Zero-current persistent potential drop across small-capacitance Josephson junctions

M. Büttiker
IBM Thomas J. Watson Research Center, P.O. Box 218, Yorktown Heights, New York 10598
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Quantum effects in a small-capacitance Josephson junction coupled capacitively to an external circuit are investigated. Capacitive coupling permits one to control the charge on the junction and to hold the junction in an equilibrium state. The equilibrium state is characterized by a persistent voltage drop without an accompanying dc current. The persistent voltage is a periodic function of the control charge with period $2e$ if Cooper pairs only are present and with period e if quasiparticles are included. In the capacitively coupled junction, Bloch oscillations are induced by increasing the control charge linearly in time. The presence of a persistent voltage drop is reflected in the small-signal impedance of the circuit which describes the response of the junction to an oscillating control charge superimposed on a static control charge. The impedance reveals features which seem closely related to experimental observations by Lambe and Jaklevic [Phys. Rev. Lett. **22**, 1371 (1969)] on a normal array of capacitively coupled junctions.

I. INTRODUCTION

In 1969 Lambe and Jaklevic[1] reported an experiment on small normal tunnel junctions with a capacitance so small that the charging energy $e^2/2C$ exceeds $k_B T$. In recent years such systems have found renewed interest,[2–14] stimulated by the prediction of new quantum effects in small-capacitance Josephson junctions[3–6] and small normal tunnel junctions.[7,11] This work[2–14] envisions a junction connected to a current source. However, the basic features of a small capacitance junction become only apparent if it is possible to externally control the charge on the junction. A possible circuit which achieves that is shown in Fig. 1. This is also the circuit which models the experiments of Ref. 1. Tunneling occurs only through the junction capacitor C but not through the external capacitor C_0. The external circuit can be used to bring a charge $Q_0(U_B)$ on the electrodes of the junction. This system, as we show below, behaves like a quantum particle in a periodic potential. A quantum particle in a periodic potential has two states of principal interest. The particle can be in a Bloch state and travel with constant velocity through the periodic lattice. If a constant force is exerted on the particle, the carrier executes, in the absence of Zener tunneling, Bloch oscillations. A small Josephson junction can be brought into a Bloch state if a time-independent control charge Q_0 is externally induced across the electrodes. In Fig. 1 such a charge difference is induced by applying a constant voltage at the battery. A small junction responds by developing a constant voltage, but without carrying a dc current. The two superconductors have differing chemical potentials despite tunneling of Cooper pairs and quasiparticles through the barrier. The small intrinsic capacitance of the Junction blocks the equilibration of the two superconductors. The voltage drop across the junction is a periodic function of the induced charge with period $2e$ (or period $2e/C_0$ in the battery voltage), if only Cooper pairs are present and with period e (or e/C_0), if quasiparticles are included. This sequence of equilibrium states is the analog of the persistent currents in a superconducting loop with a large-capacitance Josephson junction.[15] In the small-capacitance junction the externally induced charge plays the role of the Aharonov-Bohm flux in the large-capacitance junction. The key point is thus that capacitive coupling allows "parking" of the system in an equilibrium state. Changing the induced charge quasistatically brings the junction through the whole sequence of equilibrium states. The authors of Ref. 3 also discuss the equilibrium properties of small-capacitance junctions, but do not specify the circuit which allows the junction to be brought into these states. If the junction is coupled to a current source[4–14] the presence of the equilibrium states manifests itself only through the fact that for $I=0$, in the absence of a shunting resistor, the voltage is not determined, but can take any value in a limited range.[11] The main subject of Refs. 3–14 is the Bloch oscillations which occur if a constant current is fed into the junction. In the circuit of Fig. 1 Bloch oscillations are induced by increasing the control charge Q_0 linearly in time, i.e., by increasing the voltage at the battery linearly in time. A small-capacitance junction responds with an oscillating voltage and an accompanying dc current.[3,4] In the presence of in-

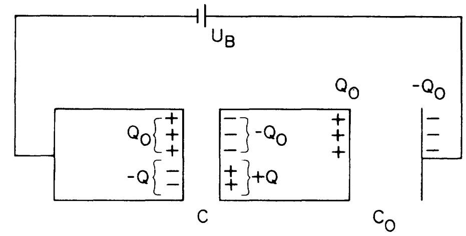

FIG. 1. Capacitively coupled Josephson junction. Q_0 is an externally induced charge difference. Q is a charge imbalance between the two superconductors.

elastic events the Bloch oscillations are accompanied by a dissipative current and give rise to a dissipative current-voltage branch in addition to the nondissipative equilibrium branch discussed above. In Ref. 1 a small-signal capacitive measurement was performed on an array of small normal tunnel junctions. It is therefore of interest to investigate the small-signal response of the circuit of Fig. 1. We find results which closely resemble the features seen in the experiment and argue that the experiments indeed provide evidence for the existence of the equilibrium states discussed above.

We have already mentioned the duality[3] of the quantum effects in the small-capacitance junction and the large-capacitance junction. Such a duality also exists between the small-capacitance junction and a disordered normal-metal ring[16–22] with electron phase coherence extending around the whole loop. A comparison of these two systems has been the subject of a recent conference paper.[23]

THE HAMILTONIAN SYSTEM

Consider first two pieces of superconductors separated by a small insulating barrier. To perturb the system we couple the system capacitively to an external circuit with a battery at a voltage U_B as shown in Fig. 1. Suppose for a moment that carriers cannot tunnel between the superconductors. If the two superconductors are in equilibrium for $U_B=0$, i.e., neutral, application of a voltage will induce a charge Q_0 across the capacitors. The energy of the circuit is then determined by

$$Q_0^2/2C_0+Q_0^2/2C-Q_0U_B \ ,$$

where C_0 is the capacitance of the external circuit. If there is now in addition a carrier imbalance Q between the two superconductors the total energy of the system is

$$Q_0^2/2C_0+(Q-Q_0)^2/2C-Q_0U_B \ .$$

Below we want to treat Q_0 as an external parameter and only treat the charge imbalance Q as a dynamical variable. To achieve this we have to take the external capacitance C_0 small compared to the junction capacitance C. In the steady state

$$Q_0/C_0-(Q-Q_0)/C=U_B \ ,$$

a small change ΔQ of the carrier imbalance changes Q_0 by

$$\Delta Q_0=[C_0/(C_0+C)]\Delta Q \cong (C_0/C)\Delta Q \ .$$

The variation ΔQ gives rise to a change $(U_0-U_B)\Delta Q_0$ in the energy of the external circuit (capacitive energy and battery work), where $U_0=Q_0/C_0$. Since U_0-U_B is independent of C_0, and ΔQ_0 is proportional to C_0, the energy variation of the external circuit is negligible in the limit of a small external capacitance. In this limit we can, therefore, consider $Q_0(U_B)$ as a control parameter. The charge imbalance Q, on the other hand, is a dynamical variable governed by the tunneling processes across the barrier of the junction. The total energy of the junction *alone* consists of the charging energy and the Josephson coupling energy,

$$H=(Q-Q_0)^2/2C+E_J(1-\cos\phi) \ . \tag{1}$$

In this Hamiltonian Q is a quantum operator and Q_0 is treated as a classical parameter.

Quantization of the Hamiltonian equation (1) can be accomplished either in the number representation, where Q is considered a pair-number difference operator, or in the more familiar definite phase representation, where Q is replaced by $(2e/i)(\partial/\partial\phi)$. We use the latter. Q then plays the role of the momentum, ϕ takes the role of the spatial dimension, and Q_0 plays the role of a (scalar) vector potential. The eigenfunction of H in Eq. (1) with eigenvalue $E_n(q)$ is the periodic function $u_{n,q}(\phi)=u_{n,q}(\phi+2\pi)$ of the Bloch state $\Psi_{n,q}(\phi)=e^{iq\phi/2e}u_{n,q}(\phi)$, which is an eigenfunction of the Hamiltonian

$$H_1=Q^2/2C+E_J[1-\cos(\phi)]$$

with an eigenvalue $E_n(q)$. The eigenvalues of Eq. (1) are, therefore, Bloch bands with periodicity $2e$, $E_n(q+2e)=E_n(q)$. A term-by-term comparison of H, Eq. (1), and H_1, yields the relation

$$q=-Q_0 \bmod 2e \ . \tag{2}$$

q therefore measures the deviation of the induced charge from an integral multiple of $2e$, $-e<q<e$. The charge Q_0 is not quantized, but charge can be transferred only in multiples of $2e$. Subsequently, we will cease to make a distinction between q and Q_0. Note that the eigenfunctions $u_{n,q}$ are single valued, in contrast to the eigenfunctions Ψ of H_1, which are multivalued when considered on the interval 0 to 2π.

The Bloch bands corresponding to Eq. (1) are shown in Fig. 2(a). Since we consider a junction of very small capacitance the Josephson coupling energy represents only a small perturbation and the gap in Fig. 2(a) is given by $E_J=\hbar I_c/2e$, where I_c is the maximum Josephson current of the junction. For a given q the possible states of the system correspond to a ladder of Bloch states, $E_n(q)$. At zero temperature the system is in the lowest band, $E_1(q)$. Below we invoke only this band and for convenience we drop the index 1 and denote the energy of this band by $E(q)$. In the presence of a time-independent induced charge difference, q, a constant

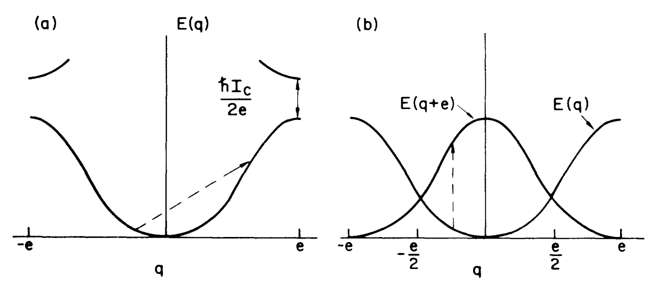

FIG. 2. (a) Energy bands of the Josephson junction as a function of the induced charge. Weak Josephson coupling opens a gap $E_J=\hbar I_c/2e$ between the first and second bands. The arrow indicates a quasiparticle transition. (b) The lowest band $E(q)$ and $E(q+e)$ of (a). The quasiparticle transition of (a) can be represented as a vertical q-conserving transition.

time-independent voltage,

$$U(q)=\frac{dE(q)}{dq}\ , \tag{3}$$

develops across the junction. Equation (3) is analogous to the expression for the group velocity of an electron moving in a periodic lattice.[3] U is thus a periodic function of the induced charge difference q with period $2e$. Since q and hence Q_0 is time independent no current flows in the circuit of Fig. 1. The junction thus develops a constant voltage without an accompanying current. This effect is the analog of the persistent current in a superconducting loop with a (large) Josephson junction. It is the key feature of a low-capacitance junction; the Coulomb energy blocks the flow of current, despite the Josephson coupling of the two conductors.

The total voltage drop across the sample is $U_B=Q_0/C_0+U(Q_0)$, where the function U is the potential drop across the junction and given by Eq. (3). This equation must now be used to find $Q_0(U_B)$. For each battery voltage U_B the function U_B-Q_0/C_0 intersects $U(Q_0)$ only once, since $C_0 \ll C$. Thus $Q_0(U_B)$ is single valued and allows one to determine the junction voltage as a function of the battery voltage, $U(U_B)$. Since U is periodic with period $2e$ in Q_0, U is periodic with period $2e/C_0$ in the battery voltage. Thus U can be expressed as a Fourier series,

$$U=\sum u_n \sin\left[2\pi n\frac{C_0}{2e}U_B\right] . \tag{4}$$

Until now we have assumed that for $U_B=0$ the chemical potentials of the superconductors forming the tunnel junction are equal. Then Q_0 is zero for $U_B=0$. But if the conductors are not shunted, neither directly nor via the external circuit, this need not be the case.[1] For $U_B=0$ the circuit of Fig. 1 can support a potential drop $U_0=Q_0/C_0$ over the external capacitor C_0 compensated by a potential drop of equal magnitude and opposite sign over the capacitor C. Classically, we have $U_0=-U=(Q_0-Q_p)/C$, where $Q=Q_p$ is the excess charge on the superconductor between the two capacitors. Quantum mechanically, we can describe such a potential drop by modifying Eq. (2). In general, we have $q=(Q_p-Q_0)\bmod 2e$, where Q_p is determined such that $Q_0/C_0=U(Q_p-Q_0)$ at $U_B=0$. The junction voltage U as a function of the battery voltage becomes

$$U=\sum u_n \sin\left[2\pi n\frac{C_0}{2e}(U_B-U_P)\right] , \tag{5}$$

where the "phase shift" U_P is determined by $U_P=Q_P/C_0$. This phase shift plays an important role in the experiments of Ref. 1: it leads to the observed memory effects. Below we discuss other physical quantities which also depend on this phase shift.

In the circuit of Fig. 1 a dc current is set up by increasing the voltage U_B linearly in time. The relation $dQ_0/dt=-dt/dt=-I$ is equivalent to Bloch's law $\hbar dk/dt=-eE$, for carriers in a periodic lattice subject to an electric field E. The induced charge $q=It$ increases linearly in time. Hence in the presence of an external current the system is driven with constant velocity through the Brillouin zone, Fig. 2(a). The voltage, Eq. (3), oscillates as a function of time with a Bloch frequency,[3]

$$\omega=2\pi I/2e\ . \tag{6}$$

This effect is the analog of the ac Josephson effect in a superconducting loop with a (large) Josephson junction. Bloch oscillations occur as long as we can neglect Zener transitions through the gap $\hbar I_c/2e$ into higher-lying states. Zener tunneling is small as long as $\hbar\omega$ is small compared to the gap, i.e., as long as $I<I_c$.

A Hamiltonian similar to Eq. (1) but with Q_0 replaced by It was proposed in Ref. 7. The authors of Ref. 7 arrived at Eq. (1) not by analyzing a specific circuit, but by arguing that the Hamiltonian of the junction should be invariant if the phase ϕ is increased by 2π. This argument has gained some acceptance,[14] but has also been criticized in Ref. 5. Since the authors of Refs. 7 and 14 envision a current source, i.e., an open system, the invariance argument is dubious and does not, in any way, specify the Hamiltonian completely. The authors of Refs. 4–7, 9, and 11 use the adiabatic coupling $I\phi$ of current and phase and thus treat a Hamiltonian which is not invariant if the phase is increased by 2π. Our derivation of Eq. (1), which is specific for the circuit of Fig. 1, suggests that the correct coupling of the current to the phase might well depend on the coupling of the junction to the external circuit. These questions do certainly warrant further efforts.

In the presence of a static induced charge difference q the voltage U is a time-independent constant. No dc current flows, $I=dq/dt=0$. The system produces no Joule heat. In the presence of a linearly increasing q, where we have a dc current $dq/dt=I$ across the barrier, the voltage oscillates. Thus the Joule heat IU, when averaged over a time $T=2e/I$, is again zero. Even if we allow for Zener tunneling, we can expect that no net Joule heat is produced.[18] The Hamiltonian system (1) can store energy, but cannot dissipate it. To obtain a resistance we have to allow for inelastic events. This is the subject of the following section, where we present an approach which closely follows Refs. 17 and 19.

THE DISSIPATIVE SYSTEM

We assume that we are dealing with an ideal junction and that the inelastic events which have to be taken into account are those that occur intrinsically in superconductors. As the temperature is raised, Cooper pairs are broken and there is, in addition to the condensate, a density of electronlike and holelike excitations in the superconductors to the left and right of the barrier. Charge transfer between the left and right superconductors occurs now not only in multiples of $2e$, but single-particle tunneling can also occur. Consider now what happens in the presence of a constant induced charge difference q. Single-particle tunneling transitions to states with $q+ne$ occur. Figure 2(a) shows a transition from q to $q+e$. Due to quasiparticle tunneling the system has now a probability $\rho(q)$ to be in the state with energy $E(q)$ and a probability $\rho(q+e)$ to be in the state with energy $E(q+e)$. Figure 2(b) shows both $E(q)$ and $E(q+e)$. The transition in Fig. 2(a) from q to $q+e$ can be represented as vertical transition in Fig. 2(b). We could, in fact, restrict q to the interval from $-e/2$ to $e/2$ and consider two new e periodic bands as in Ref. 12. That, however, complicates the investigation of the Bloch oscillations since in the absence of a gap at $\pm(e/2)$ carriers are not confined to these bands but rather follow the bands $E(q)$ and $E(q+e)$. But let us first return to the static case where the system is parked at a fixed q. Clearly, if only the states $E(q)$ and $E(q+e)$ are occupied, we must have $\rho(q)+\rho(q+e)=1$. Since

$$\rho(q) \sim \exp[-E(q)/k_B T]$$

and

$$\rho(q+e) \sim \exp[-E(q+e)/k_B T] ,$$

we immediately find

$$\rho(q)=1/\{1+\exp[-\Delta E(q)/k_B T]\} , \tag{7}$$

with

$$\Delta E(q)=E(q+e)-E(q) . \tag{8}$$

Therefore, in the presence of a time-independent q the voltage across the junction is given by

$$U=\frac{dE(q)}{dq}\rho(q)+\frac{dE(q+e)}{dq}\rho(q+e) . \tag{9}$$

Let us now first discuss Eq. (9) at $T=0$. The equilibrium distribution, Eq. (7), at $T=0$ is given by $\rho(q)=1$ if q is in the interval from $-e/2$ to $e/2$, and $\rho(q)=0$ if q is in the intervals $-e<q<-e/2$ and $e/2<q<e$. This has the consequence that the system is always in the state with lower energy, and by considering Fig. 2(b) it is obvious that this implies that $U(q)$ is periodic with period e as shown in Fig. 3. The discontinuities of the voltage at $e/2+ne$, are given by $2dE/dq\,|_{q=e/2}$; i.e., they arise because $E(q)$ and $E(q+e)$ cross $e/2+ne$ with a finite slope. For $E_J \ll 2e^2/C$ the band $E(q)$ is parabolic, except near the boundary of the Brillouin zone. The maximum (minimum) voltage in Fig. 3 is thus $\pm(e/2C)$. As the temperature increases these discontinuities disappear and $\rho(q)$ is nonzero, even in the intervals $-e<q<-e/2$ and $e/2<q<e$. Further, since the voltages $dE(q)/dq$ and $dE(q+e)/dq$ are of opposite sign, the maximum voltage decreases with increasing temperature. Since we consider only the two lowest possible states of the junction, the approach presented here is strictly valid only for temperatures small compared to E_J. It should also be noted that our result does not in an explicit way depend on the quasiparticle densities or the ability of the quasiparticles to tunnel through the barrier. This, however, is due to our assumption that the inelastic processes leave the bands in Fig. 2 unchanged. In the presence of strong inelastic processes this is not a correct assumption: Inelastic events not only change the shape of the bands[12] but also lead to a broadening of the energy levels. The discontinuity of the voltage as a function of q is also related to results of Ref. 12 for the dissipation-renormalized energy spectra of the junction.

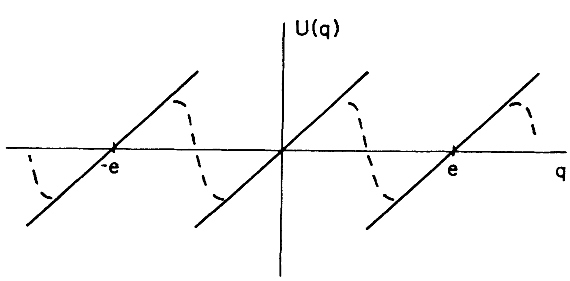

FIG. 3. The junction voltage as a function of the induced charge q in the zero-temperature limit (solid line) and for a small nonzero temperature (dashed line).

That the currentless state in the presence of a finite potential drop persists even when quasiparticle tunneling is included can be seen in the following way: Consider the transition indicated in Fig. 2. In the state with energy $E(q)$ the voltage across the junction is negative. The superconductor to the right has a higher potential then the superconductor to the left. A quasiparticle transition from right to left brings the junction into a state with energy $E(q+e)$ with a positive voltage drop across the junction. Thus a quasiparticle transition reverses the voltage across the junction and the system is eventually driven back to the state with lower energy.

The junction voltage U is now a periodic function of the battery voltage with a period $\Delta U_B = e/C_0$ instead of $2e/C_0$ as in Eqs. (4) and (5). Even in the presence of quasiparticle tunneling the chemical potentials of the superconductors forming the tunnel junction need not line up for $U_B=0$. Thus the Fourier transform of Eq. (9) also contains a phase shift U_P. We reemphasize that the equilibrium effect, Eq. (9), can be observed only if the two superconductors in Fig. 1(a) are not shunted. If the two conductors are shunted, either directly or via the external circuit, the potentials of the two superconductors will equilibrate. Indeed, in a real junction small leakage currents[3–6] might always give rise to an equilibration. Nevertheless, we can hope that the time scale for equilibration is so long that the potential drop can still be observed.[1] We return to this question in the last section of this paper.

Suppose that the system has, through the action of some force, been driven away from equilibrium. The nonequilibrium probabilities $h(q)$ and $h(q+e)$ for the system to have energy $E(q)$ and $E(q+e)$, respectively, are time dependent and must obey a master equation,

$$\frac{\partial h(q)}{\partial t}=w(q,q+e)h(q+e)-w(q+e,q)h(q) , \tag{10a}$$

$$\frac{\partial h(q+e)}{\partial t}=-w(q,q+e)h(q+e)+w(q+e,q)h(q) . \tag{10b}$$

Using $h(q)+h(q+e)=1$ and detailed balance,

$$w(q+e,q)\rho(q)=w(q,q+e)\rho(q+e) ,$$

we find

$$\frac{\partial h(q)}{\partial t}=-[w(q+e,q)+w(q,q+e)][h(q)-\rho(q)]\ , \tag{11a}$$

$$\frac{\partial h(q+e)}{\partial t}=-\ [w(q+e,q)+w(q,q+e)] \times[h(q+e)-\rho(q+e)]\ . \tag{11b}$$

Thus we obtain two equations which describe the relaxation of the actual distribution functions $h(q)$, $h(q+e)$ toward the equilibrium distribution functions $\rho(q)$, $\rho(q+e)$. The transition rates are, in general, functions of both q and $k_B T$. Below we approximate the transition rates by a relaxation time

$$\tau^{-1}=w(q+e,q)+w(q,q+e)\ ,$$

which we take to be independent of both q and $k_B T$. τ is related to the sum of the transition rates. Whereas the transition rate from the state with lower energy to the state with higher energy vanishes as the temperature tends to zero and is thus strongly temperature dependent, the rate from the state with higher energy to the state with lower energy is nonzero even at zero temperature. This latter transition rate effectively determines the relaxation rate. Thus the dynamics of the system is characterized by relaxation of the actual distribution functions $h(q)$ and $h(q+e)$ towards the instantaneous equilibrium distribution functions $\rho(q)$ and $\rho(q+e)$,

$$\frac{\partial h(q)}{\partial t}=-\tau^{-1}[h(q)-\rho(q)]\ , \tag{12a}$$

$$\frac{\partial h(q+e)}{\partial t}=-\tau^{-1}[h(q+e)-\rho(q+e)]\ . \tag{12b}$$

To obtain the voltage in a nonequilibrium situation we have to replace the equilibrium distribution functions in Eq. (9) by the actual distribution functions,

$$U=\frac{dE(q)}{dq}h(q)+\frac{dE(q+e)}{dq}h(q+e)\ . \tag{13}$$

Consider now the case where the voltage of the battery in Fig. 1 is increased linearly in time, giving rise to a charge which increases linearly in time, $q=It$. The equilibrium distribution function towards which the system is driven is now a periodic function of time, $\rho(q)=\rho(It)$, with period $T=2e/I$. Thus h must be periodic with the same period. We are interested in the voltage

$$\langle U\rangle=\frac{1}{T}\int_{-T/2}^{T/2}dt\ U=\frac{1}{e}\int_{-e}^{e}dq\frac{dE(q)}{dq}h(q)\ , \tag{14}$$

averaged over an oscillation period $T=2e/I$. We proceed as in Ref. 17 and expand the equilibrium distribution function in a Fourier series,

$$\rho(q)=\tfrac{1}{2}+\sum_{n=\text{odd}}\rho_n\cos(2\pi nq/2e)\ . \tag{15}$$

Here we have taken into account that since $\rho(q)+\rho(q+e)=1$ we must have $\rho_0=\frac{1}{2}$ and, further, only the odd harmonics can occur in the expansion. Similarly, we can expand the energy of the lowest band in Fig. 2(a) into a Fourier series,

$$E(q)=\sum\Delta_n\cos(2\pi nq/2e)\ . \tag{16}$$

Since $dh/dt=I\,dh/dq$, Eq. (12a) becomes $I\tau\,dh(q)/dq=-[h(q)-\rho(q)]$, which gives

$$h(q)=\tfrac{1}{2}+\sum_{n=\text{odd}}\frac{\rho_n}{1+n^2\omega^2\tau^2}\times[\cos(2\pi nq/2e)+n\omega\tau\sin(2\pi nq/2e)]\ , \tag{17}$$

with $q=It$. Here we have used the Bloch frequency $\omega=2\pi I/2e$. Consider for a moment the actual nonaveraged voltage equation (13). In addition to the harmonics of $2e$, a Fourier analysis of Eq. (13) using Eq. (17) also contains the period e and its harmonics. Equation (12) contains, for instance, a term

$$\sin(2\pi q/2e)\cos(2\pi q/2e)\sim\sin(2\pi q/e)\ ,$$

which has the period e. In addition to these sinusoidal voltage oscillations, quasiparticle tunneling gives rise to a time-independent voltage ($n=0$ Fourier coefficient). When averaged over an oscillation period, Eq. (14) using Eq. (17) gives

$$\langle U\rangle=-\frac{\pi}{e}\sum_{n=\text{odd}}\Delta_n\rho_n\frac{n^2\omega\tau}{1+n^2\omega^2\tau^2}\ . \tag{18}$$

Equation (18) determines the current-voltage characteristic I-U in the presence of a linearly increasing q. This branch is shown in Fig. 4 together with the constant-voltage, zero-current branch, obtained in the presence of a constant q. As shown in Fig. 4, the voltage increases for

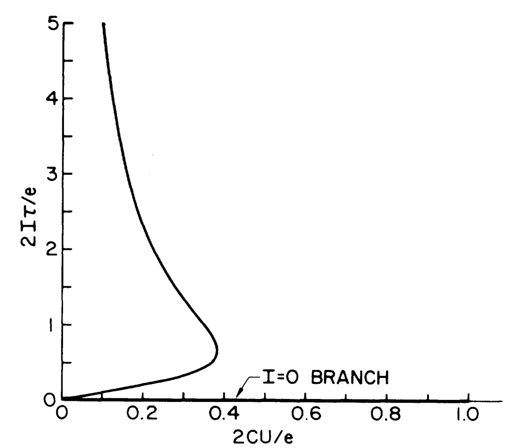

FIG. 4. Current-voltage characteristic of the small-capacitance Josephson junction. The zero-current branch is obtained by varying the induced charge quasistatically. The maximum (minimum) voltage is obtained for $q=e/2+ne$ ($q=-e/2+ne$). The nonzero-current-voltage branch given by Eq. (18) represents the dc current and dc voltage which accompany the Bloch oscillations when the induced charge is increased linearly with time at a rate $dq/dt=I$.

small currents linearly with the current, reaches a maximum for intermediate currents, and is inversely proportional to the current for large currents. If the sum in Eq. (18) is dominated by the low-order Fourier coefficients, we can expect a maximum voltage for a current $I \sim 2e/2\pi\tau$. The shape of the current-voltage characteristic is typical for systems in which Bloch oscillations provide the microscopic mode of transport, except that in these systems the role of the voltage and the current are reversed. Examples are superlattices[24,25] and ultrasmall normal loops.[17,19,22]

Instead of expanding the actual probability $h(q)$ in a Fourier series, we can also express $h(q)$ in terms of derivatives of the equilibrium function. From Eq. (12a) we find $h(q)=\rho(q)-I\tau\, d\rho(q)/dq$ to lowest order in $I\tau$. Here we have again used that $dh/dt=I\, dh/dq$. Inserting $h(q)$ into Eq. (14) yields

$$\langle U\rangle = -\frac{1}{e}\int_{-e}^{e} dq\, I\tau \frac{dE(q)}{dq}\frac{d\rho(q)}{dq} = I\tau \int_{-e}^{e} dq \frac{d^2E(q)}{dq^2}\rho(q) , \tag{19}$$

where we have used partial integration. We can now use either Eq. (17) or (19) to calculate the slope of the I-U characteristic near the origin, i.e., the resistance of such a small junction. We obtain

$$\mathcal{R}=\frac{\tau}{C_*} , \tag{20}$$

with an effective capacitance

$$C_*^{-1}=\frac{1}{e}\int_{-e}^{+e} dq \frac{d^2E}{dq^2}\rho(q)=\frac{\pi^2}{e^2}\sum_{n=\text{odd}} n^2\Delta_n\rho_n . \tag{21}$$

The inverse effective capacitance C_* is, therefore, a weighted average, over the whole Brillouin zone, of the inverse capacitance $d^2E(q)/dq^2$. We can also introduce an effective temperature and current-dependent capacitance, expressing Eq. (18) in the form $\langle U\rangle=\tau I/C_*(I)$. This current-dependent effective capacitance increases monotonically with increasing current.

In Eq. (19) we have expanded $h(q)$ only to linear order in $I\tau$. Formally, we can carry this expansion to higher order in $I\tau$. This procedure assumes that $h(q)$ is analytic in $I\tau$ at the origin. That is not correct as the following simple example shows. Consider the case $T\rightarrow 0$, where $\rho(q)$ is piecewise constant. Further, we approximate the lowest band in Fig. 2(a) by $E(q)=(q-2en)^2/2C$ for q in the interval $(n-1)e<q<(n+1)e$. This is a good approximation to the lowest band, except in the vicinity of the gap E_J at $q=e+2ne$. Using the variation of constants to obtain $h(q)$ from Eq. (12a) yields, for the average voltage, Eq. (14),

$$2C\langle U\rangle/e = y[\cosh(1/y)-1]/\cosh(1/y)$$

with $y=2I\tau/e$. This result is depicted in Fig. 4. Whereas the first derivative $d\langle U\rangle/dy\,|_{y=0}=e/2C$ exists, higher-order derivatives taken at $y=0$, i.e., $I\tau=0$, diverge. This nonanalytic behavior is typical for the zero-temperature limit only. In this limit the derivatives of the equilibrium distribution function with respect to q are not bounded. Consequently, a perturbation approach which expands in powers of $I\tau$ can give the correct answer for the resistance, Eq. (20), but will fail to find the correct nonlinear part of the I-U characteristic.

The dc I-U characteristic shown in Fig. 4 is qualitatively similar to that found in Ref. 11, but not identical. The authors of Ref. 11 find at small voltages (in the absence of a shunting resistance) a resistance which is proportional to the square of the voltage, $I\sim U^2/\tau$ and, therefore, a resistance which diverges as U tends to zero. Equation (20), in contrast, yields a finite resistance as long as the relaxation time τ is finite. For currents which are comparable or larger than I_c, Zener transitions through the gap of Fig. 2 can no longer be neglected and determine the shape of the I-U characteristic.[26,27]

IMPEDANCE OF THE JUNCTION

Next, let us follow Ref. 1 and consider the case where the battery voltage consists of an ac voltage $U_2\cos\Omega t$ superimposed on a static voltage U_1. In our circuit this implies that the induced charge is given by $q=q_1+q_2\cos\Omega t$, where q_1 and q_2 are time-independent constants. We assume that $q_2<e$ such that the system executes only small oscillations away from the static value q_1. We also assume that the frequency Ω is small, such that resonant transitions to the higher-lying bands of Fig. 2(a) do not occur. The actual probability of the system to be in the state $E(q)$ or in the state $E(q+e)$ can then also be obtained from Eq. (12) with q as given above. It is sufficient to study the response of the system to a perturbation of the form $q=q_1+q_2\exp(i\Omega t)$. We solve Eq. (12) to first order in q_2 and use Eq. (13) to determine the voltage response. This yields an impedance $Z(q_1,\Omega)=Z'+iZ''=\delta U/\delta I$ with a real part

$$Z'\equiv\mathcal{R}=\tau/C_{\text{eff}} \tag{22}$$

and an imaginary part

$$Z''=-\frac{1}{\Omega}\left[\left(\frac{d^2E}{dq^2}\right)\rho\bigg|_{q_1}+\left(\frac{d^2E}{dq^2}\right)\rho\bigg|_{q_1+e}-\frac{1}{C_{\text{eff}}}\right] . \tag{23}$$

Here we have introduced an effective small signal capacitance

$$\frac{1}{C_{\text{eff}}(q_1,\Omega)}=\frac{1}{1+\Omega^2\tau^2}\left(\frac{d\,\Delta E}{dq}\right)^2\bigg|_{q=q_1} \times\left(\frac{4}{k_BT}\right)\frac{1}{\cosh^2(\Delta E/2k_BT)}\bigg|_{q=q_1} . \tag{24}$$

The impedance Z is a periodic function of the static component of the induced charge q_1 with period e. The effective small-signal capacitance shown in Fig. 5 determines both the real and imaginary parts of the impedance: The first two terms in Eq. (23) are of the order $1/C$, where C is the bare junction capacitance. The inverse effective capacitance, the third term in Eq. (23), is of the order $e^2/C^2k_BT \gg 1/C$, for $k_BT \ll e^2/C$. The effective capacitance (24) is periodic with period e in q_1. At low temperatures the last factor in Eq. (24) peaks sharply at $e/2+n$ and is exponentially small away from these values of q. The factor $[d\,\Delta E(q)/dq]^2$ is zero at $q=ne$ and reaches a maximum near $q=e/2+n$. The inverse effective capacitance, as shown in Fig. 5, exhibits, therefore, very sharp peaks near $q=e/2+ne$ and is exponentially small away from these values of q. As a function of the applied static voltage U_1 the effective capacitance is periodic with period $\Delta U_1=e/C_0$ and, in general, exhibits a phase shift U_P.

The reactive part of the junction impedance is thus dominated by the effective capacitance and given by $Z''=1/(\Omega C_{\text{eff}})$ in the temperature range of interest. Note that the effective capacitance enters Eq. (23) with a negative sign. It is furthermore of interest that the effective capacitance and the resistance determined by Eq. (22) occur in series. This differs from the conventional resistively-shunted-junction (RSJ) model which describes the impedance of a junction in terms of a capacitor and a resistor in parallel. In the case considered here, resistance is due to quasiparticle tunneling subject to the Coulomb forces as described by the junction capacitance C. The RSJ model, on the other hand, describes a shunting resistor which permits the carriers to circumvent the Coulomb barrier. The total reactive impedance of the circuit of Fig. 1 is $Z''=-1/(\Omega C_{\text{tot}})$ with

$$\frac{1}{C_{\text{tot}}}=\frac{1}{C_0}-\frac{1}{C_{\text{eff}}}\,, \tag{25}$$

where C_0 is the external capacitance. In the limit $C_0 \ll C_{\text{eff}}$, the effective capacitance gives rise to a small modulation ΔC on top of the external capacitance C_0.

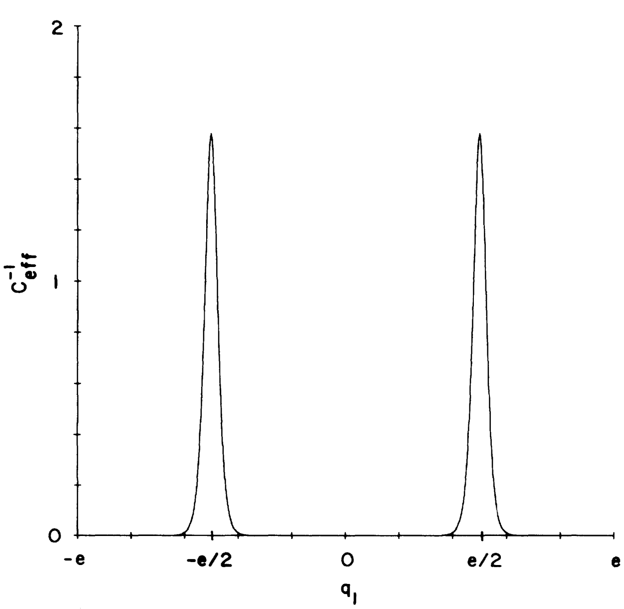

FIG. 5. Effective small-signal capacitance, Eq. (23), as a function of q for the case of a strictly sinusoidal band $E(q)$ with amplitude Δ and a temperature $k_BT=0.1\Delta$. The capacitance is given in units of $10^{-3}e^2/\Delta$.

Using Eq. (25) we find, for the modulation of the total capacitance $C_{\text{tot}}=C_0+\Delta C$,

$$\Delta C=C_0^2/C_{\text{eff}}\,. \tag{26}$$

Since C_{eff} is a periodic function of the induced charge, or a periodic function of the applied voltage U_B, the capacitance given by Eq. (26) exhibits an oscillatory behavior with a period $\Delta U_B=e/C_0$ and a phase shift U_P if the junction supports a finite voltage at $U_B=0$. Therefore, the measurement of the effective capacitance with a small signal ac voltage superimposed on a static voltage is sensitive to the potential drop given by Eq. (3), and respectively, Eq. (9), and provides a technique to detect such potential differences. This behavior of the effective capacitance in the presence of a small ac voltage superimposed on a static voltage has possibly already been seen in the experiments on normal junctions,[1] as we will argue below.

DISCUSSION

We have emphasized here that a small capacitance Josephson junction, coupled to an appropriate external circuit, exhibits *two* key phenomena. At equilibrium such a junction can maintain a potential difference between two conductors without an accompanying dc current. In the state where the junction undergoes Bloch oscillations the time lag between the actual distribution function and the instantaneous equilibrium distribution function[17] gives rise to the appearance of a dissipative current-voltage branch. We would like to point out that the approach presented in this paper can also be used to treat normal tunnel junctions. While the dissipative branch of the normal junction I-U characteristic[11] differs significantly from that of the superconducting junction (see Fig. 4), the equilibrium properties, i.e., the persistent voltage states and the small-signal impedance can be expected to be qualitatively the same. Neither of these properties relies on coherence, i.e., on Josephson coupling between the superconductors, but are even in the case of a Josephson junction determined by quasiparticle tunneling. Furthermore, for the equilibrium effects, the fact that the Josephson junction has a gap (see Fig. 2) and the normal junction has none is irrelevant. On a phenomenological level a discussion of the equilibrium effects of a normal junction can be given by taking the limit $I_c=0$ in the results presented above, i.e., by replacing the energy bands of Fig. 2 by $E(q)=q^2/2C$ reduced to a Brillouin zone of width e. The small-signal impedance is then also a periodic function of q or the applied battery voltage U_B. Thus the impedance of a normal tunnel junction is also characterized by small periodic variations as given by Eq. (26) on top of a background capacitance. Lambe and Jacklevic[1] have possibly observed the effects described here. They investigated an array of normal tunnel junctions formed by evaporating small metallic droplets on an insulating layer covering a metallic electrode. The other electrode was formed by oxidizing the metallic particles and covering them with a second metallic film. A measurement of the effective capacitance in the presence of a sinusoidally varying voltage superimposed on a static voltage exhibited two sharp peaks as function of the static component of the voltage.

Thus the measured effective capacitance of such an array is not periodic in the external voltage, in contrast to our result (26). We must, however, bear in mind that the result of Ref. 1 represents only the average behavior of many tunnel junctions. The self-capacitance C and the capacitance C_0 with which a metallic particle is coupled to the external circuit vary from particle to particle. Thus each particle exhibits an effective capacitance which is periodic in the applied voltage with a period which differs from particle to particle. If C_0 varies not too strongly from particle to particle the contributions of all these junctions will give rise to a ΔC which exhibits a fine number of peaks. Thus for such an array of junctions ΔC is an oscillatory function of the static component U_1 with an amplitude which decreases rapidly with increasing magnitude of the voltage. Precisely such a capacitance variation has been observed by Lambe and Jaklevic.[1] They observed a strong double-peak structure in ΔC centered around $U_1=0$. Another aspect of this experiment is highly interesting. The experiment described above corresponds to $U_P=0$. The particles have the same chemical potential as the electrodes. This is achieved by keeping the sample at zero battery voltage for a time long compared to the charge-redistribution relaxation time before the experiment is performed. A nonvanishing phase shift U_P is introduced by subjecting the sample to a time-independent voltage over a period of time comparable to the charge relaxation time of the sample. Small leakage currents cause the Fermi level of the particles to adjust with the upper electrode. Then the battery voltage is set to zero and the experiment is quickly repeated. The characteristic double-peak structure described above was found again, but now at a different voltage. Thus the possibility of shifting the phase in $\Delta C(U_B)$ has also been observed. This demonstrates, as mentioned in Ref. 1, that the chemical potentials of the metallic particles remain for a long time (of the order of the charge-redistribution relaxation time) at the values which they obtained in the presence of the long-term voltage. This raises our hope that the persistence of chemical-potential differences across *single* normal or superconducting junctions is an observable physical effect.

Note added in proof. Capacitively induced oscillations with period e/C_0 have recently been seen in a "single grain" experiment by Fulton and Dolan.[28] In their experiment a central electrode (connected to three junctions) is capacitively coupled to the substrate with a capacitance C_0. Instead of the capacitive measurement of Ref. 1 Fulton and Dolan observe oscillations of the Giaver and Zeller Coulomb gap (see Ref. 1) in the I-U characteristic of two of the small tunnel junctions connected to the central electrode.

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