Renormalization method for the resistive transition in Josephson networks

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By calculating the resistance of small Josephson networks we deduce the nature of the resistive transition of granular superconductors.

According to a number of experiments,[1-3] two transitions are observed in a granular superconductor as the temperature is lowered. The first occurs at a temperature T_{c0} which is close to the bulk transition temperature and evidently is the superconductor transition of the grains themselves; here the resistance rapidly decreases but remains finite. The second, at a lower temperature T_c, corresponds to the long-range phase ordered state and gives zero resistance in the system. Despite the great interest that has been shown in this phenomenon,[3-5] there lacks a good theory to describe the behavior of the conductivity at the superconducting threshold. We will introduce a new method for attacking this problem based on a real-space renormalization method.

In this paper we will consider a three-dimensional lattice of superconducting grains embedded in a normal host, forming a Josephson junction array. This differs from a bulk granular superconductor in that it lacks disorder; disorder is not believed to alter the critical behavior at T_c.[6] We assume that the Josephson coupling between the grains is weak enough that the temperature T_c at which the grains become phase locked is well separated from the bulk transition temperature T_{c0}. This allows us to ignore the BCS order-parameter fluctuations which would be important when T_c is very close to T_{c0}.[7] We will write the order parameter in polar form

$$\psi = A \exp(i\phi) . \tag{1}$$

The junction between two grains will be modeled as an ideal Josephson junction in parallel with a resistor R. The junction carries a supercurrent

$$I_s = J \sin(\Delta\phi) \tag{2}$$

and the resistor describes the normal currents. The junction voltage is related to the phase by the second Josephson equation

$$\Delta V = (\hbar/2e) d(\Delta\phi)/dt . \tag{3}$$

The temperature is modeled by a Johnson noise voltage $f(t)$ contained in the resistor with characteristics

$$\langle f(t) \rangle = 0, \quad \langle f(t) f(t') \rangle = 2Rk_B T \delta(t - t') . \tag{4}$$

The parameters J and R are implicitly temperature dependent; this does not affect our results, but may complicate comparison with experiments. In what follows we will tend to regard J as an independent variable, and study the effects of varying it at constant T. The proper thermodynamic variable is J/T; thus our results can be simply restated in the alternate language of varying T with J fixed.

Finally we neglect the intergrain capacitance C, which would introduce terms involving the second time derivative of ϕ. These may be neglected if the system is overdamped, which requires $C \ll \hbar/2eJR^2$. There is also a lower bound on the capacitance: if the capacitance is very small the zero-point phase oscillations will destroy the Josephson coupling. This requires that the charging energy $E_c = e^2/2C$ be less than $10E_J$, where $E_J = \hbar J/2e$ is the Josephson coupling energy.[8] Therefore in order for our model to work, the intergrain capacitances must satisfy the relation $e^3/10\hbar J \leq C \ll \hbar/2eJR^2$. In practice these inequalities can be achieved in granular materials.[9]

The I-V characteristics of a single junction have been determined by Ambegaokar and Halperin[10] by solving the Smoluchowski equation for the time evolution of the probability distribution of the phase difference. In the limit of small measuring current they find

$$R_{\rm eff}(T) = (\hbar/2eI)\langle d\phi/dt \rangle = R\,[I_0(\hbar J/2ek_B T)]^{-2} \equiv R_{\rm AH}(T/J) , \tag{5}$$

where I_0 is the modified Bessel function.

The essential feature of the real-space renormalization method we shall employ is the ability to represent a general two-terminal Josephson network as a single equivalent junction with parameters J' and R'. This may be done by means of two calculations: (i) the time-averaged current at constant infinitesimal $\Delta\phi$ determines J' through Eq. (2); (ii) the time-averaged voltage at constant infinitesimal measuring current determines $R_{\rm eff}$, which we equate to $R_{\rm AH}(J'/T)$ to determine the equivalent parallel resistance R'. Thus performing these calculations for a small cluster of length b defines a mapping from the original parameters J and R to the set J' and R'; repeating the transformation many times relates the microscopic parameters to the bulk behavior.

The simplest cluster to study is the Wheatstone bridge, or its generalizations with many parallel paths. We have studied these, but there is some ambiguity in assigning dimensionality and length scale b to them. We found it better to use a $b \times b \times b$ cube as the renormalization cell. The boundaries were handled by connecting together the faces of the cube, so that current leaving the cube from a grain on one face reentered to the corresponding site on

the opposite side. We imposed a phase offset between a site and its periodic images (otherwise there would never be a net current through a cube face), which is equivalent to changing the Hamiltonian to

$$H=-(\hbar J/2e)\sum_i\sum_\delta \cos(\phi_i-\phi_{i+\delta}-\Delta_\delta/b)\ , \tag{6}$$

where $i+\delta$ is the (periodically defined) neighbor of site i in direction δ, and Δ_δ are the components (λ,μ,ν) of the phase gradient.

The calculation of J' is independent of R because it is an equilibrium property (the ratio J'/J is the helicity modulus defined by Fisher *et al.*[11]). This part of the problem reduces to a study of the underlying planar rotor model. The average supercurrent through a face is given by

$$I_\delta=b^{-1}\left\langle J\sum_i \sin(\phi_i-\phi_{i+\delta}-\Delta_\delta/b)\right\rangle \simeq J'\sin\Delta_\delta \tag{7}$$

where $\langle\ \rangle$ is the usual thermodynamic average calculated by over all combinations of the phases ϕ_i using the weight function $e^{-H\beta}$. Taking a derivative with respect to Δ_δ gives

$$J'=Jb^{-2}\left\langle \sum_{i,\delta}\cos(\phi_i-\phi_{i+\delta}) -(\hbar J/2ek_BT)\left[\sum_{i,\delta}\sin(\phi_i-\phi_{i+\delta})\right]^2\right\rangle . \tag{8}$$

We evaluated the thermodynamic average by use of the Metropolis Monte Carlo algorithm for various values of b. Figure 1 shows the results, based on 15×10^6 Monte Carlo steps per site (each Monte Carlo step consists of six attempts to change the phase angle of a randomly chosen site by a randomly chosen fraction of a radian). We observe that in each case there is an unstable fixed point, where $J'=J$; for $b>2$ these estimates of the critical value

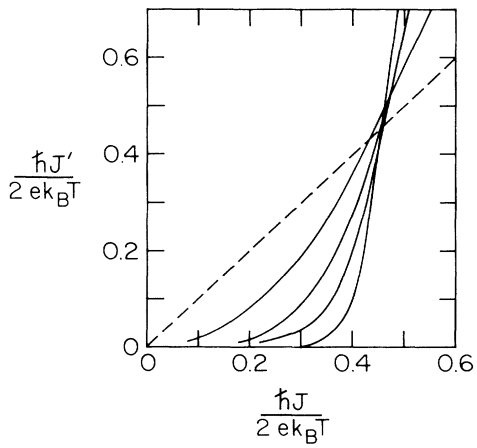

FIG. 1. The relationship between J'/T and J/T, for cubes of edge b. The four curves for $b=2$, 3, 4, and 6 are successively more rapidly rising. The dashed line is $J'=J$, which is the locus of the fixed points of the renormalization transformation. The curves for $b=3$, 4, and 6 intersect this locus at the same point, showing that they agree that the critical point is $2ek_BT_c/\hbar J=2.2$.

for J/T are in reasonable agreement with each other and with other estimates of the critical temperature (Table I). The correlation exponent is determined by

$$\left.\frac{\partial J'}{\partial J}\right|_{J_c}=b^{1/\nu} \tag{9}$$

with results that agree with other estimates (Table I).

(ii) The calculation of R' is more difficult. The current carried from grain i to a neighboring grain $j=i+\delta$ is

$$I_{ij}=(\hbar/2eR)d(\phi_i-\phi_j-\Delta_\delta/b)/dt+f_{ij}/R+F_{ij}\ , \tag{10}$$

where f_{ij} is the Johnson noise associated with the resistor and

$$F_{ij}=J\sin(\phi_i-\phi_j-\Delta_\delta/b) \tag{11}$$

is the Josephson supercurrent. The condition that the net current into each site be zero gives rise to a set of b^3 equations

$$\sum_{j\ \text{neighbor of}\ i} I_{ij}=0\ . \tag{12}$$

There is also a condition that the net current along the three coordinate axes be fixed, which has the form

$$I_\delta=\frac{1}{b}\sum_i I_{i,i+\delta}\ . \tag{13}$$

We may use the overall gauge invariance to fix one of the ϕ_i to zero (and remove its equation of motion from the set); the remaining b^3+2 equations may be solved to give each $d\phi_i/dt$ and $d\Delta_\delta/dt$ as a linear combination of all the f_{ij} and F_{ij} of general form

$$\frac{d\phi_i}{dt}=\sum_\alpha a_{i\alpha}(F_\alpha+f_\alpha/R)\ , \tag{14}$$

where the sum is over all neighbor pairs. The weighting coefficients $a_{i\alpha}$ depend only on b, and were determined by numerically inverting the matrix of coefficients appearing in Eqs. (12) and (13). Thus these equations give rise to a set of Langevin equations for ϕ_i and Δ_δ.

Since Δ_δ is the average phase difference across a cell, its time deviative corresponds to a voltage drop which we identify with $I_\delta R_{\text{eff}}$. We have found two approaches to solving these equations.

(1) *Generalization of the method of Ambegaokar and Halperin.* The time evolution of the joint probability distribution for the site phases is given by a Fokker-Planck equation, which can be derived from the Langevin equations by standard methods.[12] This is a multidimensional partial differential equation for which we were unable to find a general solution. However, for the special case that no current is applied, the system is at equilibrium and the probability distribution has the usual Boltzmann form $\exp(-\beta H)$. Then for small applied currents, we can find the steady-state behavior by using perturbation methods. This method is not practical for larger networks, because the larger number of phase variables gives rise to an enormous set of coupling terms.

(2) *Numerical integration of the Langevin equations.*

The time evolution of a particular configuration of the phase variables can be followed by directly integrating Eq. (14). For sufficiently small Δt we have

$$\Delta\phi_i = \Delta t \sum_\alpha a_{i\alpha} F_\alpha + \sum_\alpha a_{i\alpha} v_\alpha \ , \tag{15}$$

where the v_α are uncorrelated Gaussian random variables with variance $(8e^2 R k_B T \Delta t/\hbar^2)^{1/2}$. Accuracy of this equation requires that Δt be small enough that the changes in the ϕ_i give little change in the F_α. We chose $\Delta t = 0.05\hbar/(2eJR)$. By following many realizations for a sufficiently long time, the time evolution of the ensemble can be deduced. In the presence of a current, the ensemble average of Δ_δ shows a linear time dependence for large times

$$\langle \Delta_\delta \rangle \simeq (2eI_\delta/\hbar) R_{\rm eff} t \tag{16}$$

from which the I-V characteristic can be deduced. The resistance at small current is better evaluated by studying the mean-square phase gradient in the absence of a current, for which

$$\langle \Delta_\delta^2 \rangle \simeq (2e/\hbar)^2 2R_{\rm eff} k_B T t \ . \tag{17}$$

This method is adaptable to networks of arbitrary size. Its drawback is that it requires a lot of computation: we followed a hundred realizations for 5000 times steps, where each realization entailed the integration of b^3+2 equations. The greatest source of inaccuracy in our results was the failure to completely average out statistical fluctuations.

It should be noted that these two methods use different boundary conditions for the ensemble distribution function. The distribution function σ studied in the Smoluchowski approach is time independent and periodic in the phase variables. The average phase appears to have no time dependence, because the only ensemble members that are considered are those having all phase variables in the interval 0–2π. Nonetheless, the ensemble average of $d\phi_i/dt$ is not zero; as a member of the ensemble leaves the interval, a periodic image of it enters. In contrast, the Langevin approach does not attempt to restrict the range of the phase variables, and indeed it is essential to the accuracy of the results that runs be long enough that the phase has moved more than 2π: Only then can the phase diffusion be treated as a random walk from one minimum energy configuration to another.

Once $R_{\rm eff}$ has been determined, we use Eq. (5) to represent the network as a single junction having helicity modulus J' and internal resistance R'

$$R' = R_{\rm eff}[I_0(\hbar J'/2ek_B T)]^2 \ . \tag{18}$$

The ratio R'/R is small for low temperatures because phase diffusion is an activated process. In the high-temperature, or $J\to 0$, limit the supercurrents play no role and $R_{\rm eff}$ can be computed from Kirchhoff's rules; the R'/R approaches $(1/b)$, which is the transformation law for pure resistors. Thus it is convenient to define a transformation function

$$S(T/J) = bR'/R = b\,[I_0(\hbar J'/2ek_B T)]^2 R_{\rm eff}/R \tag{19}$$

which is also the scale change of resistivity produced by the renormalization.

The results of the calculations (i) and (ii) for any small network defines a transformation $(J,R)\to(J',R')$ which may be regarded as the effect of a length rescaling; performing the transformation several times gives the two-terminal characteristics of the family of networks that can be formed by repeatedly replacing bonds by the elemental network. We then assume that the parameters $(J^{(n)},R^{(n)})$ reasonably characterize the properties of a piece of Josephson junction lattice of edge b^n, where b is the length ratio between the elemental network and a single junction. (This is an approximation, since the lattice cannot actually be decomposed into nesting networks.) The bulk resistivity can be calculated as

$$\rho(T/J) = \rho_0 S(T/J) S(T/J') S(T/J'') \cdots \ , \tag{20}$$

where ρ_0 is the infinite-temperature resistivity.

There are two basic cases: below T_c, the sequence $J,J',J'',\ldots$ is nondecreasing, and $\rho=0$ because all factors are less than unity; above T_c the $J^{(n)}$ decrease to zero, and the bulk resistivity is finite. Slightly above the critical temperature many of the factors S are less than unity and ρ is small; well above T_c the sequence of $J(n)$ decrease rapidly and the bulk resistivity is well approximated by the first two terms of Eq. (20). The usual argument[13] shows that near T_c,

$$\rho \simeq (T - T_c)^s \ , \tag{21}$$

where

$$s/\nu = -\ln S(T_c/J) \ . \tag{22}$$

Figure 2 shows the results of our simulations. The dashed curve is Eq. (21) with the prefactor chosen to join $S(T/J)$ at $T = 4\hbar J/2ek_B$. Our results are summarized in Table I. We see that the results for larger cells are consistent and agree well with previous determinations of T_c and ν.

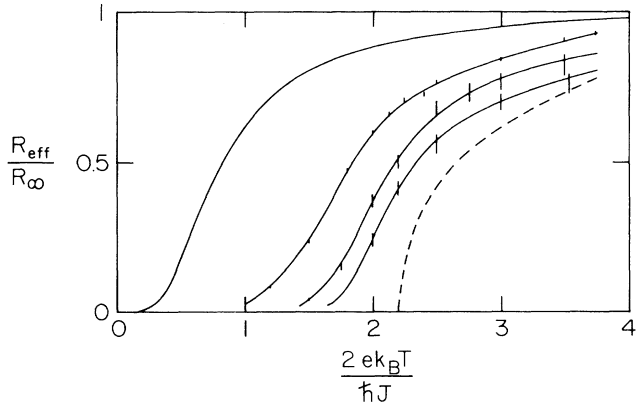

FIG. 2. Resistivity as a function of temperature. The curve farthest to the left is the resistance $R_{\rm AH}(J/T)$ of a single junction, as given by Eq. (5). The next three curves give the temperature dependence of the resistivity for cubes of edge 2, 3, and 4. The dashed curve is the bulk resistivity.

TABLE I. Results of renormalization study.

	b					
	2	3	4	6	Series[a]	Monte Carlo[b]
$2ek_BT_c/\hbar J$	2.4	2.2	2.2	2.2	2.203±0.006	2.1±0.1
ν	0.93	0.72	0.71	0.69	0.675±0.015	
s/ν	0.40±0.01	0.52±0.04	0.57±0.04			

[a]Reference 14.
[b]Reference 15.

Our result for s/ν is 0.57±0.04 and thus $s=0.38\pm0.03$. We may compare our results to the ones obtained by Halperin *et al.* for the viscosity exponent of superfluid ^{4}He by using an ϵ-expansion method.[16] We believe that our Josephson junction lattice model belongs to the same universality class as their model E for which they find $s/\nu=(4-d)/2$ in lowest order.

The behavior of the critical current near T_c can be deduced from a scaling argument. For a system of size L, the critical current density is

$$j_c = L^{-(d-1)} S_c[L^{1/\nu}(T_c - T)] , \tag{23}$$

where the prefactor is the reciprocal of the cross-sectional area. For low temperatures this should be independent of L, so $S(x)\simeq x^{\nu(d-1)}$; then for systems large enough that $L > \xi$, $j_c \simeq (T_c - T)^{\nu(d-1)}$.

The application of our renormalization method in two dimensions yields no transition. The Kosterlitz-Thouless[17] transition which does occur is due to the long-range interactions between vortices. To record the presence of a vortex in a cell requires a different set of variables than our choice (R,J). This difficulty in constructing a real-space renormalization method for the two-dimensional planar rotor model has been noted previously.[18] To find there is no transition is very nearly the right answer; indeed, there is no long-range order in two dimensions.[19]

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