Breakdown properties of quenched random systems: The random-fuse network

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An analysis of a prototypical percolation model (the fuse network) for breakdown in quenched random systems is given. The breakdown voltage and the topology of the eventual breakdown path are studied analytically and numerically. New scaling concepts, based on the most critical defect in the network, combined with standard percolation scaling ideas, lead to a complete picture of the strength of the network. The mean breakdown strength and the distribution of breakdown strengths are derived in the different concentration regimes. The breakdown path is described by new order parameters on approach to p_c. One, the number of bonds broken in the breakdown process, is studied in detail. Many models and physical systems should show an analogous behavior and simplified models for two of these problems, brittle fracture and dielectric breakdown in solids, are discussed.

I. INTRODUCTION

Electrical and mechanical breakdown are factors which limit many practical uses of modern ceramics, thin films, insulators, and other structural and electrical materials. Despite intense study,[1] there is little basic understanding of microscopic breakdown processes with most studies simply modeling experimental data at an empirical or phenomenological level. Nevertheless, it is clear that defects are of primary importance in all breakdown processes and often a few critical defects can determine the breakdown strength of the entire system. Recently, statistical-mechanical models based upon defects and percolation clusters have been introduced to study breakdown phenomena in the case of quenched random defects. For example, a percolation model for dielectric breakdown was introduced by Takayasu[2] and the nature of the breakdown paths was studied. The random-fuse network was introduced by de Arcangelis *et al.*[3] who performed numerical simulations. Models for brittle fracture have been recently introduced by Sahimi and Goddard,[4] who also consider continuous distribution functions of bond strengths. Percolation models of breakdown have been studied in detail by the present authors who presented a new formula for the size dependence of the mean breakdown strengths,[5] and for the distribution of breakdown strengths.[6] This paper contains a clarification and expansion of the results contained in Refs. 5 and 6, and in addition presents a study of an order parameter describing the geometry of the eventual breakdown path.

The random-fuse network model is also the scalar analog to the vector problem of brittle fracture or mechanical breakdown. Just as one introduces a network of fuses in the case considered here, one can introduce a network of breakable rods for the brittle fracture case. Each rod in the network would be stable up to a breaking point at which it would separate and fail.[4] In this case the current enhancement at the edges of the defects would be replaced by stress enhancement. The difference would be that the equations for the resulting stress distribution are vector equations whereas the electrical analog discussed here contains only scalar equations. In the vector case, there is also the question as to whether one considers purely central forces or whether one also includes bending forces and whether the bonds are completely elastic or not. These considerations affect the location and nature of the percolation critical point and the relevant critical exponents, but do not affect the methodology used in obtaining the scaling behavior for p near the pure limit. Thus, we believe that there is qualitatively similar behavior in the fuse and brittle fracture cases, and the discussion in the context of brittle fracture is outlined in Appendix C. In this paper, the breakdown strength of the random fuse network is studied analytically, using methods specifically designed to take into account the pronounced effect of the most critical defects. The predictions made using these methods are tested numerically by computer simulations of the two-dimensional fuse network. The predictions that are of immediate practical interest, and which are also true for dielectric breakdown and brittle fracture, have been recently reported.[5,6] The methods presented here for the fuse network are thus of much wider applicability, and with this in mind, we have written the paper in a pedagogical way to allow easy access to the methods and results for application to similar problems. In addition, two problems to which we believe the results and methods are immediately applicable are outlined in Appendix C.

The paper is arranged as follows. Section II contains the model along with its behavior in the pure limit. Sec-

tion III contains the analytic methods we have used to qualitatively study the average breakdown strengths of the fuse network in the various regions of concentration p of defects. Section IV contains a discussion of the analytic methods we have developed for the distribution of breakdown strengths expected for an ensemble of fuse networks. Section V contains the tests of these analytic behaviors through computer simulations of the model in the bond problem on a two-dimensional (2D) square lattice. Finally, Sec. VI contains our final conclusions and comments on the shortcomings of our calculations and suggests for future research directions. There are three appendices, two of which give details of certain results quoted in the body of the paper, and the third of which outlines problems that are closely related to the fuse network discussed in the main body of the paper.

II. MODEL

Make a random network of fuses. For simplicity, consider each fuse to have a resistance of 1 Ω and a breaking point of 1 A (1 V). Its current-voltage characteristic is shown in Fig. 1. Above 1 V, the fuse becomes an insulator. Now place such identical fuses at random on the bonds of a d-dimensional hypercubic lattice until a fraction (or concentration) p of the bonds are occupied. The remaining fraction $(1-p)$ of vacant bonds act as insulators. There is a fuse network only when the concentration is not below percolation threshold ($p_c \leq p \leq 1$). As illustrated in Fig. 2, above percolation threshold (in the infinite lattice) there is a connected path or network of fuses across the lattice. Now consider that two busbars are separately installed horizontally across the top and bottom of the sample (see Fig. 2) and that a voltage difference or electric field is applied vertically between the two bars. If a sufficiently small voltage (certainly anything less than 1 V is small enough) is applied then the system conducts just as a random-resistor network in the ordinary percolation problem. Now if this externally applied voltage V is sufficiently large some of the fuses will break. And if enough fuses break there will no longer be a connecting path between the busbars so we will say that there has been a breakdown of the entire network. The geometry of a typical random-fuse network is shown in Fig. 2 for an 11×11 square lattice. The "breakdown

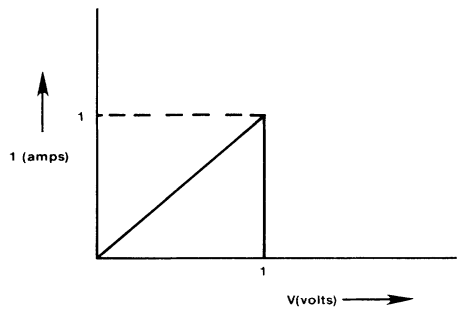

FIG. 1. I-V characteristic for a single fuse. Above a voltage of 1 V, no current flows.

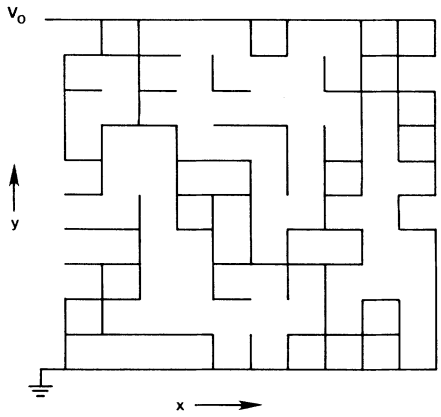

FIG. 2. Geometry of the random-fuse network on a square lattice. V_0 is the externally applied voltage and the model is defined above the percolation threshold, where a connected pathway of fuse exists.

strength" of a particular configuration is the lowest extensively applied voltage V_b or current I_b at which the network breaks down.

One useful way of calculating the breakdown voltages in the fuse network is to perform the following, two-step process iteratively[3] for each lattice configuration: (1) Solve Kirchhoff's equations (the numerical procedure we use will be described in detail in Sec. VI); (2) Find the fuse which contains the most current (the "hottest fuse") and remove it. As the set of equations to be solved is linear, it is easy for each step in the iterative process to find the external voltage required to break the hottest fuse. One then obtains a sequence of breakdown voltages or currents, which is typified in Fig. 3 for a particular configuration on a 50×50 square lattice. We introduce the notation that the external voltage at which the first fuse breaks is V_1 (the corresponding current is I_1) and the *maximum* of the external voltages (the maximum of Fig. 3) required to break a hottest fuse is V_b (the maximum of the externally applied currents required to break this fuse is I_b). We also define the number, N_b, of fuses that are broken before the network disconnects. At this point is it useful to note the difference between the case of an external voltage source as compared to the case of an external current source. In the latter case, there is no conductivity factor involved, and so in many ways this is the more natural configuration. However, the values at which the first fuse breaks are simply related by the conductivity of the percolation network, which is known to be finite for all p away from p_c and which goes to zero with exponent t on approach to p_c. These properties, and the fact that V_1 and V_b are qualitatively the same in the thermodynamic limit (see Sec. V) means that the behavior of the models used here are qualitatively the same (up to the percolation conductivity factor) for either an external current source or an external voltage source.

Let us now consider the pure limit ($p=1$) when there are no defects in the network, as illustrated in Fig. 4 for a $L\times L$ square lattice. Clearly the total resistance of an L^d hypercubic lattice is $1/L^{d-2}$ Ω. Since all the current

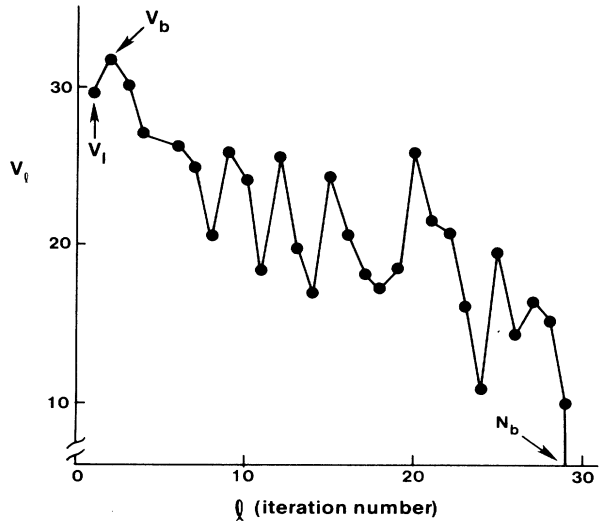

FIG. 3. Sequence of external voltages that induce failure in the hottest fuse in the network. V_l is the external voltage that need to be applied to break the hottest bond at the lth step of the iterative process defined in Sec. II; V_1 is the voltage at which the first fuse blows, and V_b is the maximum (V_l). N_b is the number of iterations to failure of the network. The calculation which lead to the figure was for one configuration on a 70×70 square lattice at $p=0.75$.

flows vertically through columns each of which is a series of fuses of total resistance L Ω, and there are L^{d-1} such columns in parallel. The external voltage needed for 1 A to flow in each column is then L V. And since the same current flows in each vertical fuse, we find that all fuses break at once so that $V_1=V_b$, and this critical external electric field and current density are given by

$$\epsilon_b=V_b/L=\epsilon_1=V_1/L=1 \tag{1a}$$

and

$$i_b=I_b/L^{d-1}=I_1/L^{d-1}=1\ , \tag{1b}$$

and all the vertical fuses break so that

$$N_b=L^d/d\ . \tag{1c}$$

A single defect placed in the horizontal hyperplane of the network has no effect on the properties of the network. However, when placed on a vertical bond (as shown for the 2D square lattice in Fig. 5) a *single* defect has a rather dramatic effect. By the superposition principle in this linear system, the voltages and currents around this single defect are equal to the sum of those in the unaltered pure network with the same external source plus those of a pure network with the external sources removed and a dipole voltage source placed along the defect bond. This second problem may then be solved in terms of the equilibrium Green's functions of the pure system. It is straightforward to show that the explicit expression for the current in a bond a vector distance r from the dipole source, $I(r)$, is;

$$I(r)=2[G(r+j)-G(r)]\ , \tag{2}$$

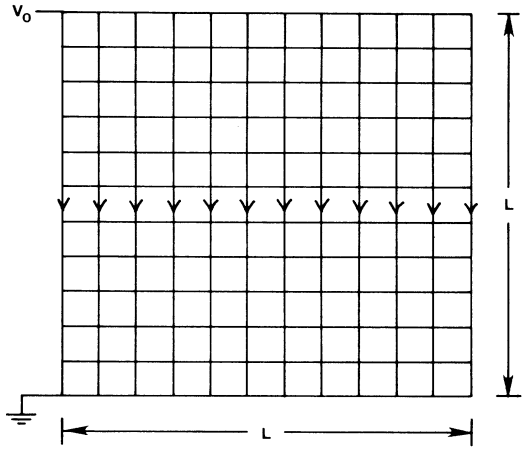

FIG. 4. Pure network for a $L\times L$ square lattice, the vertical bonds all carry the same current. The horizontal bonds carry no current.

where j is a unit displacement in the vertical direction, and the Green's function $G(r)$ is given by

$$G(r)=\frac{1}{L^d}\sum_k \exp(ik_0r)\Big/\left[1-(1/d)\sum_{i=1}^{d}\cos k_i\right] \tag{3}$$

for hypercubic lattices. It is evident from Fig. 5 that if we consider the breakdown of the network with one vertical bond removed, it is the vertical bonds horizontally adjacent to the defect bond which carry the most current and hence break first with increasing external voltage. From Eqs. (2) and (3), for the square lattice, we find

$$I=i+2[G(0,1)-G(1,1)]=i+4(1/\pi-\tfrac{1}{4})i=4i/\pi\ , \tag{4}$$

where i is the externally applied current per column ($I=iL^{d-1}$). Now, the bond labeled 1 (see Fig. 5) breaks when $I_{\text{fuse }1}=1$, so by setting Eq. (4) equal to 1, we find $i_1=\pi/4$. After the first fuse breaks the fuses adjacent to

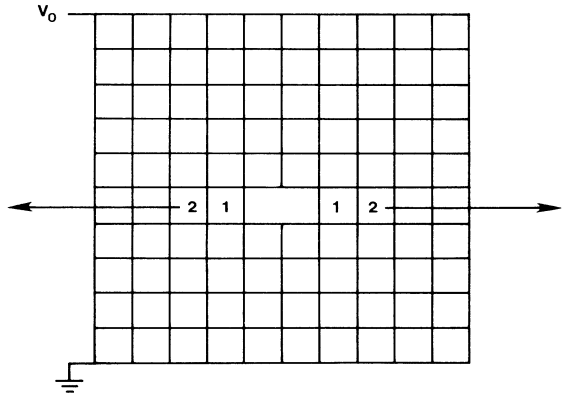

FIG. 5. Single-defect problem on a square lattice. The bonds labeled 1 break first. The bonds labeled 2 then feel an enhanced current, and hence break. A crack propagates outward in the direction of the arrows in the figure.

them (labeled 2 in Fig. 5) feel an even greater enhancement of current and hence immediately also break. Indeed, there is then a cascade throughout this horizontal hyperplane which entirely severes the network into two parts. Therefore, for a single vertical defect, we find

$$i_b = i_1 = a \tag{5a}$$

or

$$\epsilon_1 = \epsilon_b = V_1/L = V_b/L = a(1+O(1/L)) , \tag{5b}$$

where $a = \pi/4$ for the square lattice, and $a = \{2[G(0,0,1) - G(0,1,1)] - 1\}^{-1}$ for the simple cubic lattice. The order $1/L$ correction in Eq. (5b) accounts for the change in conductivity due to removing the one bond. Contrasting this result with Eq. (1), the pure limit, we see that the existence of even a *single* defect moves the breakdown strength down from $i_b = 1$ to $i_b = \pi/4$ on the square lattice in two dimensions, with a similar effect occurring on other lattices, the reduction now being from 1 to a as given above. Therefore, any finite concentration of defects will have an even more dramatic effect. And clearly the defects dominate this problem.

III. FINITE CONCENTRATIONS OF DEFECTS

A. Dilute limit

First we consider the dilute limit where $0 < 1-p \ll 1$, where we can treat percolation clusters singly and ignore interactions or interference effects between clusters. Since, from the discussion above, it was seen that a single vertical defect has a profound effect on the breakdown strength, clearly a cluster of such defects although less probable to occur, will have an even larger effect. Indeed if n fuses are moved, a large reduction in the breakdown strength occurs if these are n vertical bonds arranged together as compactly as possible so that as much current enhancement as possible appears at the edges of the defect cluster. In two dimensions, this "most critical" defect is a horizontal line of adjacent vertical bonds (see Fig. 6), whereas in three dimensions it is a horizontal "penny-shaped crack" (which is familiar from brittle-fracture theory[1] where it is the starting point of many theoretical analyses). In general the most critical defects in the fuse network problem seem to be identical to those in the brittle fracture problem, so that the results found here may in many cases apply directly to brittle fracture theory (for a detailed transcription of the breakdown strength results see Appendix C).

It can easily be seen that the most critical defect in two dimensions is a horizontal line of neighboring vertical bonds simply because in this case the dipole fields for the removed bonds add constructively to produce the maximum current enhancement at the defect edge (whereas vertical dipoles placed end to end destructively add to produce a reduced current enhancement). In order to quantify the identification of the most critical defect, it is necessary to answer two questions: (1) What is the current enhancement at the edge of the critical defect cluster; and (2) given a defect fraction $1-p$, what is the probability that the critical defect will occur somewhere within this network?

The first question is straightforward but often tedious to answer precisely on a lattice. But the important qualitative behavior may be found from the continuum limit by using an elliptical defect in two dimensions (see Fig. 7). The three-dimensional defect that we use is made from the two-dimensional one by forming a solid of revolution about the vertical direction. Then to find the current density enhancement at the tip of the elliptical defect we solve Laplace's equation is elliptical coordinates. The details are given in Appendix A where the result is $j_{\rm tip} = j(1+a/b)$, where a and b are defined in Fig. 7 and where j is the externally applied current density. To find the lattice limit of the ellipse result, we integrate equation (A9) over the lattice spacing β, and find $i_{\rm tip} \sim (a/\beta)^{1/2}$ for $b^2 \ll \beta a$; where a and b are defined in Fig. 7. Taking $\beta = 1$, we find for a horizontal row of n removed vertical defects, that the current in the vertical bond just at one end of the defect has a square-root dependence on n; namely,

$$i_{\rm tip} \sim i(1 + k_2 n^{1/2}) \quad \text{for 2D} , \tag{6a}$$

where k_2 is a constant and i is the externally applied

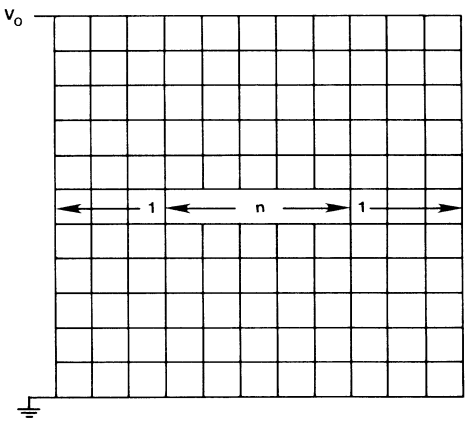

FIG. 6. "Lifshitz defect" of size n for the square lattice fuse problem. The bonds labeled 1 break first, and a crack propagates outward in the direction of the arrows.

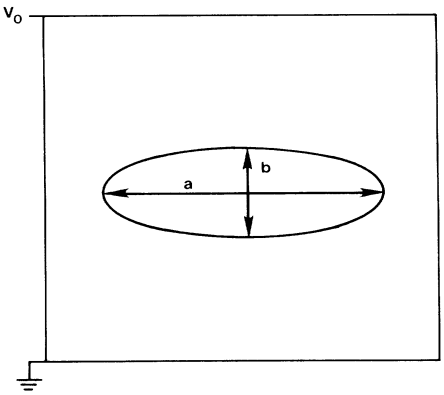

FIG. 7. Continuum representation of the "Lifshitz defect". The defect is elliptical with elongation ratio a/b.

current per bond. Essentially this formula (6a) says that the current at the tip of the defect is proportional to the square root of the total current that is being diverted by that defect. In three dimensions n defects in a disc or penny-shaped region must be removed and the corresponding result is

$$i_{\text{tip}} \sim i(1+k_3 n^{1/4}) \text{ for 3D} , \tag{6b}$$

where k_3 is a constant. In this case, the n cut bonds of the penny-shaped crack give a total diverted current proportional to $n^{1/2}$ which is spread evenly around the perimeter of size $n^{1/2}$ as it is diverted so that any single bond on the tip only carries a diverted current proportional to $n^{1/4}$. In the hypercubic lattice, the general result is

$$i_{\text{tip}} \sim i(1+k_d n^{1/2(d-1)}) . \tag{6c}$$

The single-ellipse results will certainly lead to the failure of the entire network, for as soon as the bond at the tip of the most critical defect fails, the crack is enlarged and hence feels a greater stress at its tip. The crack thus propagates and leads to the failure of the entire network. However, there may be defects that funnel a current proportional to the size of the defect cluster through the most critical bond (for example in two dimensions two horizontal cracks with one bond between them have this property). If such a defect exists, one expects $i_{\text{tip}} \sim n$. In general, this may not lead to the eventual failure of the whole network. We thus introduce two exponents, α_1 and α_b to denote the tip currents that are produced by the defect which produces the most current in a bond and the defect cluster that is first to induce complete failure, respectively. Based on the above isolated cluster arguments, the approximate inequality, $1/[2(d-1)] \leq \alpha_b \leq \alpha_1 \leq 1$ holds. A more detailed analysis of this question is deferred to a later publication. For the purpose of comparison with numerical results in Sec. V, we note that in two dimensions $\alpha_1 = 1$. The detailed analytical results of this and the next section may be discussed in terms of the exponents α_1 and α_b without reference to their specific values. In the equations that follow, we will use a generic exponent α, and it is understood that for initial breakdown quantities (such as i_1 and v_1) the appropriate exponent is α_1, while for final breakdown quantities (such as i_b and v_b), the appropriate exponent is α_b. [Here we note that if one does not take the lattice limit of the ellipse results, one obtains the approximation $\alpha_b = 1/(d-1)$, used in our previous letter on the fuse network.[5]]

The second question is what is the probability $P(n)$ that n adjacent vertical defects in a hypercube will be missing *somewhere* in the L^d network. To leading order in $L^d(1-p)^n$, the answer is

$$P(n) \sim (1-p)^n L^d , \tag{7}$$

since $(1-p)^n$ is the probability of n bonds missing and L^d measures approximately the number of places in the volume that the critical defect cluster can be placed on the lattice. The characteristic largest critical defect cluster is determined by that value of $n = n_c$ for which $p(n)$ is of order 1 or

$$(1-p)^n L^d \sim 1 , \tag{8a}$$

which implies

$$n_c \sim [-d/\ln(1-p)] \ln L . \tag{8b}$$

The important feature here is that for any nonzero $(1-p)$, a defect of size $n_c \sim \ln L$ will occur somewhere in the network. This result is true in all dimensions and for all $p_c < p < 1$ and only changes upon approach to the percolation threshold p_c when a defect of size L occurs.

With the above answers to our two primary questions we can now make a Lifshitz-type argument using Eqs. (6) and (8). Since a most critical defect of size $n_c \sim \ln L$ surely occurs somewhere in the network, and since then the current enhancement at the tip of this critical defect [Eq. (6c)] will also be of order $(\ln L)^\alpha$, then the external applied current that need be applied to produce one ampere through a bond at the tip of this defect is *reduced* by a factor of order $(\ln L)^\alpha$ from that for the pure limit. That is, setting $i_{\text{tip}} = 1$ in Eq. (6) and combining Eqs. (6) and (8) gives

$$i_1 = I_1/L \sim 1/[1+K(\ln L)^\alpha] , \tag{9a}$$

where $K = k_d[-d/\ln(1-p)]^\alpha$ is independent of sample size L. A similar dependence is found for the breakdown fields and critical voltage

$$\epsilon_1 = V_1/L \sim [\Sigma(p)]^{-1}/[1+K(\ln L)^\alpha] , \tag{9b}$$

where $\Sigma(p)$ is the conductance. Similar expressions hold for the complete breakdown current and voltage, and the number of bonds broken in the complete breakdown process is

$$N_b \sim L^{d-1} - n . \tag{10}$$

In the dilute limit, the interference of other defects or clusters can be ignored due to their low probability of occurrence in the vicinity of the rupturing cross section. We believe that Eq. (9) gives the qualitative behavior of the scaling behavior of the breakdown strength in the dilute limit for all dimensions. In particular, it is clear that in the thermodynamic limit that the breakdown fields ϵ_1 and ϵ_b; and the breakdown currents i_1 and i_b go to zero as $L \to \infty$. In Sec. V below we demonstrate that the results of the numerical simulations in two dimensions are consistent with this result.

As we all know, the breakdown strength of typical macroscopic samples is not zero. This is just an indication that even macroscopic samples are not in the thermodynamic limit since the logarithmic decay of breakdown strength is so very slow. Indeed macroscopic samples can be used to test and verify the size dependence suggested here.

The Lifshitz-type argument above is strictly applicable only in the dilute limit where the interaction between defect clusters can be ignored. At larger concentrations $(1-p)$ of defects, these interference effects must be considered. Nevertheless, the breakdown will clearly still be dominated by the extreme fluctuations rather than by the mean of the distribution of defects. So, if there are still extreme fluctuations of order $\ln L$ in an L^d sample then the decay of the mean breakdown strengths will remain logarithmic.

At percolation threshold a different behavior appears. Here a tenuous infinite cluster extends across the network. Indeed the infinite cluster at $p=p_c$ is so tenuous that it can be severed by breaking a finite number of bonds.[7] Often there is at least one bond whose breaking severs the infinite cluster (the so-called "red" bonds[7]). If a red bond exists then *all* the current in the L^d sample must go through this bond so that the critical total current is $I_1=I_b=1$ ampere since when this critical fuse breaks the network is broken. In any case (even if several bonds must carry all the current) the critical total current is of order 1 or

$$I_1=I_b=O(1) \tag{11a}$$

or

$$i_1=i_b=O(1/L)\ . \tag{11b}$$

Similarly, the critical breakdown voltage or field can be calculated since $\epsilon=\sigma^{-1}i$, but from the percolation finite-size scaling law we know that $\sigma\sim L^{-t/\nu}$. So we find

$$\epsilon_1\sim\epsilon_b\sim kL^{t/\nu-(d-1)}\ , \tag{12}$$

where k is a constant, t is the conductivity percolation critical exponent, and ν is the percolation correlation length exponent. A L^{d-2} factor has been included in Eq. (12) to correctly change from conductivity to conductance. Depending upon a comparison of t/ν with $(d-1)$ the breakdown field strength may either diverge or go to zero with increasing sample size. In three dimensions[8] $t/\nu-(d-1)>0$ so that the breakdown strength will diverge at percolation threshold. In two dimensions t and ν are approximately equal [the precise belief at present is $t/\nu=0.973$ (Ref. 9)] so that the behavior of Eq. (12) is nearly constant. This issue is discussed further in the numerical analysis of Sec. V. The scaling results are inconsistent with the fluctuation argument of Eq. (9). The resolution as p approaches p_c is clearly that there are two regimes depending upon the size of L relative to the percolation correlation length ξ. For $L>\xi$, the fluctuation argument of Eq. (9) is correct whereas for $L<\xi$ the scaling behavior of Eq. (12) dominates and the fractal geometry is important.

Since only a few fuses need to be broken to break the network at percolation threshold, we find $N_b\sim O(1)$ or

$$n_b=N_b/L^{d-1}\to 0 \text{ as } p\to p_c\ . \tag{13}$$

This result is discussed further in Sec. V below where we find n_b goes continuously to zero as a power law in $(p-p_c)$ as p approaches p_c.

IV. DISTRIBUTION OF BREAKDOWN STRENGTHS

The calculations of the previous two sections have been for the average breakdown strength and the average number of bonds broken in the breakdown process. In the case of the breakdown strengths, it is possible to make further analytic progress and to calculate the form of the full distribution of breakdown strengths occurring away from p_c. The calculation relies upon the same hypothesis as the Lifshitz-type argument presented in Sec. III above, namely, that the eventual failure of the network is dominated by the most critical defect in the network. The calculation of the full distribution function of breakdown strengths then reduces to the calculation of the distribution functions of these most critical defects. To do this, we develop a scaling argument as follows. Divide the L^d hypercubic system into $N=(L/L_1)^d$ subcubes of size L_1^d. The probability $C(n)$ that no most-critical defect of size greater than n occurs in the L_1^d cubes should in the thermodynamic limit be of the same form as this function for the L^d cubes. In addition, if the characteristic size of the largest defects is much smaller than L_1, the probability of there being no most critical defect of size greater than n on the L^d lattice is $[C(n)]^N$, using the statistical independence of the subcubes in the thermodynamic limit. To insure that the distribution functions have the same form on the L_1^d and L^d lattices, we require

$$[C(n)]^N\sim C(a_N n+b_N)\ , \tag{14}$$

where a_N and b_N are functions of N only. The functional equation (14) is equivalent to the stability postulate of the statistics of extremes,[10] and its method of solution is outlined in Appendix B. Two general solutions that are monotonic in n are

$$C(n)\sim\exp(-cL^d n^{-m})\ , \tag{15}$$

and

$$C(n)\sim\exp[-cL^d\exp(-kn)]\ , \tag{16}$$

Where c, k, and m are constants. The form (15) is the correct one when the distribution from which the extreme configurations are drawn is algebraic, while Eq. (16) is relevant when the distribution from which the extreme configurations are drawn is exponential. (See Appendix B for more details.) It is well known that the distribution of defect clusters, $P(n)$, above p_c in the percolation problem is exponential.[11] What is needed in this problem is the distribution of defect clusters of a particular most critical shape, $P_{mc}(n)$ (linear defects in two dimensions and disc or penny-shaped defects in three dimensions). It is clear that an upper bound on $P_{mc}(n)$ is $P(n)$. To construct a lower bound, we note that the set of distinct bonds visited by all Brownian walks with $2n$ steps on the lattice reproduces all percolation clusters of n bonds at least once. In particular, the most critical defect shape is produced at least once. In addition, the probability of occurrence of each shape depends on the length of its boundary according to a factor p^b, where b is the number of bonds on the defect cluster boundary. To construct the lower bound on the most critical defect probability, we consider the most critical defect cluster to be completely ramified and hence to have a boundary factor p^{2dn} associated with it. If we consider all other possible defect clusters to have no boundary factor we obtain the lower bound $p^{2dn}P(n)/(2d)^{2n}<P_{mc}(n)$. We thus find on combining the upper and lower bounds,

$$\exp(-an)<P_{mc}(n)<\exp(-bn)\ , \tag{17}$$

where $a=b+2\ln 2d-2d\ln p$, and b is a finite constant[11] that describes the rate of exponential decay of the cluster-

size distribution in the percolation problem. The distribution of defects with the most critical shape is thus exponential, and thus the distribution of largest defects of the most critical shape is given by Eq. (16) for this problem.

Since we now have in Eq. (16) the probability that no critical defect of size greater than n exists, then we also can express this distribution in terms of the breakdown strength by using the relationship between the defect size and its tip current discussed in Sec. III. We thus find for the probability that the first bond will fail upon application of an external current i_1, $F(i_1)$,

$$F(i_1)=1-\exp[-cL^d\exp(-ki_1^{-1/\alpha})] , \tag{18}$$

where k and, in this case, $\alpha=\alpha_1$, the enhancement exponent appropriate for initial failure. A similar expression may be found for the breakdown field or voltage distribution, and this voltage distribution is calculated numerically for the two-dimensional fuse network in Sec. V.

As a check of the validity of this scaling derivation and as an illustrative example we can calculate this distribution function exactly in two dimensions if we assume that only linear defects are present in the network. Specifically we wish to calculate the probability $C(n)$ that the horizontal rows of an $L\times L$ square lattice contain no critical cluster (row of adjacent defects) of greater than n defects. For simplicity we can adopt spiral periodic boundary conditions, i.e., we attach the left end of each row of the lattice to the right end of the preceding row. Then it is easy to see that the problem becomes one dimensional. If there are c distinct clusters of present bonds, then there are now exactly c distinct clusters of defects (since each cluster of fuses is separated by a cluster of vacancies and vice versa). Then the probability $C(n)$ is given by the sum

$$C(n)=\sum_{c=1}^{\infty}\prod_{i=1}^{c}\sum_{o_i=1}^{n}\sum_{u_i=1}^{\infty}p^{o_i}(1-p)^{u_i}\delta(X-L^2) , \tag{19}$$

where

$$X=\sum_{i=1}^{c}(o_i+u_i) , \tag{20}$$

where o_i is the number of sites in each occupied cluster of fuses, where u_i is the number of adjacent defects in each defect cluster, and where the δ function ensures that the total of the occupied and unoccupied sites is L^2.

If we expand the δ function according to

$$\delta(X-L^2)=\frac{1}{2\pi}\int_{-\infty}^{\infty}\exp[ik(X-L^2)]dk \tag{21}$$

and interchange the integration with all the summation signs in Eq. (19), we find the intermediate form

$$C(n)=\frac{1}{2\pi}\oint_{-\infty}^{\infty}\exp(-ikL^2)dk\sum_{c=1}^{\infty}\prod_{i=1}^{c}\sum_{o_i=1}^{n}(pe^{ik})^{o_i}\sum_{u_1=1}^{\infty}[(1-p)e^{ik}]^{u_i} . \tag{22}$$

All of the sums in Eq. (22) are geometric and hence summable with the result

$$C(n)=\frac{1}{2\pi}\oint_{-\infty}^{\infty}\frac{\exp(-ikL^2)p(1-p)\exp(2ik)\{1-[(1-p)\exp(ik)]^n\}}{1-\exp(ik)+p(1-p)^{n+1}\exp[ik(n+1)]}dk , \tag{23a}$$

which, with the change of variable $z=e^{-ik}$ becomes the contour integral about the unit circle

$$C(n)=\frac{1}{2\pi}\oint\frac{z^{L^2-3}p(1-p)[1-(1-p)^nz^{-n}]}{1-1/z+p(1-p)^{n+1}z^{-(n+1)}}dz , \tag{23b}$$

the largest pole of the integrand that lies inside the unit circle dominates in the large-L limit, and in this limit is given by $z=1-\alpha$ where

$$\alpha\approx p(1-p)^{n+1}/[1-(n+2)p(1-p)^{n+1}] . \tag{24}$$

The residue at this pole gives, for L, $n\gg 1$,

$$C(n)\sim[1-p(1-p)^{n+1}]^{L^2} , \tag{25a}$$

for $(1-p)$ sufficiently small and n sufficiently large, this is

$$C(n)\sim\exp[-p(1-p)L^2\exp(-nk)] , \tag{25b}$$

where $k=-\ln[(1-p)]$. This equation is of the form exponential of an exponential as found in Eq. (16) above, using the scaling argument.

To now get the failure distribution from the $C(n)$ we must again use in the current enhancements discussed in Sec. III with $i_{\rm tip}=1$. The probability of failure for a given applied current is then seen to be

$$F(i_1)=1-\exp[-cL^d\exp\{-ki_1^{-1/\alpha}\}] \tag{26}$$

where c and k are constants. A similar expression with related constants is also found for the distribution of breakdown fields and voltage $F(\epsilon_b)=F(V_b/L)$.

$$F(\epsilon_1)=1-\exp\{-cL^d\exp[-k\epsilon_1^{-1/\alpha}]\} \tag{27}$$

where c and k are different constants than in Eq. (26). In Eqs. (26) and (27), $\alpha=\alpha_1$. Similar expressions hold for the complete breakdown of the networks with $\alpha=\alpha_b$, and different constants. We now proceed to the numerical work where the predictions made in the analytic sections are tested using simulation studies on the two-dimensional random-fuse network.

V. NUMERICAL WORK

To perform numerical simulations of the random fuse network, one follows the procedure given in Sec. II. The basic numerical method is to solve Kirchhoff's current law at each node in the network. With a constant applied voltage source, Kirchhoff's equations are given in matrix form by

$$\underline{M}\underline{v}=\underline{b} \tag{28}$$

where the first L and last L rows of the matrix $\underline{M}$ are zero except for ones on the diagonal, and the rest of the rows of the matrix have the form

$$M_{k,l}=\sigma^{y}_{i-1,j}\delta_{k,l-L}+\sigma^{x}_{i,j-1}\delta_{k,l-1}+\sigma^{a}_{i,j}\delta_{k,l}+\sigma^{x}_{i,j}\delta_{k,l+1}+\sigma^{y}_{i,j}\delta_{k,l+L} \quad (29)$$

and

$$\underline{b}=(V_0 \cdots V_0 \;\; 0 \cdots 0) , \quad (30)$$

where the first L elements of the $\underline{b}$ vector are given by the external potential value V_0. The above matrix and vector are for a L^2 square lattice in the geometry shown in Fig. 2. $\sigma^{x}_{i,j}$ and $\sigma^{y}_{i,j}$ are the conductivities of the bond emanating from the (i,j) node in the positive x and y directions, respectively:

$$\sigma_{i,j}=(\sigma^{x}_{i,j}+\sigma^{y}_{i,j}+\sigma^{x}_{i-1,j}+\sigma^{y}_{i,j-1}) .$$

In the fuse network studied here, $\sigma^{x}_{i,j}$ and $\sigma^{y}_{i,j}$ are 0 or 1 according to whether the appropriate bond is present or absent. $\underline{v}$ is the solution vector containing the voltage at each node in the network. The matrix $\underline{M}$ is very sparse, and so iterative methods of solution to the linear set of equations are most efficient. The conjugate gradient method is well suited to this class of problems, and in fully vectorized form, we are able (on average) to solve for the node voltages on a 128×128 square lattice at p_c in 18 sec of CYBER 205 CPU time. The convergence criterion that we use is

$$R^2=\sum_{ij}(v^{l}_{i,j}-v^{l-1}_{i,j})^2<10^{-12} , \quad (31)$$

where $v^{l}_{i,j}$ is the voltage at the (i,j) node at the lth iteration.

To complete the breakdown procedure, we calculate the voltage drops across each present bond in the network and remove the hottest one. Carrying out this procedure iteratively leads to a sequence of breakdown voltages as shown in Fig. 3. V_1 and V_b are as defined in this figure. A similar sequence also occurs for the current in the hottest bond at each iteration, and so we can calculate I_1 and I_b. N_b is the number of iterations (bonds broken) before the network disconnects. Using the numerical results for the two-dimensional random-fuse network on a square lattice, we now address three questions. (i) Do V_1/L and V_b/L (and I_1/L and I_b/L) have the same scaling behavior in the thermodynamic limit, and is this behavior given correctly by Eq. (9)? (ii) Is N_b/L^{d-1} the correct order parameter for the number of bonds broken in the breakdown process, and what is its critical exponent on approach to p_c? (iii) Does the form (27) give a correct description of the distribution of breakdown strengths occurring in this model, and is it possible on the basis of numerical analysis to discriminate between this form and the Weibull form (to be defined) most often used in breakdown testing?

(i) ϵ_1 and ϵ_b are expected to scale in the same way at the percolation point if red (singly connected) bonds[7] are present in the infinite cluster. If there are no defects in the network [i.e., at the pure limit—see Eq. (1)] it is certainly true. We have therefore chosen $p=0.75$ as the dilution value that should indicate clearly any differences in the scaling behavior of V_1/L and V_b/L. The numerical results are plotted on logarithmic scales in Fig. 8. Although V_1/L and V_b/L are not equivalent, it is clear that both are decreasing functions of L. The fact that they are not equivalent is evident from Fig. 3, where the first bond clearly breaks at a lower voltage than the maximum. It is thus clear that the Lifshitz argument is an approximation, and the approximation is that the bonds that break are always at the end of the most critical defect, which implies that the breakdown path is always linear. To illustrate that this is a good approximation, especially near $p=1$, the breakdown path is shown for two values of p in Fig. 9. It is evident from this figure that near $p=1$, the breakdown path is very nearly linear, and becomes more contorted on approach to p_c. For any p away from the pure limit, there are some cooperative effects in addition to that due to the largest defect, and one of the objectives of the numerical analysis is to test whether these cooperative effects produce any qualitative change in comparison with predictions of the extreme defect calculations of the previous sections.

The fact that neither V_1/L nor V_b/L are saturating with $L\gg\xi$ (see Fig. 8), where ξ is the percolation correlation length, already implies that these quantities are not behaving as standard percolation order parameters. The percolation point is still important however, and the full behavior of the average breakdown voltage is dominated by different effects depending on the ratio of L/ξ. In particular, we expect the behavior of the breakdown voltages to be algebraic in L (influenced by the percolation fixed point) for $L\ll\xi$, and logarithmic in L (dominated by the most critical defect expressed by the Lifshitz argument) for $L\gg\xi$. The novel behavior occurs in the regime $L\gg\xi$, and to illustrate this, V_1/L is displayed as a function of L for $p=0.70$ and $p=0.90$ in Fig. 10(a). The curve does not approach a constant value for values of

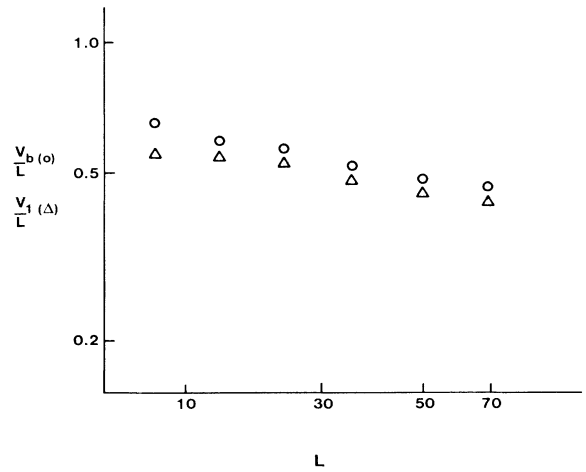

FIG. 8. Comparison of the scaling behavior of V_1/L and V_b/L for the fuse network on 50×50 square lattices at $p=0.75$. Each point is an average over 50 realizations of the random fuse network.

$L \gg \xi$, and so the curvature is not due to crossover effects. The Lifshitz argument [Eq. (9)] predicts that in this regime

$$I_1/L \sim V_1/L \sim 1/(a + b \ln L) . \quad (32)$$

An unbiased test of the data is to see if a plot of $\ln L$ against L/V_1 is a straight line. The data of Fig. 10(a) is plotted in this way in Fig. 10(b). The data is consistent with Eq. (29) although it does not give convincing evidence that the form (32) gives a better description than several other forms which may also fit the data. Another way of plotting the breakdown data is to plot V_1/L and I_1/L versus p at fixed system size, as shown in Fig. 11, for a 70×70 lattice. From a figure of this sort the authors of a previous calculation[3] (without any $p=0.50$ calculations) deduced that V_1/L diverges on approach to p_c. Their deductions are incorrect for two reasons.

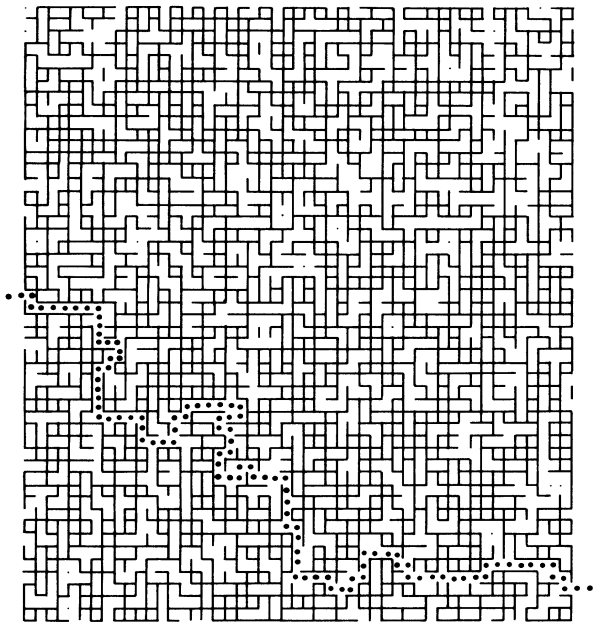

FIG. 9. Topology of the final breakdown path on 50×50 square lattices at (a) $p=0.70$ and (b) $p=0.90$.

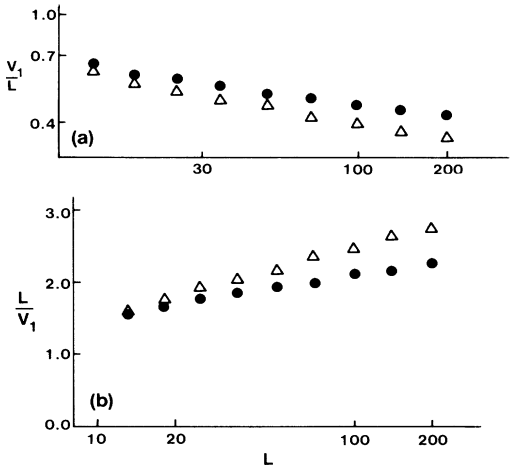

FIG. 10. (a) Scaling behavior of V_1/L at $p=0.70$ (△) and $p=0.90$ (●). (b) Plot of the same data as in (a), testing for the logarithmic behavior given in Eq. (32).

(1) At p_c, V_1/L appears to be finite. In fact the scaling argument in 2D suggests that $V_1/L \sim L^{-0.027}$ using the best estimates of t/ν available for 2D.[9] There is no evidence to suggest that it diverges at p_c.

(2) In attempting to find the critical exponents pertinent to V_1/L on approach to p_c one must take into account the logarithmic behavior depicted in Fig. 10(a). This was not done by the previous authors, and is the reason why V_1/L appears to diverge on approach to p_c.

At first sight, the breakdown voltage behavior of Fig. 11 is surprising in that the strength of the network, to an externally applied voltage, appears to *increase* on approach to p_c. This effect is due to the fact that V_1/L is a ratio of the conductivity and the current, both of which *decrease* on approach to p_c. The behavior of V_1/L is then determined by which of these two quantities decreases faster at the percolation point. In two dimensions, this ratio is especially subtle, and our numerical results suggest $V_1/L = 1.00\pm0.05$, even though the scaling argument (combined with previous numerical estimates of the conductivity exponent[9]) gives $V_1/L \sim L^{-0.027}$. It is thus clear that in the fuse network, the breakdown currents I_1 and I_b are more fundamental quantities theoretically. The numerical data for the breakdown current I_1/L as a function of system size is given on a log-log plot in Fig. 12(a). We again test for the scaling prediction (32) [Fig. 12(b)] and find that the fit to the current data is consider-

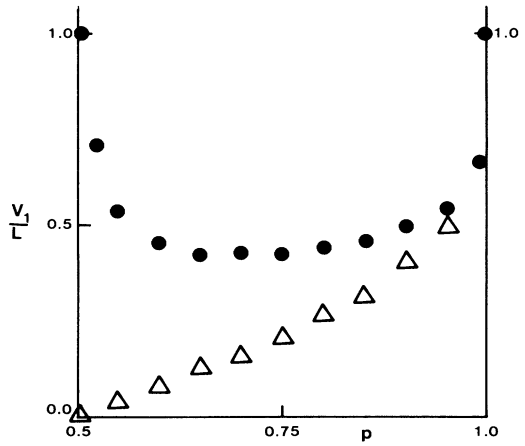

FIG. 11. Average breakdown voltage (●) and current (△) as a function of p on 70×70 square lattices. Each point is an average over 50 configurations.

ably better than that found for the voltage data and provides convincing evidence that the prediction (9) is correct.

(ii) As seen in Fig. 9, the nature of the breakdown path changes markedly as one varies p away from the pure limit. The variation may be measured in several ways. Clearly, the ratio of the vertical to horizontal bonds broken is one, the fractal dimension of the final breakdown path is another, and finally the number of bonds broken in the breakdown process is another. The number of bonds broken in the breakdown process N_b is the easiest to calculate as it is just the number of iterations to failure in the iterative process defined in Sec. II. This is the one we numerically investigate. As shown by Eqs. (1c) and (10), N_b changes radically when defects are introduced into the network. It is thus not immediately obvious which is the correct scaling quantity, in that the single defects suggest

$$n_b = N_b / L^{d-1} , \tag{33}$$

rather than the pure one N_b/L^d. The numerical analysis shows that the correct order parameter for all p away from the pure limit is n_b as defined in Eq. (33), and this is evident in Fig. 13, where n_b is shown to be saturating with L for $L \gg \xi$, as is to be expected for an order parameter. On approach to p_c, n_b goes to zero with a characteristic exponent x,

$$n_b \sim (p - p_c)^x . \tag{34}$$

From the log-log plot of n_b against $(p - p_c)$ in Fig. 13, we are able to estimate the exponent x to be

$$x = 1.2 \pm 0.2 \tag{35}$$

As argued by de Arcangelis *et al.*,[3] x may be naively expected to be equal to ν, the percolative correlation length exponent. However, this is not altogether convincing, as the number of bonds broken in the fracture process is affected by different aspects of the backbone topology than the pair connectedness. Indeed the fact that x is close to ν in two dimensions may be a peculiarity of that dimension. It would thus be interesting to do detailed calculations in three dimensions to determine the exponent x there. Before proceeding further, we note that in their calculations de Arcangelis *et al.*[3] used N_b/L^d as the order parameter to estimate x. Fortunately, this does not effect the calculation of x from a log-log plot using only one value of L (as they did), as only the intercept of the graph and not the slope is affected. The value they find,

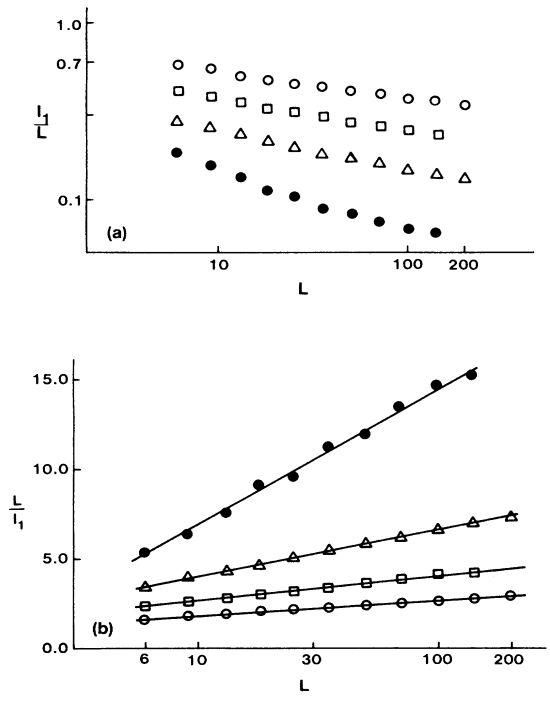

FIG. 12. (a) Average breakdown current on square lattices as a function of L for $p=0.6$ (●), $p=0.70$ (△), $p=0.80$ (□), and $p=0.90$ (○). (b) Same data as for (a) plotted on log—linear to test for the behavior predicted in Eq. (32).

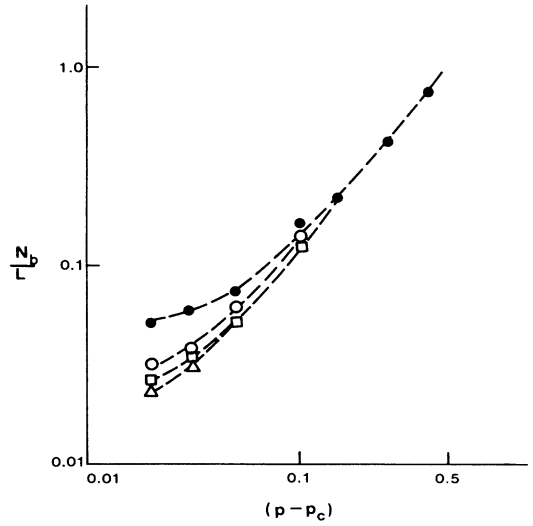

FIG. 13. Number of bonds broken in the breakdown process N_b for $L=18$ (●) 35 (○), 50 (□), and 70 (△) square lattices.

$x = 1.4 \pm 0.2$, is thus consistent with ours, to the accuracy of the calculations.

(iii) In two dimensions, the distribution of failure strengths for V_1/L is expected to be of the form [by putting $d=2$ and $\alpha_1=1$ in Eq. (27)]

$$F(V_1/L) = 1 - \exp[-cL^d \exp(-kL/V_1)] \ . \quad (36)$$

A similar expression holds for the complete breakdown voltage V_b/L. We now study this distribution in detail for V_1/L, as this is an easier quantity to calculate, and thus allows us to construct the detailed distribution functions necessary for a comprehensive analysis.

$F(V_1/L)$ is the probability that on application of an external voltage of V_1/L, the first fuse in the network will fail. For V_1/L near 0, this probability is near 0, while for V_1/L large, $F(V_1/L)$ tends toward 1. This is illustrated in Fig. 14, where the failure distribution is given for $p=0.90$ on a 50×50 square lattice using 1500 configurations. It is straightforward to fit the data in Fig. 14 with the distribution (36), as given by the solid line in Fig. 14. What is more difficult is to differentiate between the fit given by Eq. (36), and that given by the Weibull form

$$F(V_1/L) = 1 - \exp[-cL^d (V_1/L)^m] \ , \quad (37)$$

which is most often used in fitting breakdown distributions in engineering applications of materials. In Eq. (37), c and m are again constants. A direct fitting of the data using the Weibull distribution (37) gives results which to the resolution of Fig. 14 are identical with that found from Eq. (36). A more sensitive test of the data is thus required. This may be done by isolating V_1/L from Eqs. (36) and (37) and we find from Eq. (36)

$$A = \ln\{-\ln[1 - F(V_1/L)/L^d]\} = -kL/V_1 + \ln c \ , \quad (38)$$

and from (37),

$$A = \ln\{-\ln[1 - F(V_1/L)/L^d]\} = -m \ln(L/V_1) + \ln c \ . \quad (39)$$

Now we plot A against L/V_1 and $\ln(L/V_1)$ and see which plot lies closer to a straight line. These plots are given in Figs. 15(a) and 15(b), respectively, where it is seen that the new distribution function [Eqs. (36) and (38)] provides an appreciably better fit to the numerical data, than the Weibull form [Eqs. (37) and (39)]. The two fits are most different in the small V_1/L region, and this is of special engineering interest as it corresponds to the "high reliability" (in the sense that the probability of failure is small) end of the failure distribution of Fig. 14.

VI. SUMMARY AND CONCLUSIONS

The random fuse network has been studied in detail in this paper, and using analytic arguments based on the hypothesis that the most critical defect in the network determines the strength of the network, we deduce that between the pure limit and the percolation point, there is a new and novel behavior in the properties of the network. There are basically three new analytic predictions in this region of dilution.

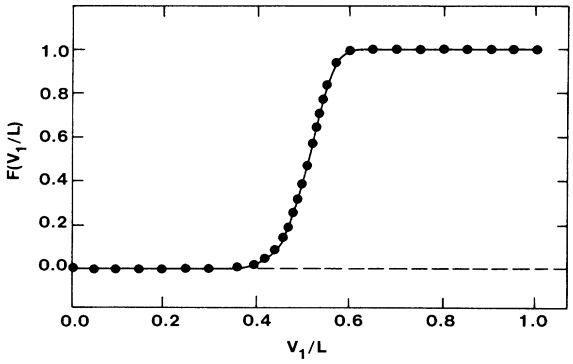

FIG. 14. Failure distribution for 50×50 square lattices at $p=0.90$, constructed from 1500 realizations. The solid line is a fit to the data using the Weibull form (37) with $c=1.826$ and $m=13.146$. A fit to the same data using the form (36) with $c=194.582$ and $k=6.889$ is, to the resolution of the figure, indistinguishable from the Weibull fit.

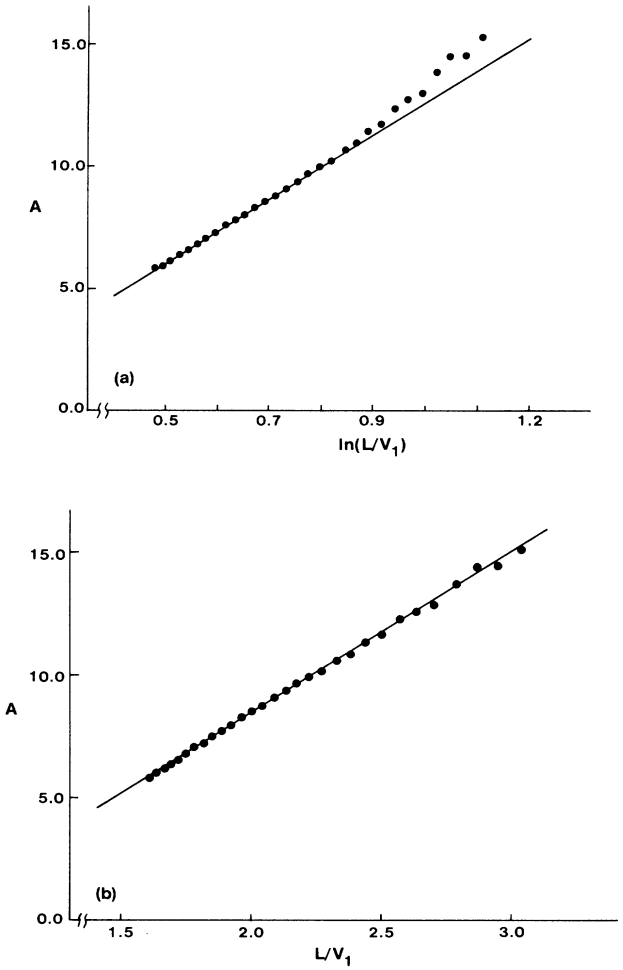

FIG. 15. (a) Plot of A [see Eq. (39) in the text] against $\ln(L/V_1)$, to test for the Weibull form. (b) Plot of the same data using A against L/V_1 to test for the new form given by Eq. (36).

(1) The average strength of the network, in comparison with the pure limit, is reduced by a factor $(\ln L)^{\alpha}$, where $\alpha=\alpha_1$ for initial breakdown and $\alpha=\alpha_b$ for complete breakdown. Arguments based on the most critical defect in the network lead to the inequality, $1/[2(d-1)]\leq\alpha_b\leq\alpha_1\leq 1$, where α_1 and α_b are the enhancement exponents for initial and complete breakdown, respectively [see Eqs. 9(a) and 9(b), and the arguments leading to them].

(2) The distribution of breakdown strengths is not of the Weibull form (37) (except at the percolation point), most often used in engineering studies of failure, but is of the form exponential of an exponential [Eqs. (26), (27), and (36)]. The implications of this result for the prediction of design criteria in high reliability applications is currently being investigated.

(3) The order parameter describing the final breakdown path changes markedly as soon as any finite fraction of defects is added to the fuse network. In the pure limit for example, the number of bonds broken in the breakdown process is $N_b=L^d/d$, while the addition of any finite fraction of defects leads to the correct order parameter being $N_b=kL^{d-1}$, for the fuse network [see Eq. (10)].

The new scaling behavior described above apply in the regime $L\gg\xi$, where ξ is the percolation correlation length. In the region $L\ll\xi$, standard percolation scaling arguments may be used, and lead to the prediction given in Eqs. (11) and (12) for the average breakdown strength. The predictions described above are tested using numerical calculations on the two dimensional random-fuse network (Sec. V), and to the accuracy of the simulations, confirmed.

In a more general context, the methods that we have used in the analytic analysis of the random-fuse network emphasize only the most critical defect in the network. This is because the bond that carries the most current in the network is the one that will undergo failure first. The breakdown problem thus depends on the very high moments of the bond current distribution of the random resistor network. In the approximation used in this paper, the infinitely high moment is the one considered. The mathematicians have also studied problems of the sort that depend only on the most extreme configurations in a distribution function, in the prediction of the likelihood of earthquakes, floods, and other natural disasters. The mathematical method developed for these problems is called the statistics of extremes,[10] and contains many results that may be taken over to the study of breakdown problems.

With the realization that essentially all breakdown problems are dominated by extreme configurations in the defect distribution function, it is straightforward to construct models of failure for many branches of physics, and to extend the extreme defect analysis described in the main body of the paper for the fuse network, to those problems. In Appendix C, this is done for the problems of (1) brittle fracture and (2) dielectric breakdown in solids.

The final comment that should be made is that in most real systems, the distribution of bond strengths is not of the percolation form used in the analysis described in this paper. This will certainly affect V_1, which in models with a continuous distribution of breakdown strengths on the bonds, will be close to zero. However, we expect the calculations described here to apply to V_b, which is the important physical quantity. The fact that V_1 behaves, in the thermodynamic limit, in the same way as V_b in the percolation models described here, is a strength of the percolation models as V_1 is an easier quantity to study theoretically, and still gives the important behavior of V_b. The same is not true in models with a continuous distribution of breakdown strengths.

APPENDIX A: CALCULATION OF THE CURRENT DENSITY AT THE TIP OF AN ELLIPTICAL DEFECT

Consider an elliptical defect in an infinite two-dimensional lattice. The equation for an ellipse centered at the origin is

$$x^2/a^2+y^2/b^2=1\ , \tag{A1}$$

which is depicted in Fig. 7 in the text. To find the current density at any point just on the boundary of the elliptical defect (ξ_0) solve Laplace's equation with the boundary condition

$$\left[\frac{\partial V}{\partial\xi}\right](\xi=\xi_0)=0\ , \tag{A2}$$

which ensures that no current flows into the insulating defect. Laplace's equation is invariant under a transformation to elliptical coordinates, and in that frame, the appropriate form of the solution is,

$$V(\xi,\eta)=Ay+Be^{-\xi}\sin(\eta+\epsilon)+Ce^{\xi}\sin(\eta+\epsilon)\ , \tag{A3}$$

where

$$x=c\cosh(\xi)\cos(\eta);\quad y=c\sinh(\xi)\sin(\eta)\ ; \tag{A4}$$

$$a=c\cosh(\xi_0);\quad b=c\sinh(\xi_0);\quad \text{and}\quad c=(a^2-b^2)^{1/2}\ ;$$

where A, B, C, and ϵ are constants. Choose $\epsilon=C=0$ and $A=-V_0$ to produce the externally applied electric field. Then,

$$V(\xi,\eta)=-V_0y+Be^{-\xi}\sin(\eta) \tag{A5}$$

$$=-V_0c\sinh(\xi)\sin(\eta)+Be^{-\xi}\sin(\eta) \tag{A6}$$

and the boundary condition (A2) implies

$$B=-V_0ce^{\xi_0}\cosh(\xi_0)\ . \tag{A7}$$

The current density in the y direction (see Fig. 7) is then[12]

$$j_y=\Sigma\frac{\partial V}{\partial y}=\frac{-\Sigma\left[\cosh(\xi)\sin(\eta)\left[\frac{\partial V}{\partial\xi}\right]+\sinh(\xi)\cos(\eta)\left[\frac{\partial V}{\partial\eta}\right]\right]}{c(\cosh^2\xi-\cos^2\eta)}\ , \tag{A8}$$

where Σ is the conductance:

$$j_y = V_0\Sigma - \Sigma B e^{-\xi}[\cosh(\xi)\sin^2(\eta) - \sinh(\xi)\cos^2(\eta)]/c[\cosh^2(\xi) - \cos^2(\eta)] \ . \tag{A9}$$

The tip of the defect is defined by $\eta = 0$, and $\xi = \xi_0$, where,

$$j_{\text{tip}} = \Sigma V_0[1 + \cosh(\xi_0)/\sinh(\xi_0)] \tag{A10}$$

$$= \Sigma V_0[1 + a/b] \tag{A11}$$

$$= j(1 + a/b) \ , \tag{A12}$$

where j is the current density at an infinite distance from the ellipse.

APPENDIX B: SOLUTIONS TO THE SCALING EQUATION (14)

The equation in question is

$$[C(n)]^N = C(a_N n + b_N) \ . \tag{B1}$$

It is straightforward to solve this equation in two cases.

(1) We have $a_N = 1$, and hence

$$[C(n)]^N = C(n + b_N) \ . \tag{B2}$$

Taking two logarithms of this equation leads to

$$\ln(N) + \ln[-\ln C(n)] = \ln[-\ln C(n + b_N)] \ , \tag{B3}$$

which is solved by

$$\ln[-\ln C(n)] = -kn \ , \tag{B4}$$

with $-kb_N = \ln N$. Which implies

$$C(n) = \exp[-cL^d \exp(-kn)] \ , \tag{B5}$$

where c is a quantity that is independent of N and n. By direct substitution, Eq. (B5) can be seen to solve (B1).

(2) The second case that is easily solved is when $b_N = 0$ and

$$[C(n)]^N = C(a_N n) \tag{B6}$$

and therefore

$$N \ln C(n) = \ln C(a_N n) \ , \tag{B7}$$

which is solved by

$$\ln C(n) = -cL^d n^{-m} \ , \tag{B8}$$

where $a_N = N^{-1/m}$. $C(n)$ is then given by

$$C(n) = \exp(-cL^d n^{-m}) \ . \tag{B9}$$

Again c is a quantity that is independent of n and N.

Now it is possible to show that the form (B5) is correct when the distribution from which the extreme values is drawn is exponential, as is the case in the percolation problem away from p_c. This is seen upon considering the tail of the defect distribution, which is of the form (see Eq. (17) of the main text of the paper, and the associated discussion)

$$P_{mc}(n) \sim \exp(-an) \ . \tag{B10}$$

Now for large n, the probability that the large defect is smaller than n is given by

$$C(n) = 1 - \int_n^\infty \exp(-an')dn' \tag{B11}$$

$$= 1 - \exp(-an)/a \quad (\text{for } n \text{ large}) \ . \tag{B12}$$

This form is the correct one in an expansion for large n of the form (B5), and shows that (B5) is the correct form for the percolation problem away from p_c. The full details and other evidence that the form (B5) is correct in this problem can be found in a work by Gumbel (Ref. 10, Chap. 6).

APPENDIX C: MODELS OF FAILURE WHICH ARE RELATED TO THE FUSE NETWORK

In this appendix we describe two problems for which one may construct percolation models that are closely related to the fuse network. They are, (1) brittle fracture and (2) dielectric breakdown in solids. For these two problems, percolation models are constructed, and the analytic predictions analogous to those contained in Secs. III and IV for the fuse network, are stated.

1. Brittle fracture

On the present bonds in a percolation network above the percolation threshold, place brittle elements (i.e., they are completely inelastic and undergo no extension) that have a threshold for breakdown upon application of bending or tensile forces. If the bending threshold is zero, the percolation point is the rigidity percolation point p_r, while if there is a bending threshold, the percolation point is the conductivity one p_c. The scaling behavior is different in these two cases, but the behavior for $L \gg \xi$ is qualitatively the same. The most critical defect is of the same form as that found in the fuse network (the penny-shaped crack), and transcribing the results to the terminology of mechanical failure, one finds that for $L \gg \xi$,

$$\sigma_b \sim 1/[a + b(\ln L)^\alpha] \ , \tag{C1}$$

where σ_b is the average failure stress of the network, with $1/[2(d-1)] \le \alpha_b \le \alpha_1 \le 1$ as for the fuse network, and

$$n_b = N_b/L^{d-1} = O(1), \quad \text{for } p \text{ away from } p_c \ . \tag{C2}$$

That is, the correct order parameter for the number of bonds broken in the breakdown process is the same as for the fuse problem. The distribution of failure strengths is (in the large L limit)

$$F(\sigma_b) \sim 1 - \exp\{-cL^d \exp[-k(1/\sigma_b)^{1/\alpha}]\} \ , \tag{C3}$$

where $F(\sigma_b)$ is the probability of failure upon application of an external load σ_b. As stated above, the scaling behavior depends upon whether one considers the rigidity case, or the percolation case, and we leave a full discussion of these questions for later work.

2. Dielectric breakdown in solids

Consider a percolation network composed of conductors and insulators, where the fraction of conductors is less than p_c, and hence no current flows initially. The distribution of voltages in the connected insulating cluster is given by the solutions to Laplace's equation. If the insulators are given a threshold beyond which they break down and become conductors, the whole network is a model for dielectric breakdown in solids. The most critical defects in this problem are different than the penny shaped crack appropriate for the fuse and brittle fracture problems in three dimensions. The nature of the most critical defect in this problem is clear when one recalls that the electric field is largest near the tips of a conductor. The most critical defect is then seen to be a long thin defect oriented in the y direction in Fig. 7. This is easily confirmed in a calculation analogous to that described in Appendix A for the fuse problem. The only change is that the boundary condition on the ellipse is now $V(\xi_0)=0$, and this changes the orientation and shape (in 3D) of the most critical defect. The remaining calculations carry through in an analogous manner, and we find that the average breakdown strength of the dielectric network for $L \gg \xi$ is

$$V_b/L \sim 1/[a + b(\ln L)^{\alpha}] , \tag{C4}$$

where $1/2 \leq \alpha_b \leq \alpha_1 \leq 1$ (note that in our previous letter[5] we used the approximation $\alpha_1=\alpha_b=1$), while the failure distribution becomes

$$F(V_b) \sim 1-\exp\{-cL^{d}\exp[-k(L/V_b)^{1/\alpha}]\} \tag{C5}$$

and the number of bonds broken in the breakdown process behaves as

$$n_b \sim N_b/L \sim O(1), \quad \text{for } p \text{ away from } p_c . \tag{C6}$$

The scaling behavior for $L \ll \xi$ is given by

$$V_b/L \sim (p_c-p)^{\nu} . \tag{C7}$$

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[1]There are many texts and reviews on this subject. Three more recent ones are S. M. Weiderhorn, Ann. Rev. Mater. Sci. **14**, 373 (1984); R. W. Davidge, *Mechanical Behavior of Ceramics, Cambridge Solid State Sciences Series* (Cambridge University Press, Cambridge, 1979); J. E. Gordon, *The New Science of Strong Materials* (Princeton University Press, Princeton, 1984).

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[12]We thank M. Stephen for pointing this out to us. It is a standard calculation that may be found in any text dealing with the applied mathematics of partial differential equations.