Internal structure of a Landau quasiparticle wave packet

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Starting from the conventional definition of the state of a Fermi system containing a quasi-particle of momentum **k**, we construct a state with a quasiparticle wave packet (QPWP) by the usual superposition. We have studied the distributions of charges and currents in such a state to first order in the interparticle interaction. The dressed QPWP state contains a charge, $\tilde{e}$, which is less than the bare charge, e. (The charge $\tilde{e}$ is different from an incorrect value previously proposed on different grounds.) The remaining charge resides at infinity. In more dimensions than one, the charge density of the QPWP also contains anisotropic components that fall off as r^{-3} away from the center of the wave packet. Similar conclusions hold for the current density. We have verified that the continuity equation is satisfied at every point in space. The relationship of these results to the classical formal Landau theory is discussed. We conclude that the Landau quasiparticle distribution function, $n(\mathbf{k},\mathbf{r})$, cannot be interpreted in terms of the QPWP's defined here. Finally, experimental consequences of the work presented in this paper are briefly discussed.

I. INTRODUCTION

Classical Landau-Fermi liquid theory is formulated in terms of the quasiparticle distribution function

$$n_t(\mathbf{k},\mathbf{r})=n_0(\mathbf{k})+n(\mathbf{k},\mathbf{r}) \ , \tag{1}$$

where

$$n_0(\mathbf{k})\equiv\begin{cases}1, & k<k_F\\ 0, & k>k_F\end{cases} \tag{2}$$

(k_F = Fermi momentum), and $n(\mathbf{k},\mathbf{r})$ describes the distribution of quasiparticles. Thermodynamic properties are formulated in terms of

$$n(\mathbf{k})=\int n(\mathbf{k},\mathbf{r})d\mathbf{r} \ ; \tag{3}$$

transport properties are obtained by solving the Landau transport equation[1] for $n(\mathbf{k},\mathbf{r})$. Conditions for the validity of this theory include the following:

$$n(\mathbf{k},\mathbf{r})=0 \quad \text{unless} \quad |\mathbf{k}-\mathbf{k}_F|/k_F \ll 1 \ , \tag{4}$$

and the dependence on r must be slow compared to k_F^{-1}.

The transport equation has exactly the form of transport equation for classical interacting particles of effective mass m and charge $e=1$. Observable quantities are unambiguously calculable from $n(\mathbf{k},\mathbf{r})$. For example, the charge density is given by

$$n(\mathbf{r})=\sum_{\mathbf{k}} n(\mathbf{k},\mathbf{r}) \ ; \tag{5}$$

the current density is given by

$$\mathbf{j}(\mathbf{r})=\sum_{\mathbf{k}} n(\mathbf{k},\mathbf{r})\mathbf{j}_{\mathbf{k}} \ , \tag{6}$$

where $j_{\mathbf{k}}$ is the current[2] associated with a quasiparticle of momentum **k**. For a translationally invariant system,

$$\mathbf{j}_{\mathbf{k}}=\frac{\mathbf{k}}{m} \ . \tag{7}$$

These results have been obtained *without* needing to interpret $n(\mathbf{k},\mathbf{r})$ as referring to wave packets of quasiparticles with average momentum **k** and center of mass **r**. Some years ago we attempted to extend the Landau theory to homogeneous systems bounded by surfaces. A microscopic derivation of such a theory is extremely difficult and therefore we attempted to obtain it by general physical arguments using quasiparticle wave packets (QPWP's) reflected from the surface. This led us, as a preliminary study, to examine in detail the internal structure of a QPWP in an *infinite* homogeneous system. The results were unexpected and are reported here.

QPWP's had been considered earlier by Falicov[3] and Stern.[4] These authors concluded that a QPWP carries a charge e^* different from unity, essentially by the following argument: the energy of a QPWP is given by

$$\epsilon(\mathbf{k})=\frac{k_F}{m^*}|\mathbf{k}-\mathbf{k}_F| \ , \tag{8}$$

where m^* is so-called effective mass. The conventional argument for group velocity then leads to the conclusion that a QPWP moves with velocity

$$\mathbf{v}_{\mathbf{k}}=\nabla_{\mathbf{k}}\epsilon(\mathbf{k})=\frac{\mathbf{k}}{m^*} \quad (k=k_F) \ . \tag{9}$$

On the other hand, a quite general argument, using momentum conservation, shows that particle current $\mathbf{j}_{\mathbf{k}}$, Eq. (7), is left unchanged by the interaction.

To obtain consistency between Eqs. (7) and (9) they assumed that the integrated particle density associated with the wave packet was m^*/m. For ease of expression we shall call the localized integrated particle density "charge," even for the case of uncharged particles. In this terminology the conclusions of Stern and Falicov were that

$$e^*=\frac{m^*}{m}(\neq 1) \ . \tag{10}$$

Both authors argued that e^*/e effects the de Haas–van Alphen periods. This was, however, found incorrect by explicit calculations[5,6] in which only the bare charge e appears.

In the present paper we put forward what, in our view, is the natural definition of a QPWP and study its density and current density distributions. Our results, obtained by explicit calculation to lowest order in the particle-particle interaction were unexpected.

We find that the particle density consists of three parts:

$$n(\mathbf{r})=n_{l=0}(\mathbf{r})+n_{l>0}(\mathbf{r})+\delta n(\mathbf{r}) \ . \tag{11}$$

Here $n_0(\mathbf{r})$ has spherical symmetry, is integrable, and integrates to a value $\tilde{e}\neq 1$. This $\tilde{e}$ is *not* equal to the e^* of Eq. (10). The next term has the expansion[7]

$$n_{l>0}(\mathbf{r})=\sum_{l=2,4,\ldots} n_l(r)P_l(\cos\theta) \ ,$$

where the n_l have the following behavior:

$$r\to 0, \quad n_l(r)\to 0 \ ,$$
$$r\to\infty, \quad n_l(r)\to\frac{a_l}{r^3} \ . \tag{12}$$

Finally,

$$\delta n(\mathbf{r})\sim O(\Omega^{-1}) \ , \tag{13}$$

where Ω is the volume, and $\int \delta n(\mathbf{r})d\mathbf{r}=(1-\tilde{e})$.

There is a similar decomposition of the current density into three parts. We have explicitly verified that total charge is conserved (although a finite fraction is at infinity) and that the continuity equation is satisfied at all points $\mathbf{r}$.

These results are independent of the detailed shape of the envelope function of the QPWP. The details of our calculation are presented in Secs. II–V and there is a discussion in Sec. VI.

II. CHARGE AND CURRENT OF A QPWP

In this section, we will give the formal definition of a QPWP. We will also define charge-density and current operators and discuss the general behavior of the expectation values of these operators.

Consider a system of N spinless fermions at $T=0$ K. We denote by Φ_0 the noninteracting ground state of this system,

$$\Phi_0=\prod_{k'<k_F} c^\dagger_{\mathbf{k}'}\Phi_{\text{vac}} \ , \tag{14}$$

where Φ_{vac} is the vacuum state and $c^\dagger_{\mathbf{k}'}$ creates a particle of momentum $\mathbf{k}'$. Let $\phi_{\mathbf{k}}$ be an eigenstate of momentum $\mathbf{k}$ obtained by adding a particle at $\mathbf{k}$ to Φ_0, i.e., $\phi_{\mathbf{k}}=c^\dagger_{\mathbf{k}}\Phi_0$, where we have assumed that $k>k_F$. We create a state $\Phi_{\mathbf{k}_0}$ containing a wave packet of bare fermions with average momentum $\mathbf{k}_0$ by superposing states $\phi_{\mathbf{k}}$,

$$\Phi_{\mathbf{k}_0}=\sum_{\mathbf{k}} A_{\mathbf{k}}\phi_{\mathbf{k}}=\sum_{\mathbf{k}} A_{\mathbf{k}}c^\dagger_{\mathbf{k}}\Phi_0 \ . \tag{15}$$

The envelope function $A_{\mathbf{k}}$ is a smooth function that is sharply peaked near $\mathbf{k}_0$ has a spread Δk, where $\Delta k \ll k_0-k_F$, and vanishes for $k<k_F$. The state Φ_{k_0} is normalized, so $\sum_{\mathbf{k}}|A_{\mathbf{k}}|^2=1$. The charge density of the state $\Phi_{\mathbf{k}_0}$ then consists of a spatially uniform part from the N electrons in the filled Fermi sphere plus a unit charge localized within a distance $\alpha=\Delta k^{-1}$. For simplicity, we shall only consider envelope functions of spherical symmetry; for example, a Gaussian, $A_{\mathbf{k}}=\text{const}\times e^{-\alpha^2(\mathbf{k}-\mathbf{k}_0)^2/2}$. However, the results in this paper do not depend qualitatively on the detailed shape of the envelope function.

We now consider a system with interparticle interaction $V(\mathbf{r})$. We take $V(\mathbf{r})$ to be a function of r only, integrable, short-ranged, repulsive, and sufficiently regular. For definiteness we assume that its Fourier transform is a monotonically decreasing function of q. An example is a Yukawa potential, $V(q)=V_0/(\kappa^2+q^2)$, with κ of the order of k_F.

To define a wave packet in the interacting system we first consider a single-momentum eigenstate, $\phi_{\mathbf{k}}$, and then turn on the interaction "adiabatically." This state will then evolve "adiabatically" into a momentum eigenstate $\psi_{\mathbf{k}}$ of the interacting system. (Of course the interaction cannot be turned on *literally* infinitely slowly since in the meantime the quasiparticles would decay. However, we show in detail in Appendix A that this limitation does not cause any difficulty. It will be ignored in what follows.) For k near k_F, the states defined this way are precisely the quasiparticle states defined by Nozières and Luttinger[8] (hereafter referred to as NL) and used in their microscopic derivation of the classical Landau-Fermi-liquid theory. Using the same $A_{\mathbf{k}}$ as in Eq. (15), we *define* the corresponding *quasi*particle wave packet as

$$\Psi_{\mathbf{k}_0}=\sum_{\mathbf{k}} A_{\mathbf{k}}\psi_{\mathbf{k}} \ . \tag{16}$$

We want to calculate the total charge and current carried by the QPWP. We will do this by studying the Fourier transforms $\hat{n}(\mathbf{q})$ and $\hat{\mathbf{j}}(\mathbf{q})$ of the charge and current density operators for small q. In the plane-wave representation, these operators are given by

$$\hat{n}(\mathbf{q})=\sum_{\mathbf{k}'} c^\dagger_{\mathbf{k}'-\mathbf{q}}c_{\mathbf{k}'} \tag{17}$$

and

$$\hat{\mathbf{j}}(\mathbf{q})=\sum_{\mathbf{k}'}\left[\mathbf{k}'-\tfrac{1}{2}\mathbf{q}\right]c^\dagger_{\mathbf{k}'-\mathbf{q}}c_{\mathbf{k}'} \ . \tag{18}$$

For simplicity, we take the volume Ω of our system to be unity and we use atomic units in which $e=m=\hbar=1$.

Since the charge-density and current-density operators commute with the full Hamiltonian, the total charge of the system and the total current are conserved by the interactions. Hence

$$n(\mathbf{q}=0)\equiv(\Psi_{\mathbf{k}_0},\hat{n}(\mathbf{q}=0)\Psi_{\mathbf{k}_0})$$
$$=N+\sum_{\mathbf{k}}|A_{\mathbf{k}}|^2(\psi_{\mathbf{k}},c^\dagger_{\mathbf{k}}c_{\mathbf{k}}\psi_{\mathbf{k}})=N+1 \tag{19}$$

and

$$\mathbf{j}(\mathbf{q}=0)\equiv(\Psi_{\mathbf{k}_0},\hat{\mathbf{j}}(\mathbf{q}=0)\Psi_{\mathbf{k}_0}) = \sum_{\mathbf{k}} |A_{\mathbf{k}}|^2\mathbf{k}(\psi_{\mathbf{k}},c_{\mathbf{k}}^\dagger c_{\mathbf{k}}\psi_{\mathbf{k}})=\mathbf{k}_0 \ . \quad (20)$$

We shall find that $n(\mathbf{q})$ and $\mathbf{j}(\mathbf{q})$ behave discontinuously at $\mathbf{q}=0$:

$$\lim_{\mathbf{q}\to 0} n(\mathbf{q})\neq 1, \quad \lim_{\mathbf{q}\to 0}\mathbf{j}(\mathbf{q})\neq\mathbf{k}_0 \ . \quad (21)$$

Moreover, in more than one dimensions, the limits depend on the direction of $\mathbf{q}$. These discontinuities at $\mathbf{q}=0$ signal the existence of delocalized charges and currents (at infinity), associated with the QPWP. Details are described in Secs. IV and V.

III. FIRST-ORDER PERTURBATION THEORY

We shall now calculate $n(\mathbf{q})$ and $\mathbf{j}(\mathbf{q})$ to first order in the interaction V by ordinary Rayleigh-Schrödinger perturbation theory. We shall apply nondegenerate perturbation theory to $\psi_{\mathbf{k}}$, although the unperturbed state $\phi_{\mathbf{k}}$ is embedded in a continuum. We shall find that divergencies, due to vanishing energy denominators, are integrable and—without having a formal proof—shall assume that this procedure is valid. Where our integrals pass through divergencies we shall interpret them as principal part integrals. The imaginary parts vanish as $(k_0-k_F)^2$ and can be ignored.

We shall also explicitly verify that the continuity equation is satisfied to first-order perturbation theory.

The Hamiltonian of the system is

$$\hat{H}=\hat{H}_0+\hat{V} = \sum_{\mathbf{k}}\epsilon_k c_{\mathbf{k}}^\dagger c_{\mathbf{k}}+\tfrac{1}{2}\sum_{\mathbf{k}_1\mathbf{k}_2\mathbf{p}} V(\mathbf{p})c_{\mathbf{k}_1-\mathbf{p}}^\dagger c_{\mathbf{k}_2}^\dagger c_{\mathbf{k}_2-\mathbf{p}}c_{\mathbf{k}_1} \quad (22)$$

with $\epsilon_k=\frac{1}{2}k^2$. With the definition

$$Q(\mathbf{q},\mathbf{k})\equiv A_{\mathbf{k}}A_{\mathbf{k}-\mathbf{q}}^* \quad (23)$$

we have, for $\mathbf{q}\neq 0$,

$$n(\mathbf{q})=\sum_{\mathbf{k}\mathbf{k}'} Q(\mathbf{q},\mathbf{k})(\psi_{\mathbf{k}-\mathbf{q}},c_{\mathbf{k}'-\mathbf{q}}^\dagger c_{\mathbf{k}'}\psi_{\mathbf{k}}) = \sum_{\mathbf{k}\mathbf{k}'} Q(\mathbf{q},\mathbf{k})((\phi_{\mathbf{k}-\mathbf{q}}+\phi_{\mathbf{k}-\mathbf{q}}^{(1)}),c_{\mathbf{k}'-\mathbf{q}}^\dagger c_{\mathbf{k}'}(\phi_{\mathbf{k}}+\phi_{\mathbf{k}}^{(1)})) \ , \quad (24)$$

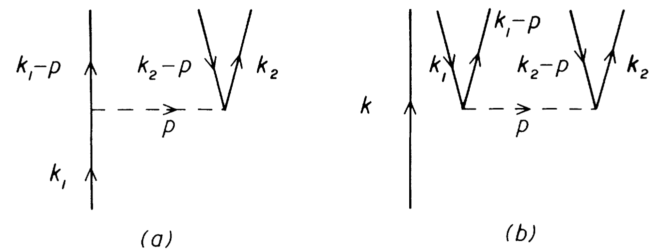

FIG. 1. First-order corrections to the state $\phi_{\mathbf{k}}$.

where

$$\phi_{\mathbf{k}}^{(1)}=(E_k-\hat{H}_0)^{-1}(1-\hat{P})\hat{V}\phi_{\mathbf{k}} \quad (25)$$

with $E_k=\hat{H}_0\phi_k$. The projection operator $1-\hat{P}$ eliminates the possibility that $\hat{V}$ scatters the state $\phi_{\mathbf{k}}$ back into the same state. Equation (25) is graphically depicted in Fig. 1. The first-order correction $\phi_{\mathbf{k}}^{(1)}$ to the state $\phi_{\mathbf{k}}$ that contributes to Eq. (24) describes an electron in the state $\phi_{\mathbf{k}_1-\mathbf{p}}$ and an electron-hole pair of momentum $\mathbf{p}$ [Fig. 1(a)]. For this to give a contribution to $n(\mathbf{q})$, Eq. (24), there are two possibilities: Either $c_{\mathbf{k}'-\mathbf{q}}^\dagger c_{\mathbf{k}'}$ can annihilate the electron-hole pair, which gives $\mathbf{p}=\mathbf{q}$, or these operators can annihilate the hole and the particle of momentum $\mathbf{k}_1-\mathbf{p}$. This gives $\mathbf{p}=\mathbf{k}'-\mathbf{k}$. [There are also other first-order corrections to $\phi_{\mathbf{k}}$; for example, the term represented by Fig. 1(b). However, these do not contribute to Eq. (24).] For $q=0$, however, the term $\hat{V}\phi_{\mathbf{k}}$ does *not* lead to states with an additional electron-hole pair. Thus for $\mathbf{q}=0$, $n(\mathbf{q})$ has no first-order correction consistent with the fact that it represents the total charge, $N+1$ which is conserved.

We define a function $Q(\mathbf{q})$ by

$$Q(\mathbf{q})\equiv\sum_{\mathbf{k}} Q(\mathbf{q},\mathbf{k})=\sum_{\mathbf{k}} A_{\mathbf{k}}A_{\mathbf{k}-\mathbf{q}}^* \ . \quad (26a)$$

This function is analytic at $\mathbf{q}=0$, and

$$\lim_{q\to 0} Q(\mathbf{q})=Q(0)=1 \ . \quad (26b)$$

Carrying out the simple calculations in Eq. (24) we then have

$$n(\mathbf{q})=\begin{cases} N+1, & \mathbf{q}=0 \\ Q(\mathbf{q})-\sum_{\mathbf{k}\mathbf{k}'} Q(\mathbf{q},\mathbf{k})\dfrac{n_\circ(\mathbf{k}'-\mathbf{q})-n_\circ(\mathbf{k}')}{\mathbf{q}\cdot(\mathbf{k}'-\mathbf{k})}[V(q)-V(\mathbf{k}'-\mathbf{k})], & \mathbf{q}\neq 0 \ . \end{cases} \quad (27)$$

From the value for $\mathbf{q}=0$ we see that the total charge is indeed conserved. The second term for $\mathbf{q}\neq 0$ in Eq. (27) is proportional to the available phase-space and inversely proportional to the energy difference for a scattering event. Both these quantities are of order q for small q. This leads to

$$\lim_{\mathbf{q}\to 0} n(\mathbf{q})=1-\sum_{\mathbf{k}\mathbf{k}'} |A_{\mathbf{k}}|^2\frac{\mathbf{q}\cdot\hat{\mathbf{k}}'\delta(k'-k_F)}{\mathbf{q}\cdot(\mathbf{k}'-\mathbf{k})}[V(0)-V(\mathbf{k}'-\mathbf{k})] \ , \quad (28)$$

where, as we will prove in the following sections, the second term is nonzero. Hence there is indeed a discontinuity in $n(\mathbf{q})$ at $\mathbf{q}=0$ (in addition to the one arising from the uniform charge density of the filled Fermi sphere) which, as we shall see, implies that part of the charge contained in the QPWP has been delocalized.

From Eqs. (18) and (27) we immediately have

$$j(\mathbf{q})=\begin{cases}\mathbf{k}_0, & \mathbf{q}=0\\ \sum_{\mathbf{k}} Q(\mathbf{q},\mathbf{k})\left[(\mathbf{k}-\tfrac{1}{2}\mathbf{q})-\sum_{\mathbf{k}'}(\mathbf{k}'-\tfrac{1}{2}\mathbf{q})\frac{n_\circ(\mathbf{k}'-\mathbf{q})-n_\circ(\mathbf{k}')}{\mathbf{q}\cdot(\mathbf{k}'-\mathbf{k})}[V(q)-V(\mathbf{k}'-\mathbf{k})]\right], & \mathbf{q}\neq 0\ .\end{cases} \tag{29}$$

Thus the total current of the system is $\mathbf{k}_0$ and is conserved, but

$$\lim_{\mathbf{q}\to 0}\mathbf{j}(\mathbf{q})=\mathbf{k}_0-\sum_{\mathbf{k}\mathbf{k}'}|A_{\mathbf{k}}|^2\mathbf{k}'\frac{\mathbf{q}\cdot\hat{\mathbf{k}}'\delta(k'-k_F)}{\mathbf{q}\cdot(\mathbf{k}'-\mathbf{k})}\times[V(0)-V(\mathbf{k}'-\mathbf{k})]\ . \tag{30}$$

In later sections we will prove that $\lim_{\mathbf{q}\to 0}\mathbf{j}(\mathbf{q})$ is *not* equal to $\mathbf{k}_0$. As with the charge we can then conclude that part of the current has been delocalized.

For $\mathbf{q}\neq 0$ we define a quantity $\Delta n(\mathbf{q})$ by

$$\Delta n(\mathbf{q})\equiv n(\mathbf{q})-Q(q)=-\sum_{\mathbf{k}}Q(\mathbf{q},\mathbf{k})\sum_{\mathbf{k}'}\frac{n_\circ(\mathbf{k}'-\mathbf{q})-n_\circ(\mathbf{k}')}{\mathbf{q}\cdot(\mathbf{k}'-\mathbf{k})}\times[V(q)-V(\mathbf{k}'-\mathbf{k})]\ . \tag{31}$$

The envelope function $A_{\mathbf{k}}$ has a length scale α and for $q>\Delta k$, $A_{\mathbf{k}}A^*_{\mathbf{k}-\mathbf{q}}$ rapidly approaches zero. The function multiplying $Q(\mathbf{q},\mathbf{k})$,

$$f(\mathbf{q},\mathbf{k})=-\sum_{\mathbf{k}'}\frac{n_\circ(\mathbf{k}'-\mathbf{q})-n_\circ(\mathbf{k}')}{\mathbf{q}\cdot(\mathbf{k}'-\mathbf{k})}[V(q)-V(\mathbf{k}'-\mathbf{k})] \tag{32}$$

contains the atomic length scales κ^{-1}, the range of the interaction, and the Fermi length k_F^{-1}. These are of similar magnitudes and, by construction, much smaller than the range α of the envelope function. Hence when $\mathbf{q}$ is sufficiently large that $f(\mathbf{q},\mathbf{k})$ starts to vary as a function of q, i.e., when q is of the order of k_F, $Q(\mathbf{q},\mathbf{k})$ is effectively zero. Therefore, in Eq. (31) we can make the replacement

$$f(\mathbf{q},\mathbf{k})\to\lim_{\mathbf{q}\to 0}f(\mathbf{q},\mathbf{k})\ . \tag{33}$$

Here the limit is taken with the direction of $\mathbf{q}$ fixed. This function of $\mathbf{k}$ varies on the scale of k_F and we can therefore make the further substitution

$$f(\mathbf{q},\mathbf{k})\to\lim_{\mathbf{q}\to 0}f(\mathbf{q},\mathbf{k}_0)=f(\theta_{\mathbf{q}},\phi_{\mathbf{q}})\ , \tag{33a}$$

where $\theta_{\mathbf{q}},\phi_{\mathbf{q}}$ are the angles of $\mathbf{q}$ relative to $\mathbf{k}_0$. The right-hand side of Eq. (31) can therefore be factored,

$$\Delta n(\mathbf{q})=Q(\mathbf{q})f(\theta_{\mathbf{q}},\phi_{\mathbf{q}})\ , \tag{34}$$

where

$$f(\theta_{\mathbf{q}},\phi_{\mathbf{q}})=-\sum_{\mathbf{k}'}\frac{\mathbf{q}\cdot\hat{\mathbf{k}}'\delta(k'-k_F)}{\mathbf{q}\cdot(\mathbf{k}'-\mathbf{k}_0)}[V(0)-V(\mathbf{k}'-\mathbf{k}_0)]\ . \tag{35}$$

For the current density we similarly define

$$\Delta\mathbf{j}(\mathbf{q})\equiv-\sum_{\mathbf{k},\mathbf{k}'}Q(\mathbf{q},\mathbf{k})(\mathbf{k}'-\tfrac{1}{2}\mathbf{q})\frac{n_\circ(\mathbf{k}'-q)-n_\circ(\mathbf{k}')}{\mathbf{q}\cdot(\mathbf{k}'-\mathbf{k})}\times[V(q)-V(\mathbf{k}'-\mathbf{k})] \tag{36}$$

for $\mathbf{q}\neq 0$. Repeating the arguments for $\Delta n(\mathbf{q})$ we can then write

$$\Delta\mathbf{j}(\mathbf{q})=\mathbf{g}(\theta_{\mathbf{q}},\phi_{\mathbf{q}})Q(\mathbf{q})\ , \tag{37}$$

where

$$\mathbf{g}(\theta_{\mathbf{q}},\phi_{\mathbf{q}})\equiv-\sum_{\mathbf{k}'}\mathbf{k}'\frac{\mathbf{q}\cdot\hat{\mathbf{k}}'\delta(k'-k_F)}{\mathbf{q}\cdot(\mathbf{k}'-\mathbf{k}_0)}[V(0)-V(\mathbf{k}'-\mathbf{k}_0)]\ . \tag{38}$$

We shall see that the quantities $\Delta n(\mathbf{q})$ and $\Delta\mathbf{j}(\mathbf{q})$ tend to finite (angle-dependent) limits as $\mathbf{q}\to 0$, thereby causing the discontinuities in $n(\mathbf{q})$ and $\mathbf{j}(\mathbf{q})$ at $\mathbf{q}=0$.

To verify the continuity equation, we consider its Fourier transform

$$\frac{\partial}{\partial t}n(\mathbf{q},t)+i\mathbf{q}\cdot\mathbf{j}(\mathbf{q},t)=0\ ; \tag{39}$$

We use the interaction representation[9] for the time development. In this representation, the time development of any operator $\hat{\mathbf{O}}$ is given by

$$\hat{\mathbf{O}}(t)=e^{i\hat{\mathbf{H}}_0 t}\hat{\mathbf{O}}e^{-i\hat{\mathbf{H}}_0 t} \tag{40}$$

and for the state $\Psi_{\mathbf{k}_0}$ we have

$$\Psi_{\mathbf{k}_0}(t)=\hat{\mathbf{U}}(t,0)\Psi_{\mathbf{k}_0} \tag{41}$$

where, to first order in the interaction,

$$\hat{\mathbf{U}}(t,0)=\hat{\mathbf{U}}^\dagger(0,t)=1-i\int_0^t dt' e^{i\hat{\mathbf{H}}_0 t'}\hat{\mathbf{V}}e^{-i\hat{\mathbf{H}}_0 t'}\ . \tag{42}$$

Combining Eqs. (40)–(42) we obtain to first order in V for $\mathbf{q}\neq 0$

$$n(\mathbf{q},t)=(\Psi_{\mathbf{k}_0}(t),\, \hat{\mathbf{n}}(\mathbf{q},t)\Psi_{\mathbf{k}_0}(t))$$

$$=\sum_{\mathbf{k}} Q(\mathbf{q},\mathbf{k})\left[e^{-i(\epsilon_{\mathbf{k}}-\epsilon_{\mathbf{k}-\mathbf{q}})t}-\sum_{\mathbf{k}'}\frac{n_\circ(\mathbf{k}'-\mathbf{q})-n_\circ(\mathbf{k}')}{\mathbf{q}\cdot(\mathbf{k}'-\mathbf{k})}[V(0)-V(\mathbf{k}'-\mathbf{k})]e^{-i(\epsilon_{\mathbf{k}}-\epsilon_{\mathbf{k}-\mathbf{q}})t}\right.$$
$$-i\sum_{\mathbf{k}'}\left[\phi_{\mathbf{k}-\mathbf{q}},e^{i\hat{\mathbf{H}}_0 t}c^\dagger_{\mathbf{k}'-\mathbf{q}}c_{\mathbf{k}'}\int_0^t dt' e^{i\hat{\mathbf{H}}_0 t'}\hat{V}e^{-i\hat{\mathbf{H}}_0 t'}\phi_{\mathbf{k}}\right]$$
$$\left.+i\sum_{\mathbf{k}'}\left[\phi_{\mathbf{k}-\mathbf{q}},\int_0^t dt' e^{i\hat{\mathbf{H}}_0 t'}\hat{V}e^{-i\hat{\mathbf{H}}_0 t'}c^\dagger_{\mathbf{k}'-\mathbf{q}}\,c_{\mathbf{k}'}e^{-i\hat{\mathbf{H}}_0 t}\phi_{\mathbf{k}}\right]\right] \tag{43}$$

In the last two summations in Eq. (43), the scattering by $\hat{V}$ is not restricted to scattering from $\phi_{\mathbf{k}}$ only to states that are different, but scattering from $\phi_{\mathbf{k}}$ back to $\phi_{\mathbf{k}}$ also contributes. All the different scattering events that contribute are depicted in Fig. 2. The contributions from Figs. 2(a) and 2(b) cancel out between the two summations, and after performing the time integrations we are left with

$$n(\mathbf{q},t)=\sum_{\mathbf{k}} Q(\mathbf{q},\mathbf{k})e^{-i(\epsilon_{\mathbf{k}}-\epsilon_{\mathbf{k}-\mathbf{q}})t}\left[1-it[\Sigma(\mathbf{k})-\Sigma(\mathbf{k}-\mathbf{q})]-\sum_{\mathbf{k}'}\frac{n_\circ(\mathbf{k}'-\mathbf{q})-n_\circ(\mathbf{k}')}{\mathbf{q}\cdot(\mathbf{k}'-\mathbf{k})}[V(q)-V(\mathbf{k}'-\mathbf{k})]\right]. \tag{44}$$

Here

$$\Sigma(\mathbf{k})=\sum_{\mathbf{k}'}n_\circ(\mathbf{k}')[V(0)-V(\mathbf{k}'-\mathbf{k})] \tag{45}$$

[cf. Figs. 2(c) and 2(d).] We can then immediately write down the expression for $\mathbf{j}(\mathbf{q},t)$:

$$\mathbf{j}(\mathbf{q},t)=\sum_{\mathbf{k}} Q(\mathbf{q},\mathbf{k})e^{-i(\epsilon_{\mathbf{k}}-\epsilon_{\mathbf{k}-\mathbf{q}})t}\left[(\mathbf{k}-\tfrac{1}{2}\mathbf{q})-it(\mathbf{k}-\tfrac{1}{2}\mathbf{q})[\Sigma(\mathbf{k})-\Sigma(\mathbf{k}-\mathbf{q})]\right.$$
$$\left.-\sum_{\mathbf{k}'}(\mathbf{k}'-\tfrac{1}{2}\mathbf{q})\frac{n_\circ(\mathbf{k}'-\mathbf{q})-n_\circ(\mathbf{k}')}{\mathbf{q}\cdot(\mathbf{k}'-\mathbf{k})}[V(q)-V(\mathbf{k}'-\mathbf{k})]\right] \tag{46}$$

From Eqs. (44) and (46) we then have

$$\frac{\partial}{\partial t}\mathbf{n}(\mathbf{q},t)+i\mathbf{q}\cdot\mathbf{j}(\mathbf{q},t)=i\sum_{\mathbf{k}} Q(\mathbf{q},\mathbf{k})e^{-i(\epsilon_{\mathbf{k}}-\epsilon_{\mathbf{k}-\mathbf{q}})t}\left[-[\Sigma(\mathbf{k})-\Sigma(\mathbf{k}-\mathbf{q})]+\sum_{\mathbf{k}'}(\mathbf{q}\cdot\mathbf{k})\frac{n_\circ(\mathbf{k}'-\mathbf{q})-n_\circ(\mathbf{k}')}{\mathbf{q}\cdot(\mathbf{k}'-\mathbf{k})}[V(q)-V(\mathbf{k}'-\mathbf{k})]\right.$$
$$\left.-\sum_{k'}(\mathbf{q}\cdot\mathbf{k}')\frac{n_\circ(\mathbf{k}'-\mathbf{q})-n_\circ(\mathbf{k}')}{\mathbf{q}\cdot(\mathbf{k}'-\mathbf{k})}[V(q)-V(\mathbf{k}'-\mathbf{k})]\right]$$
$$=i\sum_{\mathbf{k}} Q(\mathbf{q},\mathbf{k})e^{-i(\epsilon_{\mathbf{k}}-\epsilon_{\mathbf{k}-\mathbf{q}})t}\left[-[\Sigma(\mathbf{k})-\Sigma(\mathbf{k}-\mathbf{q})]-\sum_{\mathbf{k}'}[n_\circ(\mathbf{k}'-\mathbf{q})-n_\circ(\mathbf{k}')][V(q)-V(\mathbf{k}'-\mathbf{k})]\right]. \tag{47}$$

From Eq. (45) we have

$$\Sigma(\mathbf{k})-\Sigma(\mathbf{k}-\mathbf{q})=\sum_{\mathbf{k}'}n_\circ(\mathbf{k}')[V(0)-V(\mathbf{k}'-\mathbf{k})]$$
$$-\sum_{\mathbf{k}'}n_\circ(\mathbf{k}')[V(0)-V(\mathbf{k}'-\mathbf{k}+\mathbf{q})]$$
$$=\sum_{\mathbf{k}'}[n_\circ(\mathbf{k}'-\mathbf{q})-n_\circ(\mathbf{k}')][V(\mathbf{k}'-\mathbf{k})]. \tag{48}$$

Since $\sum_{\mathbf{k}'}[n_0(\mathbf{k}')-n_0(\mathbf{k}'-\mathbf{q})]=0$, it follows by inspection that the term in Eq. (47) proportional to $V(q)$ vanishes. When we now substitute Eq. (48) in Eq. (47) we obtain

$$\frac{\partial}{\partial t}n(\mathbf{q},t)+i\mathbf{q}\cdot\mathbf{j}(\mathbf{q},t)=0 \tag{39}$$

for $\mathbf{q}\neq 0$. For $\mathbf{q}=0$ the continuity equation is trivially satisfied. Hence the continuity equation is satisfied for each $\mathbf{q}$ to first order in V, and thus at each point $\mathbf{r}$.

Finally, let us for future reference write down the continuity equation for q vanishingly small. For $q\to 0$ and $\mathbf{k}\approx k_0$

$$\Sigma(\mathbf{k})-\Sigma(\mathbf{k}-\mathbf{q})\to -\mathbf{q}\cdot\nabla_{\mathbf{k}}\Sigma(\mathbf{k})=-\mathbf{q}\cdot\left[\frac{\mathbf{k}_0}{m^*}-\mathbf{k}_0\right], \tag{49}$$

where m^* is the conventional effective mass of a quasiparticle. With Eq. (49), the substitution

$$\sum_{\mathbf{k}\mathbf{k}'} Q(\mathbf{q},\mathbf{k})\mathbf{k}\frac{\mathbf{q}\cdot\hat{\mathbf{k}}'\delta(k'-k_F)}{\mathbf{q}\cdot(\mathbf{k}'-\mathbf{k})}[V(q)-V(\mathbf{k}'-\mathbf{k})]$$
$$=\mathbf{k}_0\sum_{\mathbf{q}\mathbf{k}'} Q(\mathbf{q},\mathbf{k})\frac{\mathbf{q}\cdot\hat{\mathbf{k}}'\delta(k'-k_F)}{\mathbf{q}\cdot(\mathbf{k}'-\mathbf{k}_0)}[V(q)-V(\mathbf{k}'-\mathbf{k})] \tag{50}$$

and the replacement $\mathbf{k}_0\to\mathbf{k}_F$, where $\mathbf{k}_F$ is in the direction defined by $\mathbf{k}_0$, inserted in Eq. (47), we arrive at

$$\mathbf{q}\cdot\left[\frac{\mathbf{k}_F}{m^*}-\mathbf{k}_F\right]-\mathbf{q}\cdot\Delta\mathbf{j}(\mathbf{q})+\mathbf{q}\cdot\mathbf{k}_F\Delta n(\mathbf{q})=0 \quad (q\to 0) . \tag{51}$$

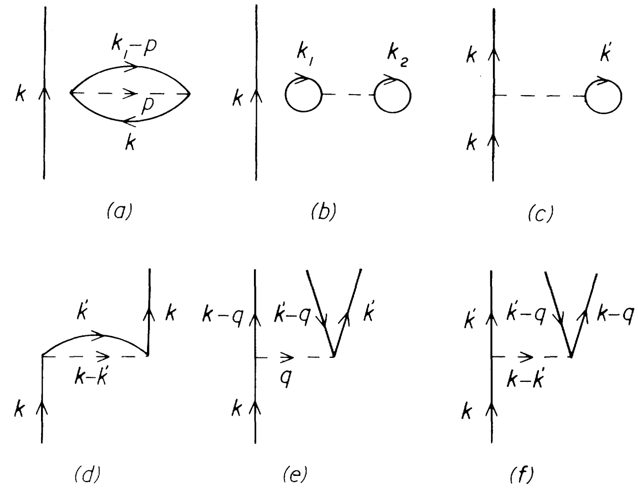

FIG. 2. Diagrams that contribute to the last two summations in Eq. (39). The first-order self-energy is given by the diagrams 2(c) and 2(d).

IV. ONE-DIMENSIONAL SYSTEM

In one dimension, we have from Eq. (34)

$$\Delta n(q)=\sum_{k'}\frac{\delta(k'+k_F)-\delta(k'-k_F)}{(k'-k_0)}[V(0)-V(k'-k_0)]Q(q)$$
$$=\frac{1}{2\pi}\left[\frac{1}{k_0-k_F}[V(0)-V(k_0-k_F)]-\frac{1}{k_0+k_F}[V(0)-V(k_0+k_F)]\right]Q(q) . \tag{52}$$

Since k_0 is very near k_F we can expand $V(k_0-k_F)$ in a Taylor series around $k_0-k_F=0$ and $V(k_0+k_F)$ in a Taylor series around $2k_F$. From our assumptions of $V(r)$ it follows that $(\partial/\partial q)V(q)|_{q=0}=0$. If we then insert the expansions in (52) and let k_0 approach k_F from above we obtain

$$\Delta n(q)=-\frac{1}{4\pi k_F}[V(0)-V(2k_F)]Q(q) . \tag{53}$$

From Eq. (53) it is clear that

$$\lim_{q\to 0}\Delta n(q)=-\frac{1}{4\pi k_F}[V(0)-V(2k_F)]<0 \tag{54}$$

since we have assumed that $V(0)>V(2k_F)$. We can then conclude that the localized charge of the QPWP is e^*, where

$$e^*=1-\frac{1}{4\pi k_F}[V(0)-V(2k_F)]<1 . \tag{55}$$

The current density, in one dimension, is

$$\Delta j(q)=\sum_{k'}\frac{k_F}{k_0-k_F}[\delta(k'-k_F)-\delta(k'+k_F)]$$
$$\times[V(0)-V(k'-k_0]Q(q)$$
$$=\frac{k_F}{2\pi}\left[\frac{1}{k_0-k_F}[V(0)-V(k_0-k_F)]+\frac{1}{k_0+k_F}[V(0)-V(k_0+k_F)]\right]Q(q) . \tag{56}$$

Following the same arguments as with $\Delta n(q)$, we obtain

$$\Delta j(q)=\frac{1}{4\pi}[V(0)-V(k_0-k_F)]Q(q) . \tag{57}$$

From this we see that

$$\lim_{q\to 0}\Delta j(q)=\frac{1}{4\pi}[V(0)-V(k_0-k_F)] . \tag{58}$$

For $k_0\to k_F$, the localized current associated with the QPWP is then

$$j^*=k_F\left[1+\frac{1}{4\pi k_F}[V(0)-V(2k_F)]\right]>k_F . \tag{59}$$

Since total charge and total current are conserved, we conclude from Eqs. (53) and (57) that there a charge $N+1-e^*$ and a current k_F-j^* at infinity.[10]

We can also easily calculate the effective mass of the QPWP. We have

$$\frac{k_0}{m^*}=\frac{\partial}{\partial k}[\epsilon_k+\Sigma(k)]_{k=k_0}$$
$$=k_0-\frac{1}{2\pi}\int dk'\frac{\partial}{\partial k'}n_0(k')V(k'-k_0) . \tag{60}$$

As k_0 approaches k_F, this gives

$$\frac{k_F}{m^*}=k_F+\frac{1}{2\pi}[V(0)-V(2k_F)] . \tag{61}$$

We now verify that the charge e^* and the current j^* are related by the continuity equation. Expanding the ratio of j^* and e^* to first order in the interaction and using

Eq. (61) we obtain

$$\frac{j^*}{e^*}=\frac{k_F}{m^*} \tag{62}$$

This is consistent with the fact that j^* is the current of a charge e^* moving with a velocity k_F/m^*.

We can summarize the one-dimensional system as follows: We imagine the system to be a ring. Injecting a localized electron at the Fermi level then gives rise to a total current of magnitude k_F. This total current has two components. One is the current of a localized charge e^* traveling with a velocity k_F/m^*. The other component comes from the charge $N+1-e^*$ at infinity and has a sign *opposite* to the current of the localized charge.

V. THREE-DIMENSIONAL SYSTEM

For definiteness, we will now take the envelope function to be a spherically symmetric Gaussian, for which $Q(q)=e^{-q^2\alpha^2/4}$ [cf. Eq. (26a)].

Returning to Eq. (34), we have in three dimensions

$$\Delta n(\mathbf{q})=\begin{cases} f(\theta_{\mathbf{q}},\phi_{\mathbf{q}})e^{-q^2\alpha^2/4}\ , & \mathbf{q}\neq 0\ ,\\ 0\ , & \mathbf{q}=0\end{cases} \tag{34'}$$

where

$$f(\theta_{\mathbf{q}},\phi_{\mathbf{q}})=-\sum_{\mathbf{k}'}\frac{\mathbf{q}\cdot\hat{\mathbf{k}}'\delta(k'-k_F)}{\mathbf{q}\cdot(\mathbf{k}'-\mathbf{k}_0)}[V(0)-V(\mathbf{k}'-\mathbf{k}_0)]\ . \tag{35'}$$

and $(\theta_{\mathbf{q}},\phi_{\mathbf{q}})$ give the direction of $\mathbf{q}$ relative to $\mathbf{k}_0$. We expand $f(\theta_{\mathbf{q}},\theta_{\mathbf{q}})$ in spherical harmonics. From symmetry it follows that

$$f(\theta_{\mathbf{q}},\phi_{\mathbf{q}})=\sum_l f_l P_l(\cos\theta_{\mathbf{q}})\ , \tag{63}$$

where l are even integers and the coefficients f_l are given by

$$f_l=\frac{2l+1}{4\pi}\int f_l(\theta_{\mathbf{q}})P_l(\cos\theta_{\mathbf{q}})d\Omega_{\mathbf{q}}\ . \tag{64}$$

The first-order correction to the charge density is

$$\Delta n(\mathbf{r})=\sum_l \Delta n_l(\mathbf{r})=\sum_{\mathbf{q}}\Delta n_l(\mathbf{q})e^{i\mathbf{q}\cdot r}$$
$$=\sum_{\mathbf{q}}\sum_l f_l P_l(\cos\theta_{\mathbf{q}})e^{-q^2\alpha^2/4}e^{i\mathbf{q}\cdot\mathbf{r}}\ . \tag{65}$$

We shall prove below that the localized parts (i.e., the parts that are *not* at infinity) have the following form: The spherically symmetric part is integrable and nonzero at the origin, whereas the higher spherical harmonics falls of as r^{-3} for $r\gg\alpha$ and goes to zero as r goes to zero. Before we proceed to prove this, let us first note that, for a fixed point $\mathbf{r}$, it is permissible to perform the Fourier transforms by converting the sums over $\mathbf{q}$ to integrals. In a volume L^3 we will at the most make errors of order L^{-3} if we convert the summations to integrals. As $L\to\infty$ these errors can then be ignored.

The spherical part of $\Delta n(\mathbf{r})$ is then

$$\Delta n_0(\mathbf{r})=\frac{f_0}{2\pi^2}\int_0^\infty e^{-q^2\alpha^2/4}j_0(qr)q^2dq$$
$$=\frac{f_0}{(\alpha\sqrt{\pi})^3}e^{-r^2/\alpha^2} \tag{66}$$

and is thus nonzero at the origin and integrable. For $l\geq 2$ and $f_l\neq 0$ we have

$$\Delta n_l(\mathbf{r})=\frac{2}{\pi^{3/2}}i^l f_l P_l(\cos\theta)\frac{\Gamma(l/2+\frac{3}{2})}{\Gamma(l+\frac{3}{2})}\frac{r^l}{\alpha^{l+3}}$$
$$\times {}_1F_1\left[\frac{l}{2}+\frac{3}{2},l+\frac{3}{2};-\frac{r^2}{\alpha^2}\right]\ , \tag{67}$$

where ${}_1F_1(a,b;z)$ is the confluent hypergeometric function. The asymptotic form $(z\gg 1)$ of this function is given by

$$\frac{\Gamma(a)}{\Gamma(b)}\,{}_1F_1(a,b;-z)\sim\frac{\Gamma(a)}{\Gamma(b-a)}(-z)^{-a}\ ; \tag{68}$$

so for $r\gg\alpha$,

$$\Delta n(r)=\sum_{l\geq 2}\frac{2i^l f_l}{\pi^{3/2}}P_l(\cos\theta)\frac{\Gamma(l/2+\frac{3}{2})}{\Gamma(l/2)}r^{-3}\ . \tag{69}$$

Moreover, when $r\ll\alpha$ $\Delta n_l(r)$ goes to zero as r^l.

To study the charge at infinity, we separate from $\Delta n(\mathbf{q})$ the term containing f_0:

$$\Delta n(\mathbf{q})=\Delta n_0(\mathbf{q})+\sum_{l\geq 2}\Delta n_l(\mathbf{q})\ , \tag{70}$$

where

$$\Delta n_0(\mathbf{q})=\begin{cases} f_0 e^{-q^2\alpha^2/4}\ , & \mathbf{q}\neq 0\\ 0\ , & \mathbf{q}=0\end{cases} \tag{71a}$$

$$\equiv f_0 e^{-q^2\alpha^2/4}-\delta_{\mathbf{q},0}f_0 \tag{71b}$$

and

$$\Delta n_l(q)=\begin{cases} f_l P_l(\cos\theta_{\mathbf{q}})e^{-q^2\alpha^2/4}\ , & \mathbf{q}\neq 0\\ 0\ , & \mathbf{q}=0\end{cases}\quad l\geq 2\ . \tag{72}$$

Referring to Eq. (71b), let us consider the density distribution, $\Delta n_0(\mathbf{r})$, corresponding to $\Delta n_0(\mathbf{q})$ in a large volume L^3:

$$\Delta n_0(\mathbf{r})=\frac{f_0}{L^3}\sum_{\mathbf{q}}e^{-q^2\alpha^2/4}e^{i\mathbf{q}\cdot\mathbf{r}}-\frac{f_0}{L^3}\ . \tag{73}$$

For fixed $\mathbf{r}$, we replace the summation over $\mathbf{q}$ and $\mathbf{k}$ by integrals as $L\to\infty$. The first term then represents a localized charge distribution of total charge $f_0(\neq 0)$ and shape identical to the unperturbed QPWP. Since, by Eq. (71a), $\Delta n_0(\mathbf{q})=0$ for $\mathbf{q}=0$, it follows that $\int\Delta n_0(\mathbf{r})d\mathbf{r}=0$, so that a charge of $-f_0$ must reside at infinity.

From the second line of Eq. (72) we see that $\Delta n_l(\mathbf{q})$ $(l\geq 2)$ does not represent any net charge in the system. For these functions the discontinuities at $\mathbf{q}=0$ give rise to the r^{-3} tails of the corresponding charge densities. In Appendix A we show that $f_0<0$, so that there is an excess of charge at infinity.

We now turn to the current density

$$\Delta\mathbf{j}(\mathbf{r})=\sum_{\mathbf{q}}\Delta\mathbf{j}(\mathbf{q})e^{i\mathbf{q}\cdot\mathbf{r}}$$

with $\Delta\mathbf{j}(\mathbf{q})$ given by Eq. (37). We denote the Cartesian components of $\Delta\mathbf{j}(\mathbf{r})$ by $\Delta j_x(\mathbf{r})$, $\Delta j_y(\mathbf{r})$, and $\Delta j_z(\mathbf{r})$, and those of $\Delta\mathbf{j}(\mathbf{q})$ by $\Delta j_x(\mathbf{q})$, $\Delta j_y(\mathbf{q})$, and $\Delta j_z(\mathbf{q})$, respectively. We expand the components of $\Delta j(\mathbf{q})$ in spherical harmonics, and from symmetry arguments the expansions have the forms

$$\Delta j_z(\mathbf{q})=\sum_l g_l P_l(\cos\theta_{\mathbf{q}})e^{-q^2\alpha^2/4}\ , \tag{74a}$$

$$\Delta j_x(\mathbf{q})=\cos\phi_{\mathbf{q}}\sum_l g_{\perp,l}P_l^1(\cos\theta_{\mathbf{q}})e^{-q^2\alpha^2/4}\ , \tag{74b}$$

$$\Delta j_y(\mathbf{q})=\sin\phi_{\mathbf{q}}\sum_l g_{\perp,l}P_l^1(\cos\theta_{\mathbf{q}})e^{-q^2\alpha^2/4}\ , \tag{74c}$$

where l are even integers. If we transform to $\Delta\mathbf{j}(\mathbf{r})$ we obtain

$$\Delta j_z(\mathbf{r})=\frac{g_0}{(\alpha\sqrt{\pi})^3}e^{-r^2/\alpha^2}+\frac{2}{\pi^{3/2}}\sum_{l\geq 2}g_l P_l(\cos\theta)\frac{i^l r^l}{\alpha^{l+3}}\frac{\Gamma(l/2+\frac{3}{2})}{\Gamma(l+\frac{3}{2})}\times{}_1F_1\left[\frac{l}{2}+\frac{3}{2},l+\frac{3}{2};-\frac{r^2}{\alpha^2}\right]\ , \tag{75a}$$

$$\Delta j_x(\mathbf{r})=\frac{2}{\pi^{3/2}}\cos\phi\sum_{l\geq 2}g_{\perp,l}P_l^1(\cos\theta)\frac{i^l r^l}{\alpha^{l+3}}\frac{\Gamma(l/2+\frac{3}{2})}{\Gamma(l+\frac{3}{2})}\times{}_1F_1\left[\frac{l}{2}+\frac{3}{2},l+\frac{3}{2};\frac{r^2}{\alpha^2}\right]\ , \tag{75b}$$

$$\Delta j_y(\mathbf{r})=\frac{2}{\pi^{3/2}}\sin\phi\sum_{l\geq 2}g_{\perp,l}P_l^1(\cos\theta)\frac{i^l r^l}{\alpha^{l+3}}\frac{\Gamma(l/2+\frac{3}{2})}{\Gamma(l+\frac{3}{2})}\times{}_1F_1\left[\frac{l}{2}+\frac{3}{2},l+\frac{3}{2};-\frac{r^2}{\alpha^2}\right]\ . \tag{75c}$$

From Eq. (75a) we see that $\Delta j_z(\mathbf{r})$ has a spherically symmetric, integrable part. For $r\gg\alpha$, the components of $\mathbf{j}(\mathbf{r})$ have the asymptotic forms

$$\Delta j_z(\mathbf{r})=\frac{2}{(r\sqrt{\pi})^3}\sum_{l\geq 2}i^l g_l P_l(\cos\theta)\frac{\Gamma(l/2+\frac{3}{2})}{\Gamma(l/2)}\ , \tag{76a}$$

$$\Delta j_x(\mathbf{r})=\frac{2}{(r\sqrt{\pi})^3}\cos\phi\sum_{l\geq 2}i^l g_{\perp,l}P_l^1(\cos\theta)\frac{\Gamma(l/2+\frac{3}{2})}{\Gamma(l/2)}\ , \tag{76b}$$

$$\Delta j_y(\mathbf{r})=\frac{2}{(r\sqrt{\pi})^3}\sin\phi\sum_{l\geq 2}i^l g_{\perp,l}P_l^1(\cos\theta)\frac{\Gamma(l/2+\frac{3}{2})}{\Gamma(l/2)}\ .$$

Thus the higher spherical harmonics of $\mathbf{j}(\mathbf{r})$ have r^{-3} tails, similar to the higher harmonics of $\Delta n(\mathbf{r})$.

From conservation of total current and Eq. (74a) (in the limit $\mathbf{q}\to 0$) it follows that there is a current density $-g_0\hat{\mathbf{z}}$ at infinity. In Appendix A we show that $g_0>0$ so this particle current is, as in the one-dimensional system, in the direction opposite to the propagation of the wave packet.

The following considerations refer to charge and current densities at arbitrary fixed position vectors $\mathbf{r}$ in the limit $\Omega^{1/3}/r\to\infty$.

In view of the r^{-3} behavior of the charge and current densities there is no obvious way to define a localized charge e^* and a localized current j^*. One would be tempted to define $e^*\equiv 1+f_0$ and $\mathbf{j}^*\equiv(k_0+g_0)\hat{\mathbf{z}}$, i.e., as the spherical parts of $n(\mathbf{r})$ and $\mathbf{j}(\mathbf{r})$, but we shall now show that these definitions do *not* satisfy the integrated form of the continuity equation, $\mathbf{k}_0 e^*/m^*=\mathbf{j}^*$. From Eqs. (44), (46), and (49) we obtain

$$\frac{\mathbf{k}_0}{m^*}-\mathbf{k}_0-\lim_{\mathbf{q}\to 0}\Delta\mathbf{j}(\mathbf{q})+\mathbf{k}_0\lim_{\mathbf{q}\to 0}\Delta n(\mathbf{q})=\sum_{k'}\frac{\mathbf{q}\times(\hat{\mathbf{k}}'\times\mathbf{k}_0)}{(\mathbf{k}'-\mathbf{k}_0)}[V(0)-V(\mathbf{k}'-\mathbf{k}_0)]\ . \tag{77}$$

The right-hand side of Eq. (77) is a vector perpendicular to $\mathbf{q}$. By forming the scalar product of Eq. (77) and $\mathbf{q}$ we arrive at the continuity equation for q small, Eq. (51). If the spherical average of the right-hand side of Eq. (77) were zero, then the relationship $\mathbf{k}_0 e^*/m^*=\mathbf{j}^*$ would hold. But this spherical average is *not* zero, so the spherical parts alone of $n(\mathbf{r})$ and $\mathbf{j}(\mathbf{r})$ do *not* satisfy the continuity equation. In fact, if we multiply Eq. (51) by $\cos\theta_{\mathbf{q}}$ and integrate over solid angle we obtain

$$k_0\left[\frac{1}{m^*}+f_0\right]-(k_0+g_0)=\tfrac{2}{5}g_2+\tfrac{6}{5}g_{\perp,2}-\tfrac{2}{5}k_0 f_2\neq 0\ . \tag{78}$$

This reflects the fact that whereas the $l=2$ components of charge and current densities behave as r^{-3} asymptotically, the time derivative and divergence of these components have terms that vanish as re^{-r^2/α^2} for $r\gg\alpha$, as well as terms that go as r^{-4} asymptotically.

For r very large, the leading terms to $n(\mathbf{r})$ and $\mathbf{j}(\mathbf{r})$ are the r^{-3} tails. These terms have a common r dependence and therefore must satisfy the continuity equation by themselves. This leads us to the following definitions: For every point $\mathbf{r}$, except $\mathbf{r}=0$, we *define* a long-range charge density, $n_L(\mathbf{r})$, and a long-range current density, $\mathbf{j}_L(\mathbf{r})$, as the asymptotic forms of $n(\mathbf{r})$ and $\mathbf{j}(\mathbf{r})$ taken all the way to the origin. Explicitly,

$$n_L(\mathbf{r})=\frac{2}{(r\sqrt{\pi})^3}\sum_{l\geq 2}i^l f_l\frac{\Gamma(l/2+\frac{3}{2})}{\Gamma(l/2)} \tag{79}$$

and

$$\mathbf{j}_L(\mathbf{r})=\frac{2}{(r\sqrt{\pi})^3}\sum_{l\geq 2} i^l\frac{\Gamma(l/2+\frac{3}{2})}{\Gamma(l/2)}[\hat{\mathbf{z}}g_lP_l(\cos\theta)+(\hat{\mathbf{x}}\cos\phi+\hat{\mathbf{y}}\sin\phi)g_{\perp,l}P_l^1(\cos\theta)] \ . \tag{80}$$

We define, for $\mathbf{r}\neq 0$, a short-range charge density, $n_s(\mathbf{r})$ and a short-range current density, $\mathbf{j}_s(\mathbf{r})$, as

$$n_s(\mathbf{r})\equiv n(\mathbf{r})-n_L(\mathbf{r})=\frac{1+f_0}{(\alpha\sqrt{\pi})^3}e^{-r^2/\alpha^2} + \frac{2}{\pi^{3/2}}\sum_{l\geq 2} i^l f_l P_l(\cos\theta)\Gamma\left[\frac{l}{2}+\frac{3}{2}\right]\left[\frac{r^l}{\Gamma(l+\frac{3}{2})\alpha^{l+3}}\,{}_1F_1\left[\frac{l}{2}+\frac{3}{2},l+\frac{3}{2};-\frac{r^2}{\alpha^2}\right]-\frac{1}{\Gamma(l/2)r^3}\right] \tag{81}$$

and

$$\mathbf{j}_s(\mathbf{r})\equiv\mathbf{j}(\mathbf{r})-\mathbf{j}_L(\mathbf{r})=\hat{\mathbf{z}}\frac{k_0+g_0}{(\alpha\sqrt{\pi})^3}e^{-r^2/\alpha^2} + \frac{2}{\pi^{3/2}}\sum_{l\geq 2} i^l\Gamma\left[\frac{l}{2}+\frac{3}{2}\right]\left\{\left[\frac{r^l}{\Gamma(l+\frac{3}{2})\alpha^{l+3}}\,{}_1F_1\left[\frac{l}{2}+\frac{3}{2},l+\frac{3}{2},\frac{r^2}{\alpha^2}\right]-\frac{1}{\Gamma(l/2)r^3}\right]\times\{\hat{\mathbf{z}}g_lP_l(\cos\theta)+[\hat{\mathbf{x}}\cos\phi+\hat{\mathbf{y}}\sin\phi g_{\perp,l}P_l^1(\cos\theta)]\}\right\} \ . \tag{82}$$

We shall now verify that the long-range charge and current densities and the short-range charge and current densities satisfy the continuity equations separately for $\mathbf{r}\neq 0$. With the substitution $\mathbf{r}\rightarrow\mathbf{r}+\mathbf{k}_0 t$ the continuity equation becomes

$$\begin{aligned}0={}&P_1(\cos\theta)\frac{r}{\alpha^5}e^{-r^2/\alpha^2}\left[\frac{k_0}{m^*}-k_0+k_0(f_0+\tfrac{2}{5}f_2)-g_0-\tfrac{2}{5}g_2-\tfrac{6}{5}g_{\perp,2}\right]\\&-P_3(\cos\theta)\frac{2r^3\Gamma(\frac{7}{2})}{\alpha^7\Gamma(\frac{9}{2})}\,{}_1F_1\left[\frac{7}{2};\frac{9}{2};-\frac{r^2}{\alpha^2}\right]\tfrac{3}{5}(k_0f_2-g_2+\tfrac{6}{5}g_{\perp,2})\\&+\sum_{l\geq 4} i^l\frac{2r^{l-1}}{\alpha^{l+3}}\left[\frac{l+1}{2l+1}P_{l+1}(\cos\theta)\frac{r^2\Gamma(l/2+\frac{5}{2})}{\alpha^2\Gamma(l+\frac{5}{2})}\,{}_1F_1\left[\frac{l}{2}+\frac{5}{2};l+\frac{5}{2};-\frac{r^2}{\alpha^2}\right]\right.\\&\left.-\frac{l}{2l+1}P_{l-1}(\cos\theta)\frac{\Gamma(l/2+\frac{3}{2})}{\Gamma(l+\frac{1}{2})}\,{}_1F_1\left[\frac{l}{2}+\frac{3}{2};l+\frac{1}{2};-\frac{r^2}{\alpha^2}\right]\right](k_0f_l-g_l)\\&+\sum_{l\geq 4} g_{\perp,l}i^l\frac{l(l+1)}{2l+1}\frac{2r^{l-1}}{\alpha^{l+3}}\left[P_{l+1}(\cos\theta)\frac{r^2}{\alpha^2}\frac{\Gamma(l/2+\frac{5}{2})}{\Gamma(l+\frac{5}{2})}\,{}_1F_1\left[\frac{l}{2}+\frac{5}{2},l+\frac{5}{2};-\frac{r^2}{\alpha^2}\right]\right.\\&\left.+P_{l-1}(\cos\theta)\frac{\Gamma(l/2+\frac{3}{2})}{\Gamma(l+\frac{1}{2})}\,{}_1F_1\left[\frac{l}{2}+\frac{3}{2};l+\frac{1}{2};-\frac{r^2}{\alpha^2}\right]\right] \ .\end{aligned} \tag{83}$$

For $r^2/\alpha^2\gg 1$, we obtain from Eq. (83)

$$\begin{aligned}0={}&-P_3(\cos\theta)\frac{\Gamma(\frac{7}{2})}{\Gamma(\frac{1}{2})}\frac{1}{r^4}[\tfrac{3}{5}(k_0f_2-g_2)+\tfrac{6}{5}g_{\perp,2}]\\&+\sum_{l\geq 4}\frac{i^l}{r^4}\left[\frac{l+1}{2l+1}P_{l+1}(\cos\theta)\frac{\Gamma(l/2+\frac{5}{2})}{\Gamma(l/2)}-\frac{l}{2l+1}P_{l-1}(\cos\theta)\frac{\Gamma(l/2+\frac{3}{2})}{\Gamma(l/2-1)}\right](k_0f_l-g_l)\\&+\sum_{l\geq 4} i^l\frac{g_{\perp,l}}{r^4}\frac{l(l+1)}{2l+1}\left[P_{l+1}(\cos\theta)\frac{\Gamma(l/2+\frac{5}{2})}{\Gamma(l/2)}+P_{l-1}(\cos\theta)\frac{\Gamma(l/2+\frac{3}{2})}{\Gamma(l/2-1)}\right] \ .\end{aligned} \tag{84}$$

It is then easy to check that $(\partial/\partial t)n_L(\mathbf{r},t)+\nabla\cdot j_L(\mathbf{r},t)$ is precisely equal to the right-hand side of Eq. (84), from which it immediately follows that $n_L(\mathbf{r})$ and $\mathbf{j}_L(\mathbf{r})$ satisfy the continuity equation. From Eqs. (78), (83), and (84) we then conclude that $n_s(\mathbf{r})$ and $\mathbf{j}_s(\mathbf{r})$ also satisfy the continuity equation for $\mathbf{r}\neq 0$.

Summarizing, we have in this section shown that in three dimensions the charge and current densities have the form stated in Eqs. (11) and (12). There is, however, no obvious way to define an e^* and a j^* such that they satisfy the continuity equation, but we have succeeded in separating $n(\mathbf{r})$ and $\mathbf{j}(\mathbf{r})$ into short-range, integrable parts and long-range parts that satisfy the continuity equation separately.

VI. CONCLUSIONS AND DISCUSSION

In this paper, we have given what we believe is a natural definition of a dressed QPWP and studied its structure. We found two unexpected features brought about by the interparticle interactions: (1) Finite fractions of the noninteracting charge e, and current, $\mathbf{k}/m$, are displaced to infinity, and (2) even though the density distribution of the unperturbed wave packet is spherical and exponentially localized, the dressed density contribution has long-range (r^{-3}) anisotropic tails.

We have no reason to believe that these results are in contradiction with the results of classical formal Landau- Fermi-liquid theory. We do, however, believe that our work has implications on the interpretation of the quasiparticle distribution function, $n(\mathbf{k},\mathbf{r})$, of the Landau theory. Nozières and Luttinger[8] (NL) have shown how to obtain the distribution function, $n(k,r)$, of the Landau- Fermi-liquid theory from microscopic calculations. Their derivation clearly shows that the Landau quasiparticles couple with the bare charge e to an external field. In view of this fact, it would seem natural, and has indeed been common,[11] to interpret the Landau distribution function $n(\mathbf{k},\mathbf{r})$ semiclassically as describing QPWP's of mean momentum $\mathbf{k}$, mean position $\mathbf{r}$, and charge e. The definition of the single quasiparticle states, $\psi_{\mathbf{k}}$, used in this paper, is identical to that of NL.[8] Nevertheless the natural wave packet, $\Psi_{\mathbf{k}_0}$, formed from these states, does not carry a localized charge e. The distribution function $n(\mathbf{k},\mathbf{r})$, must therefore not be simply interpreted in terms of the wave packets $\Psi_{\mathbf{k}_0}$, without mutual interactions.

The fact that particle-particle interactions can displace localized charge to infinity may be seen in a different system, analyzed by Landau- Fermi-liquid theory. Consider a weak external localized potential $v(r)$ varying slowly compared to k_F^{-1}. Without interactions the eigenfunctions can be described by the BWK approximation and the *localized* charge density, $\delta n_0(\mathbf{r})$, obtained in the Thomas-Fermi approximation:

$$\delta n_0(\mathbf{r}) = -\frac{3}{2}\frac{n_0}{E_F}v(r) , \tag{85}$$

where $E_F = k_F^2/2$ and $n_0(\mathbf{r})$ is the unperturbed density. When the interaction is turned on the effective potential is given by $v(r) + D(\epsilon_F)F_0 n(\mathbf{r})$, where $D(\epsilon_F)$ is the density-of-states at the Fermi surface and $D(\epsilon_F)F_0$ is the $l=0$ component of the Landau interaction function. Hence the dressed localized charge is given by

$$\delta n(\mathbf{r}) = m^* \frac{\delta n_0(\mathbf{r})}{1+F_0} . \tag{86}$$

The difference is, of course, displaced to infinity. Note, however, that here the fractional charge displaced to infinity by the interaction is given, to first order in $V(r)$, by

$$1 - \frac{m^*}{1+F_0} = \frac{k_F}{2\pi^2} V(0) . \tag{87}$$

It is *not* the same as the fractional charge displaced to infinity by the interaction in the case of a QPWP, which to first order in $V(\mathbf{r})$ is given by

$$1 - \frac{\tilde{e}}{e} = \frac{k_F^2}{2\pi^2} \int \frac{q\cdot\hat{\mathbf{k}}'}{\mathbf{q}\cdot(\mathbf{k}'-\mathbf{k}_0)} [V(0) - V(\mathbf{k}'-\mathbf{k}_0)] d\Omega_{\mathbf{k}'} \frac{d\Omega_{\mathbf{q}}}{4\pi} , \tag{88}$$

where $|\mathbf{k}'| = k_F$.

An interesting question is what direct experimental consequences the work presented in this paper has. The simplest case in which the theory presented here would be applicable would be in a one-dimensional conductor at very low temperature. Injecting a single electron at the Fermi level would then lead to delocalization of some of its charge and the occurrence of a "solenoidal" current, as explained in Sec. IV. We hope that these effects can be experimentally realized.

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APPENDIX A

In this Appendix, we will consider the effect of switching on the interaction at a finite rate s. We will show that in this case the delocalized charge will be spread over a finite volume of dimension R', where R' depends on s. However, by using quasiparticles sufficiently close to the Fermi surface, R' can be made arbitrarily much larger than the spread R of the remaining localized charge. Thus the effect of the delocalization of charge is physically observable.

The decay rate of a quasiparticle state $\psi_{\mathbf{k}}$ is proportional to $(|\mathbf{k}| - k_F)^2$, so the condition that s must be much greater than the quasiparticle decay rate becomes

$$s \gtrsim A(\delta k)^2 , \tag{A1}$$

where $\delta k = k_0 - k_F$ and A is a constant characteristic of the Fermi liquid and of dimension L^2T^{-1}. The noninteracting wave packet can be localized over a sphere of radius

$$R \gtrsim (\delta k)^{-1} . \tag{A2}$$

By describing the switching on explicitly by a factor e^{st}, one finds that the discontinuity in $n(\mathbf{q})$, due to the vanishing energy denominator in Eq. (35), is eliminated and the displaced charge spreads over a volume of radius

$$R' \approx \frac{v_F}{s} \lesssim \left[\frac{v_F}{A}\right](\delta k)^{-2} . \tag{A3}$$

Because of the different powers in (A2) and (A3), it is clear that, for sufficiently small δk,

$$R'/R \gg 1 \ , \tag{A4}$$

i.e., in the sense of physics, and possible observability, the delocalized charge goes "infinity" before the quasiparticles decay. The same argument can readily be applied to the current density.

APPENDIX B

In this appendix, we will prove that $f_0 < 0$ and that $g_0 > 0$. To show that $f_0 < 0$, we start with

$$\begin{aligned} f(\theta_q) &= -\sum_{\mathbf{k}'} \frac{\mathbf{q}\cdot\hat{\mathbf{k}}'\delta(k'-k_F)}{\mathbf{q}\cdot(\mathbf{k}'-\mathbf{k}_0)}[V(0)-V(\mathbf{k}'-\mathbf{k}_0)] \\ &= -\frac{k_F}{(2\pi)^3}\int \frac{\cos\theta_{\mathbf{q}\mathbf{k}'}}{\cos\theta_{\mathbf{q}\mathbf{k}'}-(k_0/k_F)\cos\theta_{\mathbf{q}\mathbf{k}_0}}[V(0)-V(k_F\hat{\mathbf{k}}'-\mathbf{k}_0)]\sin\theta_{\mathbf{k}'}d\theta_{\mathbf{k}'}d\phi_{k'} \ , \end{aligned} \tag{B1}$$

where $\theta_{\mathbf{q}\mathbf{k}'}$ is the angle between $\mathbf{q}$ and $\mathbf{k}'$, and $\theta_{\mathbf{q}\mathbf{k}_0}$ is the angle between $\mathbf{q}$ and $\mathbf{k}_0$. To expand in Legendre polynomials, we have to multiply by $[(2l+1)/4\pi]P_l(\cos\theta_\mathbf{q})$ and integrate over $d\Omega_\mathbf{q}$. We can reverse the order of integration and do the $d\Omega_\mathbf{q}$ integral first, with $\hat{\mathbf{k}}'$ as preferred direction. This integral is then

$$\int_0^\pi \int_0^{2\pi} \frac{\cos\theta_\mathbf{q} P_l(\cos\theta_\mathbf{q})\sin\theta_\mathbf{q} d\theta_\mathbf{q} d\phi_\mathbf{q}}{\cos\theta_\mathbf{q}-(k_0/k_F)[\sin\theta_\mathbf{q}\sin\theta_{\mathbf{k}_0}\cos(\phi_\mathbf{q}-\phi_{\mathbf{k}_0})+\cos\theta_\mathbf{q}\cos\theta_{\mathbf{k}_0}]} \ . \tag{B2}$$

The integral is even under inversion, so we integrate over the upper hemisphere $0 \le \theta_\mathbf{q} \le \pi/2$ and multiply by 2. The integral over $\phi_\mathbf{q}$ can be done in the complex plane or by straightforward integration, and gives

$$2\pi \left[\cos^2\theta_\mathbf{q} + \frac{k_0^2}{k_F^2}\cos^2\theta_\mathbf{q} - \frac{2k_0}{k_F}\cos^2\theta_\mathbf{q}\cos\theta_{\mathbf{k}_0} - \frac{k_0^2}{k_F^2}\sin^2\theta_{\mathbf{k}_0}\right]^{-1/2} \operatorname{sgn}\left[1-\frac{k_0}{k_F}\cos\theta_{\mathbf{k}_0}\right] \tag{B3}$$

for

$$x_0^2 \equiv \frac{k_0^2\sin^2\theta_{\mathbf{k}_0}}{k_F^2+k_0^2-2k_0k_F\cos\theta_{\mathbf{k}_0}} < \cos^2\theta_\mathbf{q} < 1 \tag{B4}$$

and zero otherwise.

Note that $k_F^2+k_0^2-2k_0k_F\cos\theta_{\mathbf{k}_0} \ge (k_F-k_0)^2 \ge 0$ and that

$$1-\frac{k_0^2\sin^2\theta_{\mathbf{k}_0}}{k_0^2+k_F^2-2k_0k_F\cos\theta_{\mathbf{k}_0}} = \frac{(k_F-k_0\cos\theta_{\mathbf{k}_0})^2}{k_F^2+k_0^2-2k_0k_F\cos\theta_{\mathbf{k}_0}} \ge 0 \tag{B5}$$

so that the limits of integration over $\theta_\mathbf{q}$ are well defined.

The integral over $\theta_\mathbf{q}$ is then

$$\begin{aligned} \int_{x_0}^1 \frac{xP_l(x)dx}{\left[x^2\left[1+\frac{k_0^2}{k_F^2}-2\frac{k_0}{k_F}\cos\theta_{\mathbf{k}_0}\right]-\frac{k_0^2}{k_F^2}\sin^2\theta_{\mathbf{k}_0}\right]^{1/2}} &= \frac{\left|1-\frac{k_0}{k_F}\cos\theta_{\mathbf{k}_0}\right|}{\left|1+\frac{k_0^2}{k_F^2}-2\frac{k_0}{k_F}\cos\theta_{\mathbf{k}_0}\right|} \\ &\quad - \frac{1}{\left[1+\frac{k_0^2}{k_F^2}-2\frac{k_0}{k_F}\cos\theta_{\mathbf{k}_0}\right]^{1/2}}\int_{x_0}^1 P_l'(x)(x^2-x_0^2)^{1/2} \ . \end{aligned} \tag{B6}$$

For $l=0$, we then have when we let k_0 approach k_F from above

$$f_0 = -\frac{k_F}{16\pi^3}\int [V(0)-V(k_F\hat{\mathbf{k}}'-k_F\hat{\mathbf{z}})]d\Omega_{\mathbf{k}'} \ . \tag{B7}$$

Since $V(0) > V(q)$ for $q \ne 0$ it then follows that $f_0 < 0$. From these results it is easy to show that $g_0 > 0$. We have

$$g_0=-\frac{k_F^2}{4\pi(2\pi)^3}\int\int\cos\theta_{\mathbf{k}'}\frac{\cos\theta_{\mathbf{qk}'}}{\cos\theta_{\mathbf{qk}'}-(k_0/k_F)\cos\theta_{\mathbf{qk}_0}}[V(0)-V(k_F\hat{\mathbf{k}}'-\mathbf{k}_0)]d\Omega_{\mathbf{k}'}d\Omega_{\mathbf{q}}$$

$$=-\frac{k_F^2}{2(2\pi)^3}\int\cos\theta_{\mathbf{k}}'[V(0)-V(k_F\hat{\mathbf{k}}'-\mathbf{k}_F\hat{\mathbf{z}})]d\Omega_{k'}\ . \tag{B8}$$

Since $V(k_F\hat{\mathbf{k}}'-k_F\hat{\mathbf{z}})$ is a function only of $|k_F\hat{\mathbf{k}}'-k_F\hat{\mathbf{z}}|=2k_F(1-\cos\theta_{\mathbf{k}'})$, Eq. (B8) is

$$g_0=\frac{k_F^2}{8\pi^2}\int_{-1}^{1}xV(2k_F(1-x))dx\ . \tag{B9}$$

We have assumed that $V(q)$ is a monotonically decreasing function of q. The absolute value of the integrand is then larger for $x>0$ than for $x<0$ and we have $g_0>0$.

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[10]Because of the finite lifetime of the quasiparticles, this charge is not literally at infinity but can be spread out over an arbitrarily large volume, by using quasiparticles sufficiently close to the Fermi surface (see Appendix A).

[11]See, for example, A. A. Abrikosov, *Introduction to the Theory of Normal Metals* (Academic, New York, 1972).