac{a}{L}\right)^{d-2}\left(\frac{R_\parallel}{L}\right)^{(2-d')/2}\times\left[\left(\frac{2\xi}{L}\right)^{(2-d')/2}K_{(2-d')/2}(R_\parallel/\xi)+\sum_{\mathbf{n}(d^*)}{}'\ \frac{\cos[2\pi(\mathbf{n}\cdot\mathbf{R}_\perp)/L]K_{(2-d')/2}(2\pi nR_\parallel/L)}{(n\pi)^{(2-d')/2}}\right] \ . \qquad (119)$$

We now make the following observations on this expression.

(i) If we keep **R** fixed and let L (and along with it ξ) increase indefinitely, then the first term of this expression reduces to

$$\Gamma\left(\frac{2-d'}{2}\right)\frac{Ta^{d-2}\xi^{2-d'}}{2(4\pi)^{d'/2}JL^{d-d'}} \quad (0<d'<2) \ , \qquad (120)$$

which, with the help of Eq. (58), reproduces exactly the long-range order term $1-(T/T_c)$. In the second part, we may, under these conditions, replace the summation over $\mathbf{n}(d^*)$ by an integration, which yields precisely the isotropic term characteristic of the d-dimensional bulk system, viz.

$$\Gamma\left(\frac{d-2}{2}\right)\frac{Ta^{d-2}}{8\pi^{d/2}JR^{d-2}} \quad [(R/\xi)\to 0] \ . \qquad (121)$$

For $d'=2$, the approach has to be slightly different. Now, for the first part we use the standard approximation[15]

$$K_0(z) \approx -[\ln(\tfrac{1}{2}z)+\gamma_E] \quad (z \to 0) , \tag{122}$$

where γ_E is Euler's constant, while for the sum over n (which, in the present case, can only be one-dimensional because, with $d'=2$, $0<d^*<2$) we use a tabulated result,[20] namely

$$\sum_{n=1}^{\infty} \cos(pn)K_0(qn) \approx \frac{\pi}{2(p^2+q^2)^{1/2}} + \frac{1}{2}\left[\ln\left[\frac{q}{4\pi}\right]+\gamma_E\right] \quad [(p,q)\to 0] . \tag{123}$$

Expression (119) then reduces to

$$\frac{Ta}{4\pi JL}\left[\ln\left[\frac{\xi}{L}\right]+\frac{L}{2R}\right] \quad (d=3,\ d'=2) . \tag{124}$$

Using (61), this becomes precisely the result expected of a three-dimensional bulk system, viz.

$$\left[1-\frac{T}{T_c}\right]+\frac{Ta}{8\pi JR} . \tag{125}$$

(ii) If, on the other hand, we keep L fixed and increase $R_\parallel$ indefinitely, we encounter a very different situation. Now, as $R_\parallel$ becomes much larger than L, the second term of expression (119) becomes insignificant; the correlation function is then dominated by the first part which yields a result characteristic of a *d'-dimensional* bulk system; see Eq. (17) of the appendix. Thus, quite generally (for $0<d'\le 2$), we obtain what may be called the "splitting theorem" for the correlation function, namely

$$G(\mathbf{R},T;L^{d^*}\times\infty^{d'}) \approx \left[\frac{a}{L}\right]^{d^*} G(R_\parallel,T:\infty^{d'}) \quad [L \ll R_\parallel = O(\xi)] . \tag{126}$$

The physical interpretation of this theorem is straightforward; it must, however, be emphasized that the correlation length ξ appearing in the function $G^{(d')}$ pertains to the actual system in geometry $L^{d^*}\times\infty^{d'}$ and *not* to the d'-dimensional bulk system.

The exceptional case of the "block" geometry ($d'=0$) is worthy of a special note. Here, $R_\parallel$ is irrelevant, so the foregoing considerations (and those of case 3 of the preceding section) do not apply. The appropriate results for this geometry are given under cases 1 and 2 above, and the underlying feature of these results is that long-range order at $T<T_c$ pervades the *whole* system. Finite-size effect in the correlation function may in this case stay as a small correction to the R-dependent bulk term ($R \ll L$) or become comparable to it [$R=O(L)$] but does not, under any circumstances, mollify the $\mathbf{R}$-independent term, $1-(T/T_c)$, representing the long-range order in the system.

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APPENDIX

In this appendix we propose to examine the behavior of a d'-dimensional *bulk* system ($d'\le 2$) as $T\to T_c(d')=0$. The constraint equation for the system in zero field is given by the standard expression

$$\sum_{\{n_j\}} \frac{1}{\lambda-2J\sum_j \cos(2\pi n_j/N_j)} = 2N\beta \quad [n_j=0,1,\ldots,(N_j-1)] , \tag{A1}$$

where various symbols have their usual meanings; in particular, $\beta=1/T$. Following a straightforward procedure,[11] Eq. (A1) can be put in the form

$$2N\beta J = \tfrac{1}{2}\sum_{\{n_j\}} \int_0^\infty e^{-(\lambda/2J)x} \prod_j e^{x\cos(2\pi n_j/N_j)} dx , \tag{A2}$$

which, in the bulk limit, becomes

$$2\beta J = \tfrac{1}{2}\int_0^\infty e^{-(\lambda/2J)x}\left[\frac{1}{2\pi}\int_0^{2\pi} e^{x\cos\theta} d\theta\right]^{d'} dx = W_{d'}(\phi)$$

where

$$W_{d'}(\phi) = \tfrac{1}{2}\int_0^\infty e^{-\phi x/2}[e^{-x}I_0(x)]^{d'}dx \quad \left[\phi=\frac{\lambda}{J}-2d'\right] , \tag{A3}$$

$I_\nu(z)$ being the other modified Bessel functions. For $\phi \ll 1$ and $d'<2$, the integral in (A3) may be evaluated by substituting[15]

$$I_0(x) \approx \frac{e^x}{\sqrt{2\pi x}}\left[1+\frac{1}{8x}+\frac{9}{128x^2}+\cdots\right] , \tag{A4}$$

with the result that

$$2\beta J = \Gamma\left[\frac{2-d'}{2}\right]\frac{1}{(4\pi)^{d'/2}}\phi^{-(2-d')/2}+O(\phi^{d'/2}) . \tag{A5}$$

It follows that, in the limit $T\to 0$,

$$\phi \sim T^{2/(2-d')} . \tag{A6}$$

The correlation length, ξ, the zero-field susceptibility, χ, and the singular part of the specific heat per unit "volume," $c^{(s)}$, are then given by

$$\xi = a/\sqrt{\phi} \sim T^{-1/(2-d')} , \tag{A7}$$

$$\chi = 1/(2Ja^{d'}\phi) \sim T^{-2/(2-d')} , \tag{A8}$$

and

$$c^{(s)} = -(J/a^{d'})(d\phi/dT) \sim T^{d'/(2-d')} , \tag{A9}$$

with respective exponents

$$\dot{\nu}=\frac{1}{(2-d')},\quad \dot{\gamma}=\frac{2}{(2-d')},\quad \dot{\alpha}=-\frac{d'}{(2-d')}\quad (d'<2)\ . \tag{A10}$$

For $d'=2$, on the other hand, one obtains[11]

$$2\beta J=\frac{1}{4\pi}\ln(32/\phi)+O(\phi\ln\phi)\ , \tag{A11}$$

with the result that

$$\phi\sim 32\exp(-8\pi J/T) \tag{A12}$$

whence

$$\xi\sim\exp(4\pi J/T),\quad \chi\sim\exp(8\pi J/T)\ ,$$
$$c^{(s)}\sim T^{-2}\exp(-8\pi J/T)\quad (d'=2)\ . \tag{A13}$$

The correlation function of the system is given by

$$G(\mathbf{R},T)=\frac{T}{2N}\sum_{\{n_j\}}\frac{\cos(\mathbf{k}\cdot\mathbf{R})}{\lambda-2J\sum_j\cos(k_ja)}\quad\left[k_j=\frac{2\pi n_j}{N_ja}\right]\ . \tag{A14}$$

Proceeding as before, we obtain in the bulk limit

$$G(\mathbf{R},T)=\frac{T}{4J}\int_0^\infty e^{-\phi x/2}\prod_j[e^{-x}I_{R_j/a}(x)]dx\ . \tag{A15}$$

For the functions $I_{\nu_j}(x)$ we may use the asymptotic expression[1]

$$I_{\nu_j}(x)\approx\frac{e^{x-\nu_j^2/2x}}{\sqrt{2\pi x}}\left[1+\frac{1}{8x}+\frac{9-32\nu_j^2}{128x^2}+\cdots\right]\ , \tag{A16}$$

with the result that, for $\phi\ll 1$,

$$G(\mathbf{R},T)=\frac{T}{2(2\pi)^{d'/2}J}\left[\frac{R\xi}{a^2}\right]^{(2-d')/2}K_{(2-d')/2}(R/\xi)\ , \tag{A17}$$

where $\xi(=a/\sqrt{\phi})$ is the correlation length of the system. For the special cases $d'=1$ and 2,

$$G(\mathbf{R},T)=\frac{T}{J}\times\begin{cases}(\xi/4a)\exp(-R/\xi)\quad (d'=1)\ , & \text{(A18)}\\ (1/4\pi)K_0(R/\xi)\quad (d'=2)\ . & \text{(A19)}\end{cases}$$

[1]S. Singh and R. K. Pathria, Phys. Rev. B **31**, 4483 (1985), herein referred to as I.

[2]V. Privman and M. E. Fisher, Phys. Rev. B **30**, 322 (1984).

[3]S. Singh and R. K. Pathria, Phys. Rev. Lett. **55**, 347 (1985).

[4]J. Shapiro, Phys. Rev. Lett. **56**, 2225 (1986).

[5]S. Singh and R. K. Pathria, Phys. Rev. Lett. **56**, 2226 (1986).

[6]A preliminary version of these results has already appeared; see S. Singh and R. K. Pathria, Phys. Rev. B **33**, 672 (1986).

[7]T. D. Schultz, E. H. Lieb, and D. C. Mattis, Rev. Mod. Phys. **36**, 856 (1964).

[8]G. S. Joyce, in *Phase Transitions and Critical Phenomena*, edited by C. Domb and M. S. Green (Academic, New York, 1972), Vol. 2, pp. 375–442.

[9]M. E. Fisher, M. N. Barber, and D. Jasnow, Phys. Rev. A **8**, 1111 (1973).

[10]M. E. Fisher and V. Privman, Phys. Rev. B **32**, 447 (1985).

[11]M. N. Barber and M. E. Fisher, Ann. Phys. (N.Y.) **77**, 1 (1973).

[12]M. E. Fisher and V. Privman, Commun. Math. Phys. **103**, 527 (1986).

[13]J. M. Luck, Phys. Rev. B **31**, 3069 (1985).

[14]E. Brézin, J. Phys. (Paris) **43**, 15 (1982).

[15]M. Abramowitz and I. A. Stegun, *Handbook of Mathematical Functions* (Dover, New York, 1970).

[16]G. H. Hardy, Mess. Math. **49**, 85 (1919); M. L. Glasser, J. Math. Phys. **14**, 409, 701 (1973); I. J. Zucker, *ibid.* **15**, 187 (1974).

[17]A. N. Chaba and R. K. Pathria, J. Phys. A **9**, 1411 (1976), herein referred to as II.

[18]V. Privman and M. E. Fisher, J. Stat. Phys. **33**, 385 (1983).

[19]A. N. Chaba and R. K. Pathria, J. Phys. A **10**, 1823 (1977).

[20]I. S. Gradshteyn and I. M. Ryzhik, *Table of Integrals, Series and Products* (Academic, New York, 1980), p. 978.