Weak localization in inhomogeneous magnetic fields

J. Rammer
*Nordisk Institut for Teoretisk Atomfysik, NORDITA, Blegdamsvej 17, DK-2100 Copenhagen Ø, Denmark and Department of Physics and Materials Research Laboratory, University of Illinois at Urbana-Champaign, 1110 West Green Street, Illinois 61801**

A. L. Shelankov
Nordisk Institut for Teoretisk Atomfysik, NORDITA, Blegdamsvej 17, DK-2100 Copenhagen Ø, Denmark and A. F. Ioffe Physical-Technical Institute, Leningrad 194021, Union of Soviet Socialist Republics†
(Received 27 January 1987)

A theoretical analysis of weak localization in inhomogeneous magnetic fields is presented. The weak-localization correction to the classical conductivity is the result of interference effects characteristic of quantum-mechanical motion in a disordered conductor. The phase-coherence length, set by inelastic collisions, describes the spatial limit to single-particle quantum-mechanical interference. When this length exceeds the length scale over which the magnetic field is uniform the standard theory of the weak-localization magnetoresistance in a uniform magnetic field is no longer applicable. The coherent quantum-mechanical motion probes the inhomogeneity of the magnetic field, and the conductivity thus exhibits its nonlocal dependence on the magnetic field. The inhomogeneity of the magnetic field is shown to lead to observable changes in macroscopic quantities; one possible experimental method of testing the effects of weak localization in inhomogeneous magnetic fields is presented.

I. INTRODUCTION

During the past few years substantial progress in the understanding of the physics of disordered conductors has been achieved.[1] Since the original work of Anderson[2] it has been known that the quantum-mechanical description of the motion of a particle in a random medium does not give rise to a diffusive motion if the disorder is sufficiently strong, but leads to localization of the particle. In the case where the disorder is weak the transport properties are essentially classical and may be calculated by the Boltzmann theory. One achievement of the recent progress has been the recognition of a method for calculating the quantum corrections to the Boltzmann result in perturbation theory and an interpretation of the theory in terms of a simple physical picture leading to a comprehensive physical understanding of moderately disordered conductors. The quantum correction to the Drude conductivity σ_0 is the first correction in the expansion parameter $\hbar/p_F l$ where $p_F = mv_F$ is the Fermi momentum and $l = v_F\tau$ is the mean free path; when this perturbation expression is sufficient we are in the weak-localization regime. In order to clarify the investigation presented in this paper as well as the interpretion of the results we shall briefly remind the reader of the physics of weak localization.

The origin of the weak-localization effect is the general phenomena of the interference of waves propagating in a random medium. In our case of electronic motion its wave character is due to quantum effects represented by interference between alternative probability amplitudes for electron propagation between space-time points. The weak-localization effect, however, may be understood at the quasiclassical level with the effect of quantum-mechanical interference taken into account.[3] The phases of interfering probability amplitudes for the traversal of different classical paths will be random due to the random positions of the impurities, except for the case where a trajectory and its time-reversed trajectory interfere. It is now well established that the interference between the two phase-coherent alternative ways of traversing, in the disordered material, a classical path that returns to its starting point is responsible for the quantum correction to the conductivity.[4,5]

Consequently, the weak-localization correction to the Drude conductivity describing the coherent backscattering is expressed through the interference term

$$\Delta\sigma = -\frac{2e^2 D}{\pi\hbar}\left\langle \sum_c \cos(\phi_c - \bar{\phi}_c)\exp(-t_c/\tau_\phi)\right\rangle \qquad (1.1)$$

where ϕ_c and $\bar{\phi}_c$ are the phases of the probability amplitudes acquired by traversing of a classical path c that returns to its starting point along its two respective directions as depicted in Fig. 1 and D is the diffusion constant. The sum over all such closed loops should be performed taking into account their probability, which we express by use of the angular brackets. The interference phenomena requires phase coherence and the factor $\exp(-t_c/\tau_\phi)$ ac-

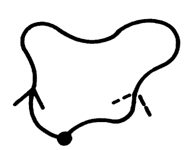

FIG. 1. A classical diffusive path c that returns to its starting point can be traversed in two possible directions.

counts for processes that destroy the phase coherence at a phase-breaking rate $1/\tau_\phi$. In the case of time-reversal invariance the phases ϕ_c and $\bar{\phi}_c$ are equal, so that, as noted above, the averaging over the disorder does not lead to cancellation of this type of interference term. The number of relevant paths will depend on the phase coherence time τ_ϕ, as paths with too long a duration t_c do not contribute due to the factor $\exp(-t_c/\tau_\phi)$. Since τ_ϕ is a function of temperature, controlled by inelastic processes, the weak-localization effect gives rise to a temperature dependence of the conductivity which in the case of a two-dimensional sample is given by

$$\Delta\sigma = -\frac{e^2}{2\pi^2\hbar}\ln(\tau_\phi/\tau) \ . \tag{1.2}$$

An especially important way of influencing the interference process is to break the time-reversal symmetry by a magnetic field. In the presence of a magnetic field the quantum-mechanical amplitude for traversing a loop counterclockwise around the magnetic field lines contains the additional phase factor $\exp(i2\pi\Phi_c/\Phi_0)$ where Φ_c is the magnetic flux enclosed by the loop c and $\Phi_0 = h/2|e|$ is the flux quantum. Since the phase difference $\phi_c - \bar{\phi}_c$ is equal to $2\pi\Phi_c/\Phi_0$ the weak-localization contribution to the magnetoconductivity is given by

$$\Delta\sigma(B) - \Delta\sigma(0) = \frac{2e^2D}{\pi\hbar}\left\langle \sum_c \left[1 - \cos 2\pi\frac{\Phi_c}{\Phi_0}\right]\exp(-t_c/\tau_\phi)\right\rangle \ . \tag{1.3}$$

The theory of weak-localization predicts for the case of a homogeneous magnetic field B a magnetoconductivity[6]

$$\Delta\sigma(B) - \Delta\sigma(0) = \frac{e^2}{2\pi^2\hbar} f_2\left[\frac{B}{B_\phi}\right] , \tag{1.4}$$

where the function f_2 is related to the digamma function ψ by

$$f_2(x) = \ln x + \psi\left[\frac{1}{2} + \frac{1}{x}\right] \tag{1.5}$$

and $B_\phi = \Phi_0/4\pi L_\phi^2$, $L_\phi = (D\tau_\phi)^{1/2}$. The applicability of formula (1.4) assumes that the sample is two dimensional with respect to the weak-localization effect, which requires the transverse sample dimension to be much smaller than the typical length L_ϕ the electron traverses in its diffusive motion within the phase coherence time τ_ϕ. Furthermore, it is assumed that the magnetic field is weak enough that bending of trajectories is unimportant; that is, $\omega_c\tau \ll 1$ where $\omega_c = |e|B/m$ is the cyclotron frequency and $1/\tau$ is the impurity scattering rate. Numerous measurements have shown that Eq. (1.4) is in very good agreement with the experimental data and that the sensitivity of the weak-localization effect to a magnetic field is indeed the property that enables one to separate out the effect by suppression of the coherent backscattering. Magnetoresistance measurements are thus a most important method for the observation of weak-localization effects and due to their accuracy and simplicity they have turned out to be a useful tool for the experimental determination of characteristic times.[7]

The above physical description of the by now standard theory of magnetoresistance in the weak-localization regime has been undertaken in order to assess its range of validity in the context of inhomogeneous magnetic fields, and shows that the quantum correction to the conductivity at a given point is sensitive to the magnetic field structure within an area of order L_ϕ^2 so that the range of validity of Eq. (1.4) is restricted to magnetic fields that are uniform on the length scale L_ϕ. In this case we have a local situation where the conductivity $\sigma(\mathbf{r})$ at a point $\mathbf{r}$ depends only on the magnetic field at this point. The magnitude of L_ϕ may, at low temperature, be very large (typically in the micrometer range) and it is therefore of interest to investigate the weak-localization magnetoresistance in magnetic fields that are nonuniform on this length scale, since we then are in the nonlocal limit where Eq. (1.4) no longer applies. It is the purpose of the present paper to consider this physical situation and to calculate the nonlocal dependence of the conductivity on the magnetic field.[8] Before we continue with the presentation of the theory of weak localization in inhomogeneous magnetic fields we propose and discuss a physical system relevant for testing the results of the following.

We shall restrict ourselves to the experimentally most interesting case of two- or quasi-two-dimensional conductors [a metal-oxide-semiconductor field-effect transistor (MOSFET) or a thin film]. For these systems there is a rather simple way of imposing a nonuniform magnetic field. It is sufficient to place a superconducting film near (and insulated from) the normal film under study or to use a superconducting gate in the case of a MOSFET.[9] For the sake of completeness, we briefly review the behavior of a superconducting film in a magnetic field. A magnetic field creates the intermediate or the mixed state depending on whether the superconductor is of type I or type II.[10] In the intermediate state the magnetic field penetrates as a series of lamina or flux tubes, depending on field strength and film properties. In the latter case each tube contains an integral number of flux quanta Φ_0. In the mixed state the field penetration is accomplished by the formation of a two-dimensional lattice of vortices.

The presence of a superconductor in a uniform external magnetic field leads to nonuniformity in the magnetic field within and therefore near the surface of the superconductor as depicted in Fig. 2. We let $b_z(\mathbf{r},z)$ denote the z component of the local magnetic field $\mathbf{b}$ at a point $\mathbf{r}=(x,y)$ in

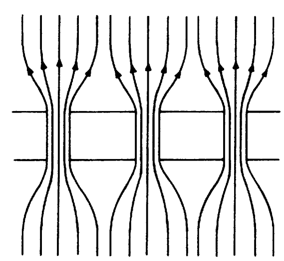

FIG. 2. Magnetic field lines showing the nonuniformity within and near the superconductor.

the plane parallel to the surface of the superconductor at a distance z outside the superconductor and expand it in its Fourier series

$$b_z(\mathbf{r},z)=\sum_{\mathbf{q}} b_z(\mathbf{q},z)e^{i\mathbf{q}\cdot\mathbf{r}} . \quad (1.6)$$

From the Maxwell equations we then get that

$$b_z(\mathbf{q},z)=b_z(\mathbf{q},z=0)\exp(-|\mathbf{q}|z) . \quad (1.7)$$

The nonuniformity of the magnetic field therefore exists only over distances of the order of the period of the magnetic structure.

The period is determined by the applied magnetic field. In the case of a type-II superconductor the period is equal to the magnetic length $l_B=(\Phi_0/B)^{1/2}$. Thus for a small enough magnetic field and $z \ll l_B$, the magnetic field penetrates the normal film or the MOSFET channel as an array of magnetic tubes, i.e., the field is highly nonuniform. In this case the dependence of the conductivity on the average magnetic field differs significantly from the prediction of Eq. (1.4), as discussed in Sec. IV. In the case where the magnetic field is nearly uniform the modulation of $\Delta\sigma(r)$ is too small to be observed by means of a conductivity measurement as discussed in Sec. IV. However, we wish to point out the important feature that the system under consideration allows one to extract experimentally the spatially dependent part of the conductivity even for the case where the modulation of the magnetic field is small. We shortly discuss the circumstances under which this can be brought about.

It is well known,[10] that passing an electric current through the superconductor can make the magnetic field structure in the superconductor move. The motion of an inhomogeneous magnetic field structure will thereupon induce a dc electric current in any magnetoresistive film placed near the surface of the superconductor. To illustrate the idea we consider the following simple example. We imagine that the magnetoresistive film is placed in a magnetic field of the form $\mathbf{B}(\mathbf{r},t)=\hat{\mathbf{z}}B_0\Theta(x-ut)$, where Θ is the step function. This moving magnetic field structure induces electric fields and currents as depicted in Fig. 3. If the conductivity depends on the magnetic field the currents at point 1 and 2 do not cancel and a resulting dc current flows through the film. Thus, an experiment with a current through the superconducting film allows one to extract the spatial dependence of the magnetoconductivity of the normal film. The effect exists in any magnetoresistive material irrespective of the origin of the magnetoresistivity and has long been recognized[11] in connection with the physics of the Giaever transformer.[12] However, to our knowledge, no quantitative theory of the effect has been developed. As discussed in Sec. IV this brings about the opportunity to test the weak-localization theory for inhomogeneous magnetic fields even in the case of a small modulation of the magnetic field.

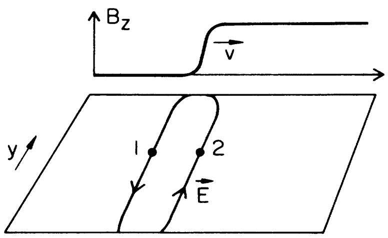

FIG. 3. Electric field lines as a result of the moving magnetic field structure.

Besides the quantum correction to the conductivity due to weak localization, there are, as recently established, corrections of equal order of magnitude, the so-called mesoscopic effects. These effects, however, vanish upon averaging over the impurity potential and since we shall calculate the influence of the magnetic field on macroscopic properties, i.e., properties measured by probes far apart, we can neglect all mesoscopic effects.

In outline, the paper is organized as follows: In Sec. II we study the case of a highly inhomogeneous magnetic field where the film is penetrated by widely separated magnetic-flux tubes. The Aharonov-Bohm model problem of a magnetic string is considered as well as effects of finite tube size. In Sec. III we calculate the spatially dependent part of the quantum correction to the conductivity for the case of a small but arbitrarily rapid modulation of the magnetic field. In Sec. IV we consider the effects of an inhomogeneous field on macroscopic quantities. The proposed physical realization of the inhomogeneous magnetic field allows one, by generating flux flow in the superconductor, to induce a dc current in the normal film, thus enabling one to study the spatially dependent part of the conductivity. Furthermore, the classical magnetoconductivity and Hall effect are discussed in the limit of classically weak fields since they, in the limit of magnetic fields larger than the characteristic field B_ϕ for the saturation of the quantum correction, may be of more importance. Finally, Sec. V contains a summary and conclusions.

II. THE WEAK LOCALIZATION MAGNETOCONDUCTIVITY IN A HIGHLY INHOMOGENEOUS MAGNETIC FIELD

In this section we consider the case of a sufficiently weak applied magnetic field so that the presence of the superconductor generates widely separated magnetic-flux tubes. It is then possible to consider each tube separately; that is, the distance between the tubes penetrating the normal film is much larger than L_ϕ. For simplicity, we assume that the tubes have a circular cross section with radius r_0 within the film and that the magnetic field has the constant magnitude b_0 inside the circles and is zero outside.

For the trivial case when $r_0 \gg L_\phi$ the spatially dependent part of the conductivity $\delta\sigma$ inside the tube is given by

$$\delta\sigma=\frac{e^2}{2\pi^2\hbar}f_2\left[\frac{b_0}{B_\phi}\right] \quad (2.1)$$

and outside the tube we have that $\delta\sigma=0$. Throughout we use the diffusion approximation and assume therefore that all lengths are much larger than the mean free path l, in particular for Eq. (2.1) to be valid $r_0 \gg l$.

We then turn to consider the highly nonlocal situation when $r_0 \ll L_\phi$. To start with, we shall assume that r_0 is the shortest length in the diffusion problem and in the limit where r_0 goes to zero the tube shrinks into a magnetic string.

A. The magnetic string

We are thus led to study the problem of a magnetic string intersecting the film at the origin of the (x,y) coordinate system in the film plane and containing a magnetic flux Φ.

The weak-localization correction to the dc conductivity is given by[6]

$$\Delta\sigma(\mathbf{r}) = -\frac{2e^2 D}{\pi\hbar} C(\mathbf{r},\mathbf{r}) \tag{2.2}$$

where the particle-particle diffusion propagator satisfies the equation[13]

$$\left[D\left(-i\nabla - \frac{2e}{\hbar}\mathbf{A}\right)^2 + \frac{1}{\tau_\phi}\right] C(\mathbf{r},\mathbf{r}') = \delta(\mathbf{r}-\mathbf{r}') \ . \tag{2.3}$$

Formally Eq. (2.3) is identical to the equation for the Green's function of the Schrödinger equation. The physical interpretation of the particle-particle diffusion propagator was discussed in detail in the Introduction.

To describe the magnetic field configuration of the magnetic string we choose the vector potential $\mathbf{A}$ to be, in cylindrical coordinates, given by

$$\mathbf{A} = \frac{\Phi}{2\pi r}\hat{\boldsymbol{\phi}} \ , \tag{2.4}$$

where $\hat{\boldsymbol{\phi}}$ is the azimuthal unit vector. To our benefit, the magnetic string problem has been extensively studied in connection with the Aharonov-Bohm effect[14,15] and referring to these results we write down the solution C_α of Eq. (2.3) for the magnetic string problem where the vector potential is given by Eq. (2.4)

$$C_\alpha(\mathbf{r},\mathbf{r}) = \frac{1}{2\pi}\int_0^\infty dt \sum_{n=-\infty}^{\infty} \int_0^\infty dk\, k\, J^2_{|n+\alpha|}(kr) \times \exp\left[-\left(Dk^2 + \frac{1}{\tau_\phi}\right)t\right] \tag{2.5}$$

where J_ν denotes the Bessel function and $\alpha = \Phi/\Phi_0$. Evidently, $C_\alpha(\mathbf{r},\mathbf{r})$ is a periodic function in Φ with the period Φ_0. One can now, by use of a table of integrals and series,[16] perform the integration over k and the summation over n to obtain

$$C_\alpha(\mathbf{r},\mathbf{r}) - C_0 = -\frac{1}{4\pi^2}\int_0^\infty \frac{dt}{Dt}\exp\left[-\frac{r^2}{2Dt} - \frac{t}{\tau_\phi}\right] \times Q\left[\alpha, \frac{r^2}{2Dt}\right] , \tag{2.6}$$

where C_0 is the particle-particle diffusion propagator for the case $\alpha = 0$ and Q is the periodic function in α defined for $|\alpha| \le 1$ by

$$Q(\alpha,z) = \sin(\pi|\alpha|)\int_0^\infty dx \frac{e^{|\alpha|x} + e^{(1-|\alpha|)x}}{e^x + 1} e^{-z\cosh x} \ . \tag{2.7}$$

We can then expand $Q(\alpha,z)$ in its Fourier series with respect to α to finally obtain

$$C_\alpha(\mathbf{r},\mathbf{r}) - C_0 = -\frac{1}{4\pi}\int_0^\infty \frac{dt}{Dt}\exp\left[-\frac{r^2}{2Dt} - \frac{t}{\tau_\phi}\right] q_1\left[\frac{r^2}{2Dt}\right] \sum_{n=1}^{\infty} P_n\left[\frac{r^2}{2Dt}\right][1 - \cos(2\pi n\alpha)] , \tag{2.8}$$

where

$$q_n(z) = \frac{2}{\pi}\int_0^\infty dx \frac{\exp\{-z\cosh[(2n-1)\pi x]\}}{1+x^2} \tag{2.9}$$

and

$$P_n(z) = \frac{1}{q_1(z)}[q_n(z) - q_{n+1}(z)] \ . \tag{2.10}$$

Let us pause to analyze the expression in Eq. (2.8) in terms of the physical description of the quantum correction to the conductivity given in the Introduction. The magnetic field is zero everywhere in the plane of the two-dimensional electron gas except for the singular point where the string intersects the plane. The only trajectories that contribute to $C_\alpha - C_0$ are therefore those which encircle the string. Each relevant path can therefore be ascribed a number characterizing its topological property of how many times it encircles the string. Let us denote this winding number by n. A path with winding number n encloses the flux $n\Phi$ and such an n-type path contributes therefore to the magnetoconductivity and the particle-particle diffusion propagator with a factor $(1-\cos 2\pi n\alpha)$. Furthermore, the total contribution of all n-type paths is proportional to the relative probability of the n-type paths.

From this consideration we infer that the quantity

$$\frac{e^{-(r^2/2Dt)}}{Dt} q_1\left[\frac{r^2}{2Dt}\right] p_n\left[\frac{r^2}{2Dt}\right]$$

appearing in Eq. (2.8) has the following interpretation: It is proportional to the probability that the diffusing particle at a distance r from the string in time t returns to this point with the constraint that its path encircles the string n times. The function p_n is the distribution function over the winding number and has the property $\sum_{n=1}^{\infty} p_n = 1$. The number $q_n(z)/q_1(z) = \sum_{m=n}^{\infty} p_m$ gives the conditional probability to have at least n circulations.

It is important to emphasize that the quantities p_n and q_n are of purely statistical and geometrical origin. They are determined by the statistics of diffusion, i.e., random walk, in two dimensions. We note that the above investigation may therefore be considered as a method for calculating these statistical distributions; in fact, we know of no alternative method.

The function $q_n(z)$, Eq. (2.9), may be expressed in terms of the modified Bessel function K_0

$$q_n(z) \simeq \frac{2}{\pi^2(2n-1)} K_0(z) \tag{2.11}$$

if either of the three criteria $z \ll 1$ and $n \gg |\ln z|$, $z \sim 1$ and $n \gg 1$, or $z \gg 1$ and n arbitrary are fulfilled. Using Eqs. (2.10) and (2.11) we then in turn get

$$p_n(z) \sim n^{-2} \quad \text{for } z \text{ arbitrary and } n \to \infty \ , \tag{2.12}$$

$$p_n(z) = \frac{2}{4n^2-1}, \quad z \gg 1 \tag{2.13}$$

$$p_n(z) \simeq \pi^{-2} n^{-2} |\ln z| \ , \quad z \ll 1, \quad n \gg |\ln z| \ . \tag{2.14}$$

In the course of the derivation of Eq. (2.14) we have taken into account that $q_1(z \ll 1) \simeq 1$.

We note, that the average winding number $\bar{n}(z) = \sum_{n=1}^{\infty} n p_n(z)$, diverges for any z. This is particularly surprising for the case of large z, $r^2/2Dt \gg 1$, where the particle has a short time available to reach the singular point but nevertheless spends time revolving around the point.

The weak-localization "magnetoconductivity" $\Delta\sigma(r,\Phi) - \Delta\sigma(\Phi=0)$, i.e., the quantum correction to the conductivity due to the presence of the magnetic string, can be calculated from Eqs. (2.2) and (2.8)–(2.10) for arbitrary distances r to the string. If this distance is much larger than the phase coherence length L_ϕ, $r \gg L_\phi$, then by using Eqs. (2.8) and (2.13) we obtain

$$\Delta\sigma(r,\Phi) - \Delta\sigma(\Phi=0) = \frac{e^2}{2\pi^2\hbar} \left| \sin\left(\pi \frac{\Phi}{\Phi_0}\right) \right| \frac{L_\phi}{r} e^{-(2r/L_\phi)} \ . \tag{2.15}$$

We note that the magnetoconductivity behaves nonanalytically at the flux Φ equal to zero and at the equivalent values $\Phi/\Phi_0 = \pm 1, \pm 2, \dots$. The nonanalyticity is attributed to the divergence of the average winding number.

The Aharonov-Bohm effect in the weak-localization theory was predicted in the famous paper by Altshuler, Aronov, and Spivak.[17] They considered the Aharonov-Bohm effect in a cylindrical geometry. Certainly the flux periodicity of $\Delta\sigma$ does not depend on the kind of doubly connected sample geometry but the flux dependence of $\Delta\sigma(\Phi)$ does. The reason for this is the difference in the statistics of the paths which are allowed in different geometries and our results differ therefore from those of Ref. 17.

B. The effects due to finite tube size

The quantum magnetoresistance vanishes when the flux in the string is equal to an integral number of flux quanta as we have demonstrated above. This result is evident from the description presented in the Introduction since all trajectories enclose zero or an integral number of flux quanta leading to an inessential phase difference equal to a multiple of 2π.

We shall now take into account the finite size of the tube assuming that the flux is an integral number of flux quanta. In this case trajectories traversing the tube may enclose an arbitrary flux. In the considered case where the tube contains an integral number of flux quanta these trajectories will be the only source of magnetoresistance. An intuitive argument it seems, would lead to the conclusion that the finite-size correction should approach zero proportionally to the area of the tube as the area approaches zero. This estimate is based on the assumption that the probability for a diffusing particle to reach a small region in the plane is proportional to the area of the region. However, this intuitive argument fails as a consequence of the divergence of the average winding number discussed above. For our model of a uniform magnetic field of total flux $N\Phi_0$ inside a tube of radius r_0, we have according to Appendix A in the leading approximation $(r \gg r_0)$

$$C(r,r) - C_0 = -\frac{1}{2\pi} \int_{t_0}^{\infty} \frac{dt}{Dt} e^{-t/\tau_\phi} \frac{e^{-(r^2/2Dt)}}{\ln(Dt/r_0^2)} K_0\left(\frac{r^2}{2Dt}\right) , \tag{2.16}$$

where $t_0 \gtrsim r_0^2/D$.

Since only trajectories passing through the tube contribute to Eq. (2.16) we conclude that the integrand excluding the factor e^{-t/τ_ϕ}, has the following interpretation: It is proportional to the probability that the particle at a distance r from the center of the tube in time t returns to this point with the constraint that it passes through the circle with radius r_0, $r \gg r_0$. We note, that indeed this probability goes to zero when r_0 goes to zero, but very slowly, only logarithmically. This is again a peculiarity of diffusion in two dimensions and this logarithmic dependence on r_0^2 is not restricted to paths which return to their starting point.

From Eqs. (2.2) and (2.16) we then obtain for the quantum correction to the conductivity, due to the presence of the tube with an integral number of flux quanta, at distances $r \gg r_0$

$$\Delta\sigma(r,N\Phi_0) - \Delta\sigma(r,0) \simeq \frac{2e^2}{\pi\hbar^2} \frac{1}{\ln(S/r_0^2)} K_0^2\left(\frac{r}{L_\phi}\right) \tag{2.17}$$

where $S \sim \max(L_\phi^2, rL_\phi)$.

From Eq. (2.17) it follows that in the leading approximation the magnetoconductivity does not depend on the magnetic field in the tube, i.e., the flux $\Phi = N\Phi_0$. This dependence exists only in the next approximation, namely where we take into account terms of order $1/(\ln S/r_0^2)^2$, r_0^2/S.

In the following we shall need $\Delta\sigma(r)$ for $r \lesssim r_0$. Having solved Eq. (2.3) for the particle-particle diffusion propagator it is in principle possible to calculate $\Delta\sigma(r)$ for the model under consideration. Evidently, the results would have only qualitative significance since they depend essentially on the magnetic field distribution and in this respect the tube model is certainly unrealistic. For this reason, we shall restrict ourselves to give only a simple estimate of $\Delta\sigma(r<r_0)$ using the qualitative arguments presented in the Introduction.

We therefore consider diffusive paths which return to their starting point inside the tube. If the time of flight for a path is so short that it does not enclose considerable flux (of the order of Φ_0) it will not contribute to the magnetoconductivity. The magnitude of the magnetic field inside the tube is b_0, so $b_0 D t_{\min} \sim \Phi_0$ determines the order of magnitude of the time of flight for the shortest paths which contribute to the magnetoconductivity and we have[18]

$$\Delta\sigma(r<r_0,N\Phi_0)-\Delta\sigma(r,0)\simeq\frac{e^2}{2\pi^2\hbar}\int_{t_{\min}}^{\tau_\phi}\frac{dt}{t} \simeq\frac{e^2}{2\pi^2\hbar}\ln\frac{b_0}{B_\phi} . \tag{2.18}$$

We note that the result of Eq. (2.18) is of the same order of magnitude as $\Delta\sigma(r\sim r_0)$ evaluated by Eq. (2.17).

In the case under consideration, where $r_0^2 \ll L_\phi^2$ we obtain $b_0 \gg B_\phi$. Since $f_2(x)\simeq \ln x$, $x\gg 1$ we observe that the result in Eq. (2.18) can be expressed in terms of the function f_2. In the opposite limiting case where $r_0 \gg L_\phi$ we obtained as well [see Eq. (2.1)], that the magnetoconductivity was described by $f_2(b_0/B_\phi)$. We therefore conclude that for the magnetoconductivity inside the tube we have approximately

$$\Delta\sigma(r\lesssim r_0,N\Phi_0)-\Delta\sigma(r,0)=\frac{e^2}{2\pi^2\hbar}f_2\left[\frac{b_0}{B_\phi}\right] \tag{2.19}$$

irrespective of the relation between r_0 and L_ϕ. We shall in the following make use of the expression in Eq. (2.19) for $\Delta\sigma(r<r_0)$.

III. THE WEAK LOCALIZATION MAGNETOCONDUCTIVITY IN A SLIGHTLY INHOMOGENEOUS MAGNETIC FIELD

In this section we consider the case where the two-dimensional electron gas is situated in a nearly uniform magnetic field $b_z(\mathbf{r},z)=B+\delta b(\mathbf{r},z)$, so that $|\delta b| \ll B$ and the spatial average $\langle \delta b\rangle=0$. We then calculate the spatially dependent part of the conductivity $\delta\sigma$ due to the small modulation δb of the magnetic field.

In the case where the field varies smoothly on the length scale L_ϕ the problem is trivial and we have

$$\delta\sigma=\frac{e^2}{2\pi^2\hbar}f_2'\left[\frac{B}{B_\phi}\right]\frac{\delta b}{B_\phi} . \tag{3.1}$$

Here f_2' denotes the derivative of the function f_2 defined in the Introduction. We assume that the (x,y) components of the field b_x and b_y are small and that they have no influence on τ_ϕ and consequently on B_ϕ.

We then turn to consider the problem of an arbitrarily rapid but small variation of the field. We therefore split the vector potential $\mathbf{A}$ describing the magnetic field $\mathbf{b}$ into two parts

$$\mathbf{A}(\mathbf{r},z)=\mathbf{A}_0(\mathbf{r},z)+\mathbf{A}_1(\mathbf{r},z) ,$$

where $\mathbf{A}_0$ describes the uniform magnetic field $B\hat{\mathbf{z}}$ and $\mathbf{A}_1$ describes the small modulation. We then expand the vector potential $\mathbf{A}_1$ in a Fourier series with respect to its coordinate $\mathbf{r}$ in the plane of the electron gas

$$\mathbf{A}_1(\mathbf{r},z)=\sum_{\mathbf{q}}\mathbf{a}_{\mathbf{q}}(z)e^{i\mathbf{q}\cdot\mathbf{r}} , \tag{3.2}$$

where $\mathbf{q}$ is a two-dimensional vector in the plane of the electron gas.

Without loss of generality, we can assume that the vector potential $\mathbf{A}_1$ satisfies $\nabla\cdot\mathbf{A}_1=0$ and has only x and y components in the plane of the film so that $\mathbf{q}\cdot\mathbf{a}_{\mathbf{q}}(z_0)=0$.

In this gauge the Fourier components $b_{\mathbf{q}}(z)$ of $\delta b_z(\mathbf{r},z)$

$$\delta b_z(\mathbf{r},z)=\sum_{\mathbf{q}} b_{\mathbf{q}}(z)e^{i\mathbf{q}\cdot\mathbf{r}} \tag{3.3}$$

are related to $\mathbf{a}_{\mathbf{q}}(z)$ by

$$\mathbf{a}_{\mathbf{q}}(z)=\frac{i}{q^2}(\mathbf{q}\times\hat{\mathbf{z}})b_{\mathbf{q}}(z) . \tag{3.4}$$

Since we assume that the modulation is small we only need to solve the equation for the particle-particle diffusion propagator Eq. (2.3) to first order in $\mathbf{A}_1$. For a given Fourier component of $\mathbf{A}_1$, specified by $\mathbf{q}$, it is always possible to choose the coordinate system so that $\mathbf{q}=q\hat{\mathbf{y}}$ and for this choice it is convenient to use the Landau gauge $\mathbf{A}_0=xB\hat{\mathbf{y}}$ for the uniform part of the field. We then solve Eq. (2.3) for the particle-particle diffusion propagator $C=C^{(0)}+C^{(1)}$ and get for the Fourier component $C^{(1)}(\mathbf{q})$ of the first-order correction in $\mathbf{A}_1$

$$C^{(1)}(\mathbf{r},\mathbf{r})=\sum_{\mathbf{q}}C^{(1)}(\mathbf{q})e^{i\mathbf{q}\cdot\mathbf{r}} \tag{3.5}$$

the expression

$$C^{(1)}(\mathbf{q})=-\frac{1}{4\pi D}F\left[\beta,\frac{B}{B_\phi}\right]\frac{b_{\mathbf{q}}(z_0)}{B} , \tag{3.6}$$

where $\beta=\beta(q,B)=(2\sqrt{\pi})^{-1}ql_B$ and the function F is defined by

$$F\left[\beta,\frac{B}{B_\phi}\right]=-\frac{D^2}{\beta L_\phi^3}\left[\frac{B}{B_\phi}\right]^{3/2}\int_{-\infty}^{\infty}dx\int_{-\infty}^{\infty}dx'\,\tilde{C}_0^{(0)}(x,x')\frac{\partial}{\partial x'}\tilde{C}_{-q}^{(0)}(x',x) \tag{3.7}$$

and given in terms of the particle-particle diffusion propagator

$$C^{(0)}(\mathbf{r},\mathbf{r})=\int \frac{dk}{2\pi} e^{ik(y-y')}\tilde{C}_k^{(0)}(x,x')$$

for the uniform problem corresponding to $\mathbf{A}=\mathbf{A}_0$

$$\tilde{C}_k^{(0)}(x,x')=\sum_{n=0}^{\infty} \frac{\psi_n(x-(\hbar/2eB)k)\psi_n(x'-(\hbar/2eB)k)}{4\,|\,e\,|\,BD\hbar^{-1}(n+\frac{1}{2})+1/\tau_\phi} \tag{3.8}$$

where ψ_n are the harmonic oscillator wave function.

The function F can be reduced to the form

$$F(\beta,x)=\frac{1}{2\beta^2}\sum_{n,m=0}^{\infty}\frac{t_{nm}(\beta)}{D_n(x)D_m(x)}[(n+m-\beta^2)t_{nm}(\beta) -2\sqrt{nm}\,t_{n-1,m-1}(\beta)], \tag{3.9}$$

where $D_n(x)=n+\frac{1}{2}+1/x$ and the harmonic oscillator matrix elements

$$t_{nm}(\beta)=\int_{-\infty}^{\infty} dx\,\psi_n(x)\psi_m\left[x-\frac{1}{\sqrt{\pi}}\beta l_B\right]$$

satisfy the recurrence relations

$$\sqrt{m}\,t_{nm}=\beta t_{n,m-1}+\sqrt{n}\,t_{n-1,m-1}\ , \tag{3.10}$$

$$\sqrt{n}\,t_{nm}=-\beta t_{n-1,m}+\sqrt{m}\,t_{n-1,m-1}\ , \tag{3.11}$$

and

$$t_{0n}(\beta)=(-1)^n(n!)^{-1/2}\beta^n e^{-\beta^2/2},t_{mn}=(-1)^{m+n}t_{nm}\ .$$

For the Fourier component $\sigma_1(\mathbf{q})$ of the conductivity $\delta\sigma(\mathbf{r})=\sum_{\mathbf{q}}\sigma_1(\mathbf{q})e^{i\mathbf{q}\cdot\mathbf{r}}$ we then have

$$\sigma_1(\mathbf{q})=\frac{e^2}{2\pi^2\hbar}F\left[\beta,\frac{B}{B_\phi}\right]\frac{b_{\mathbf{q}}(z_0)}{B}\ . \tag{3.12}$$

In the limit where β goes to zero

$$F(\beta,x)\simeq 1-\frac{1}{x}\sum_{n=0}^{\infty}\frac{1}{(n+\frac{1}{2}+1/x)^2},\quad \beta\simeq 0 \tag{3.13}$$

so that $F(\beta\simeq 0,x)=xf_2'(x)$. As expected, Eq. (3.12) thus reduces to Eq. (3.1) for the case of slow and small modulation. In the limit of large B we have $F\simeq 1$. We note that the recurrence relations Eqs. (3.10) and (3.11) enable one to obtain the function F by a straightforward numerical calculation.

IV. THE MACROSCOPIC CONDUCTIVITY

In the previous sections we have derived the expressions for the spatially dependent part of the conductivity in a nonuniform magnetic field. In principle, an experimental test of this theory could be a direct measurement of the local conductivity. This, however, would be a rather sophisticated experiment since the theory has nontrivial predictions only for sufficiently rapid variations of the magnetic field and as a consequence the probes for measuring the local conductivity would have to have a separation smaller than the distance between vortices or the period of modulation of $b(\mathbf{r})$. We note that mesoscopic effects could influence the results in this case.[19]

In a typical experiment a voltage or a macroscopic current is measured and we are therefore interested in the relation between the local conductivity and macroscopic quantities measured by probes far apart. In the following we shall show that nontrivial effects of the inhomogeneous magnetic field survives the spatial averaging over the inhomogeneity.

A. The macroscopic magnetoconductivity

We shall now consider how the nonuniformity of the magnetic field influences the macroscopic magnetoconductivity σ_M, i.e., the conductivity measured by probes distanced much further apart than the period of the structure in the magnetic field. Neglecting terms of relative order $(\Delta\sigma/\sigma_0)^2$ we have

$$\Delta\sigma_M=\langle\Delta\sigma(\mathbf{r})\rangle\ , \tag{4.1}$$

where the angular brackets $\langle\ \rangle$ denote spatial averaging.

We first consider the simplest case, where the magnetic field varies smoothly on the scale L_ϕ and the local magnetoconductivity $\Delta\sigma(\mathbf{r})$ is just $(e^2/2\pi^2\hbar)f_2(b_z(\mathbf{r})/B_\phi)$, then

$$\Delta\sigma_M=\frac{e^2}{2\pi^2\hbar}\left\langle f_2\left[\frac{b_z(\mathbf{r})}{B_\phi}\right]\right\rangle\ . \tag{4.2}$$

In the case where the magnetic field pattern has the form of an array of wide magnetic tubes with flux $N\Phi_0$ and radii r_0 satisfying $r_0\gg L_\phi$, Eq. (4.2) transforms into

$$\Delta\sigma_M=\frac{e^2}{2\pi^2\hbar}\frac{B}{b_0}f_2\left[\frac{b_0}{B_\phi}\right]\ , \tag{4.3}$$

where b_0 is the field inside the tube and B is the macroscopic field $B=\langle b\rangle$. If the magnetic flux in the tubes is not too high then the condition $r_0\gg L_\phi$ ensures that $b_0\ll B_\phi$. We can then use that $f_2(x)\simeq\frac{1}{24}x^2$, $x\ll 1$ to obtain from Eq. (4.3)

$$\Delta\sigma_M=\frac{e^2}{2\pi^2\hbar}\frac{1}{24}\frac{B}{B_\phi}\frac{b_0}{B_\phi},\quad L_\phi\ll\frac{r_0}{\sqrt{N}}\ . \tag{4.4}$$

We note that the macroscopic magnetoconductivity is linear in the average magnetic field B in contrast to the quadratic dependence for a weak uniform field and $b_0/B(>1)$ times stronger.

In the case of small tubes, $r_0\ll L_\phi$, each containing the flux $N\Phi_0$ and having a mutual separation larger than L_ϕ, we obtain from Eq. (2.17) in the region $B\ll NB_\phi$

$$\Delta\sigma_M=\frac{e^2}{2\pi^2\hbar}\frac{1}{N\ln(b_0/NB_\phi)}\frac{B}{B_\phi}\ . \tag{4.5}$$

In the region $B\gg 4\pi NB_\phi$ a typical path encircles many tubes. The nonuniformity of the magnetic field is then unimportant and for this case $\Delta\sigma_M$ depends on B in the usual manner.[6]

B. The current induced by a moving magnetic field

As mentioned in the Introduction, even for the case where the modulation of the magnetic field is small it is possible to measure the effect of the spatially dependent magnetoconductivity. This can be achieved by inducing a dc current by moving a nonuniform magnetic field structure.

We first consider the case of a weak magnetic field so that the magnetoconductivity is purely of quantum origin and the Hall effect is negligible. Starting from Ohm's law for the local current density

$$\mathbf{j}(\mathbf{r},t)=\sigma(\mathbf{r},t)\mathbf{e}(\mathbf{r},t)\ , \tag{4.6}$$

where σ and $\mathbf{e}$ are the local conductivity and electric field, respectively, we may represent σ as

$$\sigma(\mathbf{r},t)=\langle\sigma\rangle+\sigma_1(\mathbf{r},t)\ , \tag{4.7}$$

where $\langle\sigma_1\rangle=0$. Analogously, we have

$$\mathbf{e}(\mathbf{r},t)=\mathbf{E}+\mathbf{e}_1(\mathbf{r},t)\ , \tag{4.8}$$

where $\langle\mathbf{e}_1\rangle=\mathbf{0}$ and $\mathbf{E}$ is the macroscopic electric field. We then obtain for the macroscopic electric current density $\mathbf{J}$

$$\mathbf{J}=\langle\mathbf{j}\rangle=\langle\sigma\rangle\mathbf{E}+\langle\sigma_1\mathbf{e}_1\rangle\ . \tag{4.9}$$

The field $\mathbf{e}_1$ consists of two components. The first one is proportional to $\mathbf{E}$, since an appropriate field is necessary to maintain charge conservation in an inhomogeneous system. This field is of no interest to us since it only gives a renormalization of $\langle\sigma\rangle$ of the order $\langle\sigma_1^2\rangle/\langle\sigma\rangle$ (Ref. 20) which in our case is negligibly small. The other component $\mathbf{e}_{\rm ind}$ is induced by the flux movement and the induced current is given by

$$\mathbf{J}_{\rm ind}=\langle\sigma_1(\mathbf{r},t)\mathbf{e}_{\rm ind}(\mathbf{r},t)\rangle\ . \tag{4.10}$$

The two factors on the rhs of Eq. (4.10) are correlated in space and time since they originate in the same source, the nonuniform magnetic field. This correlation leads to a nonvanishing induced dc current.

According to Appendix B we have

$$\mathbf{e}_{\rm ind}(\mathbf{r},t)=\sum_{\mathbf{q}}\mathbf{e}_{\mathbf{q}}e^{i\mathbf{q}\cdot(\mathbf{r}-\mathbf{u}t)}\ , \tag{4.11}$$

$$\mathbf{e}_{\mathbf{q}}=\frac{1}{q^2}[q\times(\mathbf{v}\times\mathbf{q})]b_z(\mathbf{q})\ , \tag{4.12}$$

where $\mathbf{v}=\hat{\mathbf{z}}\times\mathbf{u}$ and $\mathbf{u}$ is the velocity of the moving magnetic structure. Inserting in Eq. (4.10) we then have for the induced current

$$\mathbf{J}_{\rm ind}=\sum_{\mathbf{q}}\sigma_1(\mathbf{q})b_z(-\mathbf{q})\frac{\mathbf{q}\times(\mathbf{v}\times\mathbf{q})}{q^2}\ . \tag{4.13}$$

In the case where the magnetic structure is a system of lamina (intermediate state of a type-I superconductor) the vectors $\mathbf{q}$ have only a component along the direction perpendicular to the lamina, then $\mathbf{v}$ is perpendicular to $\mathbf{q}$ and Eq. (4.13) becomes

$$\mathbf{J}_{\rm ind}=\mathbf{v}\sum_{q}\sigma_1(q)b_z(-q)\ . \tag{4.14}$$

In the case where the magnetic structure is a two-dimensional lattice with C_4 or C_6 symmetry or fully disordered, averaging over the direction of $\mathbf{q}$ in (4.13) yields

$$\mathbf{J}_{\rm ind}=\tfrac{1}{2}\mathbf{v}\sum_{\mathbf{q}}\sigma_1(\mathbf{q})b_z(-\mathbf{q}) \tag{4.15}$$

or alternatively

$$\mathbf{J}_{\rm ind}=\tfrac{1}{2}\mathbf{v}\langle(\sigma-\langle\sigma\rangle)(b_z-B)\rangle\ . \tag{4.16}$$

In the case where the motion of the magnetic structure is due to flux flow in a superconductor as described in the Introduction, the macroscopic electric field in the superconductor and the flux-flow velocity $\mathbf{u}$ are related by[10]

$$\mathbf{E}_s=\mathbf{B}\times\mathbf{u}=B\mathbf{v}\ . \tag{4.17}$$

We can then rewrite Eq. (4.16) as

$$\mathbf{J}_{\rm ind}=\sigma'\mathbf{E}_s\ , \tag{4.18}$$

where

$$\sigma'=\tfrac{1}{2}\langle(\sigma-\langle\sigma\rangle)(b_z/B-1)\rangle\ . \tag{4.19}$$

For the model with small magnetic tubes we get

$$\sigma'\simeq\frac{\gamma}{2}\frac{e^2}{2\pi^2\hbar}f_2\left[\frac{b_0}{B_\phi}\right]\ , \tag{4.20}$$

where $\gamma=B/b_0$ is the relative area of the plane with nonzero magnetic field. In the course of the derivation of Eq. (4.20) we have taken advantage of Eq. (2.19), which is valid inside the tube.

In the case of small modulation of the magnetic field we get from Eqs. (3.12), (4.13), (4.17), and (4.18) that

$$\sigma'=\frac{e^2}{4\pi^2\hbar}\sum_{\mathbf{q}}F\left[\beta,\frac{B}{B_\phi}\right]\left|\frac{b_{\mathbf{q}}(z_0)}{B}\right|^2\ . \tag{4.21}$$

Here we have assumed that the modulation possesses C_4 or C_6 symmetry or is fully disordered, otherwise σ' is a tensor.

The magnitude of the modulation falls off exponentially away from the surface of the superconductor according to Eq. (1.7).[21] For practical purposes we can therefore restrict ourselves to consider only the smallest $\mathbf{q}$ vectors.

A convenient way to measure the effect of the induced current would be to measure the voltage built up across a disconnected sample. In this case the induced current is compensated and the total macroscopic current is zero in accordance with $\mathbf{J}=\langle\sigma\rangle\mathbf{E}+\mathbf{J}_{\rm ind}=\mathbf{0}$, and thus the electric field

$$\mathbf{E}=-\frac{\sigma'}{\langle\sigma\rangle}\mathbf{E}_s \tag{4.22}$$

arises. Assuming that the films have the same length a voltage $V=-(\sigma'/\langle\sigma\rangle)V_s$ appears, where V_s is the voltage across the superconductor.

C. Classical magnetoconductance

For sufficiently weak magnetic fields the main source for the magnetoconductivity is the quantum correction due to weak localization. In the limit where $B \gg B_\phi$, the quantum correction saturates and the classical magnetoconductivity and the Hall effect may be of more importance.

We therefore turn to a discussion of the classical magnetoconductance in a nonuniform field. The origin of classical magnetoconductance is the bending of particle trajectories due to the Lorentz force. Since the Hall effect has the same origin the two effects should be treated on an equal footing.

Then the local current density has the form

$$\mathbf{j}(\mathbf{r})=\sigma(\mathbf{r})\mathbf{e}+\sigma_H(\mathbf{r})\hat{\mathbf{z}}\times\mathbf{e} \ . \tag{4.23}$$

For definiteness we consider the Boltzmann results in the relaxation-time approximation although the following considerations are generally valid as we shall comment upon at the end of the section. The dissipative conductivity is then given by

$$\sigma=\frac{\sigma_0}{1+(\omega_c\tau)^2} \tag{4.24}$$

and the Hall conductivity equals

$$\sigma_H=\omega_c\tau\sigma \ . \tag{4.25}$$

Here $\omega_c=|e|b_z/m$ denotes the local cyclotron frequency corresponding to the local field $b_z=B+\delta b$ where $\langle\delta b\rangle=0$. We let $\omega_c^0=|e|B/m$ denote the cyclotron frequency corresponding to the uniform field B and assuming $\omega_c^0\tau \ll 1$ we shall in the following take into account only terms of lowest order in $\omega_c^0\tau$. We note that Eqs. (4.24) and (4.25) lead to a diagonal resistivity independent of B.

The macroscopic current density $\mathbf{J}$ is given by

$$\mathbf{J}=\langle\sigma\rangle\mathbf{E}+\langle\sigma_H\rangle(\hat{\mathbf{z}}\times\mathbf{E})+\langle\sigma_1\mathbf{e}_1\rangle+\hat{\mathbf{z}}\times\langle\sigma_H^1\mathbf{e}_1\rangle \ , \tag{4.26}$$

where we in accordance with our previous notation have introduced $\sigma_H=\langle\sigma_H\rangle+\sigma_H^1$.

For the same reason as above we are interested only in the electric field $\mathbf{e}_{\text{ind}}$ induced by the moving magnetic field giving rise to an induced current $\mathbf{J}_{\text{ind}}$ given by

$$\mathbf{J}_{\text{ind}}=\langle\sigma_1\mathbf{e}_{\text{ind}}\rangle+\hat{\mathbf{z}}\times\langle\sigma_H^1\mathbf{e}_{\text{ind}}\rangle \ . \tag{4.27}$$

Since $\sigma_1\sim(\omega_c^0\tau)^2(\delta b/B)$ while $\sigma_H^1\sim(\omega_c^0\tau)(\delta b/B)$ we need to calculate the electric field including the term to leading order in $\omega_c^0\tau$. According to Appendix B we have

$$\mathbf{e}_{\text{ind}}(\mathbf{q})=q^{-2}\{\mathbf{q}\times(\mathbf{v}\times\mathbf{q})-\omega_c^0\tau[\mathbf{q}\cdot(\hat{\mathbf{z}}\times\mathbf{v})]\mathbf{q}\}b_z(\mathbf{q}) \ . \tag{4.28}$$

To leading order in $\omega_c^0\tau$ we then obtain

$$\mathbf{J}_{\text{ind}}=-\tfrac{1}{2}\sigma_0(\omega_c^0\tau)^2\left\langle\left[\frac{\delta b}{B}\right]^2\right\rangle\mathbf{E}_s+\tfrac{1}{2}\sigma_H^0\left\langle\left[\frac{\delta b}{B}\right]^2\right\rangle\hat{\mathbf{z}}\times\mathbf{E}_s \ , \tag{4.29}$$

where σ_H^0 is the Hall conductivity corresponding to the uniform field B, $\sigma_H^0=\langle\sigma_H\rangle$. Equation (4.29) is valid provided the moving magnetic structure has C_4 or C_6 symmetry or is fully disordered. In the case of a lamina structure with $\mathbf{E}_s$ perpendicular to the lamina direction, Eq. (4.29) is valid provided the rhs of the equation is multiplied by 2. One may note that the main part of the induced current is in the Hall direction.

In order to measure the effect of the induced current $\mathbf{J}_{\text{ind}}$ it is sufficient to measure the voltage built up across a disconnected sample. In this case the total macroscopic current $\mathbf{J}$ is zero and a macroscopic field $\mathbf{E}$ arises in accordance with

$$\mathbf{J}=\langle\sigma\rangle\mathbf{E}+\langle\sigma_H\rangle\hat{\mathbf{z}}\times\mathbf{E}+\mathbf{J}_{\text{ind}}=\mathbf{0} \ . \tag{4.30}$$

It follows from Eqs. (4.29) and (4.30) that the field $\mathbf{E}$ has only a Hall component

$$\mathbf{E}=-\tfrac{1}{2}\omega_c^0\tau\left\langle\left[\frac{\delta b}{B}\right]^2\right\rangle\hat{\mathbf{z}}\times\mathbf{E}_s \ . \tag{4.31}$$

The absence of the longitudinal component along $\mathbf{E}_s$ is obviously due to the absence of magnetoresistance resulting from the simplicity of the chosen model. Very often real samples show magnetoresistance due to inhomogeneities. We shall not consider this question in detail here since such a discussion necessitates use of a particular model. However, the above analysis could equally well have been carried through quite generally in which case a longitudinal component of the electric field would appear. We note, therefore, that the effect under consideration can be used as a method for experimental study of the inhomogeneities since the longitudinal voltage should depend on the relation between the period of the magnetic structure and the scale of sample inhomogeneities.

V. SUMMARY AND CONCLUSIONS

We have studied the weak-localization effects in inhomogeneous magnetic fields. As shown in the Introduction the theory of weak-localization magnetoresistance in a uniform magnetic field ceases to be applicable when the field is no longer uniform on the scale of the phase-coherence length. In Sec. II we considered the case of a highly inhomogeneous magnetic field and derived the Aharonov-Bohm–type quantum correction to the conductivity for the magnetic-string problem. The result was conveniently interpreted in terms of a topological quantity, the number of windings around the flux line. The average winding number was shown to be a divergent quantity due to the nature of diffusion in two dimensions resulting in an Aharonov-Bohm–type conductivity expression being a nonanalytic function of the flux. For the same reason we obtained nonanalytic results for the finite tube size effects. In Sec. III we studied the case of a small but arbitrary rapid modulation of the magnetic field.

The effects of the inhomogeneity of the magnetic field on macroscopic quantities were worked out in Sec. IV. In the case where the magnetic field pattern forms an array of magnetic tubes we showed that, when the average magnetic field is small compared to the characteristic field B_ϕ, the quadratic dependence of the weak-localization magnetoconductivity characteristic of the uniform case was changed to a linear dependence on the average magnetic field.

One possible way of imposing a nonuniform magnetic field was suggested, namely to superimpose the two-dimensional normal film on a superconductor. In this system it is possible, by passing a current through the superconductor, to create flux flow and thereby induce an electric field in the normal film. In Sec. IV we calculated the macroscopic induced current resulting from the weak-localization magnetoconductivity. The proposed experiment allows one to extract the spatially dependent part of the conductivity even for the case where the modulation of the magnetic field is small. In conclusion, we have presented the theory of weak localization in magnetic fields inhomogeneous on the scale of the phase-coherence length and demonstrated its observable consequences for macroscopic currents and voltages.

ACKNOWLEDGMENTS

One of us (A.L.S.) would like to thank Nordisk Institut for Teoretisk Atomfysik, København, for the hospitality extended to him during a visit. One of us (J.R.) acknowledges travel support by North Atlantic Treaty Organization (NATO) and this research was supported in part by the John D. and Catherine T. MacArthur Foundation at the University of Illinois under Grant No. 0-6-40129 and the National Science Foundation under Grant No. DMR-86-12860.

APPENDIX A: CALCULATION OF THE PARTICLE-PARTICLE DIFFUSION PROPAGATOR FOR THE CASE OF A FLUX TUBE OF FINITE SIZE

In this appendix we seek the Green's function C of Eq. (2.3) for the case of a magnetic flux tube of radius r_0 containing the flux $\Phi=N\Phi_0$ in the gauge where the vector potential $\mathbf{A}=\hat{\boldsymbol{\phi}}A_\phi$ is specified by

$$A_\phi(r)=\begin{cases}\dfrac{N\phi_0}{2\pi r_0^2}r, & r<r_0\\[2mm] \dfrac{N\phi_0}{2\pi r}, & r>r_0.\end{cases} \tag{A1}$$

The Green's function (particle-particle diffusion propagator) can be expressed as

$$C(\mathbf{r},\mathbf{r}')=\sum_\lambda \frac{\psi_\lambda(\mathbf{r})\psi_\lambda^*(\mathbf{r}')}{\omega_\lambda+\dfrac{1}{\tau_\phi}} = \sum_\lambda \psi_\lambda(\mathbf{r})\psi_\lambda^*(\mathbf{r}')\int_0^\infty dt\, e^{-t(\omega_\lambda+1/\tau_\phi)} , \tag{A2}$$

where ψ_λ are the eigenfunctions of the "Schrödinger equation"

$$H\psi_\lambda=\omega_\lambda\psi_\lambda \tag{A3}$$

with

$$H=-D\left[\frac{1}{r}\frac{\partial}{\partial r}\left[r\frac{\partial}{\partial r}\right]+\frac{1}{r^2}\left[\frac{\partial}{\partial\phi}-\frac{2ie}{\hbar}rA_\phi\right]^2\right] . \tag{A4}$$

The solution of Eq. (A3) has the form $\psi_\lambda(r)=\psi_{n,k}(r)e^{in\phi}$; $n=0,\pm1,\pm2,\ldots$ where $Dk^2=\omega_\lambda$ and for $r>r_0$, $\psi_{n,k}$ satisfies the equation

$$\left[\frac{1}{r}\frac{\partial}{\partial r}\left[r\frac{\partial}{\partial r}\right]-\frac{1}{r^2}(n+N)^2+k^2\right]\psi_{n,k}(r)=0 . \tag{A5}$$

Equation (A5) is Bessel's equation with the general solution

$$\psi_{n,k}(r)=a_n(k)[J_{|n+N|}(kr)+b_n(k)N_{|n+N|}(kr)] \quad (n=0,\pm1,\pm2,\ldots) , \tag{A6}$$

which is regular for large r. The coefficients a_n and b_n are to be found from continuity of the logarithmic derivative

$$(\ln\psi_{n,k})'\equiv r_0\frac{d}{dr}(\ln\psi_{n,k})\bigg|_{r=r_0}$$

in terms of which we have

$$b_n=\frac{\rho_0 J'_{|n+N|}(\rho_0)-(\ln\psi_{nk})'J_{|n+N|}(\rho_0)}{(\ln\psi_{n,k})'N_{|n+N|}(\rho_0)-\rho_0 N'_{|n+N|}(\rho_0)} , \tag{A7}$$

where $\rho_0=kr_0$ and the quantity $(\ln\psi_{n,k})'$ should be found from the solution of Eq. (A3) for $r\le r_0$. The essential values of k are of order r^{-1} or L_ϕ^{-1} so that $\rho_0\sim\max(r_0/r,\,r_0/L_\phi)$. In the following we shall assume that $\max(r_0/r,\,r_0/L_\phi)\ll 1$ so that $\rho_0\ll1$. We then get the following expressions

$$b_{n\neq -N}=\pi\frac{\left[\dfrac{\rho_0}{2}\right]^{2|n+N|}}{|n+N|!(|n+N|-1)!}\,\frac{(\ln\psi_{n,k})'-|n+N|}{(\ln\psi_{n,k})'+|n+N|} + O(\rho_0^{2|n+N|+2}) , \tag{A8}$$

$$b_{n=-N}=-\frac{\pi}{2}\,\frac{(\ln\psi_{-N,k})'}{(\ln\psi_{-N,k})'[\ln(\rho_0/2)+C]-1}+O(\rho_0^2) , \tag{A9}$$

where C denotes the Euler number. It follows from Eqs. (A8) and (A9) that we can neglect all the b_n's except the one where $n=-N$. The term with $n=-N$ gives the main correction to $C-C_0$, which is nonanalytic, in the limit where r_0 goes to zero. One point, however, needs attention namely that $(\ln\psi_{n,k})'$ is different from zero in the limit where k goes to zero since we would otherwise have that $b_{-N}\sim\rho_0^2$. We have checked this point in the for us most interesting case where $N=1$. In this case, the function

$$\psi(x)=x[I_0(\tfrac{1}{2}x^2)-I_1(\tfrac{1}{2}x^2)],\quad x=\frac{r}{r_0} \tag{A10}$$

satisfies the Schrödinger equation (A3) for $r<r_0$ and is

regular at $r=0$ since I_0 and I_1 are modified Bessel functions. Furthermore, the function in (A10) has a positive logarithmic derivative at $x=1$. When N approaches infinity so do $(\ln\psi_{-N,0})'$ since the problem then is equivalent to that of an impenetrable cylinder. We expect that there is no physical reason for $(\ln\psi_{-N,0})'$ to be zero for intermediate N.

We note that all information about the flux $N\Phi_0$, $N\geq 1$, in the tube enters only through $(\ln\psi_{-N,0})'$ in (A9) and this information disappears in the limit where ρ_0 goes to zero.

In the main approximation where $|\ln\rho_0|\gg 1$ we have

$$b_{-N}=\frac{\pi}{\ln\rho_0^{-2}}+O((\ln\rho_0)^{-2}) \ . \tag{A11}$$

Neglecting all the b_n's except b_{-N} we obtain from Eqs. (A2) and (A6) for all $N\geq 1$

$$C(r,r)-C_0=\int_0^\infty dt\, e^{-t/\tau_\phi}\int_0^{k_{\max}} dk\, k\frac{e^{-Dk^2t}}{\ln[(kr_0)^{-2}]}J_0(kr)N_0(kr) \ , \tag{A12}$$

where $k_{\max}\sim r_0^{-1}$ in accordance with the range of validity of Eq. (A11). In the slowly varying logarithm we can substitute for k the value $k=(Dt)^{-1/2}$ and then perform the integration over k to obtain

$$C(r,r)-C_0=-\frac{1}{2\pi}\int_{t_0}^\infty \frac{dt}{Dt}e^{-(r^2/2Dt)}K_0\left[\frac{r^2}{2Dt}\right] \times\frac{e^{-t/\tau_\phi}}{\ln(Dt/r_0^2)} \ , \tag{A13}$$

where $t_0\gtrsim r_0^2/D$.

The expression in Eq. (A13) has been used in Eq. (2.16) and is valid at distances $r\gg r_0$ provided $L_\phi\gg r_0$, with an accuracy of relative order $\{\ln[(r,L_\phi)/r_0]\}^{-1}\ll 1$.

APPENDIX B: CALCULATION OF THE INDUCED ELECTRIC FIELD BY A MOVING MAGNETIC FIELD STRUCTURE

In this appendix we consider the electrodynamics for the case of a moving magnetic field structure. The electric field induced in the film is due to the moving magnetic structure $\mathbf{b}(\mathbf{r},t)=\mathbf{B}+\delta\mathbf{b}(\mathbf{r}-\mathbf{u}t)$, where $\mathbf{u}$ is the velocity of the moving structure. We assume that the magnetic field generated by the induced current can be neglected compared to $\mathbf{b}$ (i.e., the film thickness is much smaller than the skin depth, which is always the case for small velocity $\mathbf{u}$). We can then assume $\mathbf{b}(\mathbf{r},t)=[B+\delta b(\mathbf{r}-\mathbf{u}t)]\hat{\mathbf{z}}$ to be a given quantity.

We are interested only in the component of the electric field $\mathbf{e}$ in the (x,y) plane since e_z has no influence and in fact is essentially screened. In the following we therefore consider $\mathbf{e}$ to be a two-dimensional vector in the (x,y) plane determined by the equations

$$(\nabla\times\mathbf{e})_z=-\frac{\partial b_z}{\partial t} \ , \tag{B1}$$

$$\langle\mathbf{e}\rangle=\mathbf{0} \ , \tag{B2}$$

$$\nabla\cdot\mathbf{j}=0 \ . \tag{B3}$$

In order to solve these equations it is convenient to expand the electric field $\mathbf{e}$ in its Fourier series

$$\mathbf{e}(\mathbf{r},t)=\sum_{\mathbf{q}}\mathbf{e}(\mathbf{q})e^{i\mathbf{q}\cdot(\mathbf{r}-\mathbf{u}t)} \ . \tag{B4}$$

The transverse component $\mathbf{e}_t(\text{div}\mathbf{e}_t=0)$ of the field $\mathbf{e}$ is given by

$$\mathbf{e}_t(\mathbf{q})=\frac{\mathbf{q}\times(\mathbf{v}\times\mathbf{q})}{q^2}b_z(\mathbf{q}) \ , \tag{B5}$$

where the abbreviation $\mathbf{v}=\hat{\mathbf{z}}\times\mathbf{u}$ has been introduced.

In the weak-localization regime to zeroth order in the nonuniformity of σ_1 we have from $\mathbf{j}=\sigma\mathbf{e}$ that $\text{div}\mathbf{j}=0$ leads to $\text{div}\mathbf{e}=0$. The longitudinal part of $\mathbf{e}$ has then no source and is consequently equal to zero. In the weak localization case, therefore, the electric field $\mathbf{e}$ can be taken as purely transverse and described by Eq. (B5).

We then turn to consider the case of classical magnetoconductance. In the zeroth-order approximation with respect to the nonuniformity of the magnetic field we have for the current

$$\mathbf{j}=\langle\sigma\rangle\mathbf{e}+\langle\sigma_H\rangle\hat{\mathbf{z}}\times\mathbf{e} \ , \tag{B6}$$

where $\langle\sigma\rangle$ and $\langle\sigma_H\rangle$ denote the dissipative and Hall conductivity, respectively, in the uniform field. Then, from the continuity equation $\text{div}\mathbf{j}=0$ we get

$$\langle\sigma\rangle\nabla\cdot\mathbf{e}-\langle\sigma_H\rangle\text{rot}_z\mathbf{e}=0 \ . \tag{B7}$$

This means that the field $\mathbf{e}$ now has a longitudinal component $\mathbf{e}_l$ determined by

$$\nabla\cdot\mathbf{e}_l=\frac{\langle\sigma_H\rangle}{\langle\sigma\rangle}(\nabla\times\mathbf{e}_t)_z \ , \tag{B8}$$

$$\nabla\times\mathbf{e}_l=0 \ . \tag{B9}$$

With $\mathbf{e}_t$ given by Eq. (B5) we then have

$$\mathbf{e}_l=\frac{\mathbf{q}(\mathbf{q}\cdot\mathbf{u})}{q^2}\frac{\langle\sigma_H\rangle}{\langle\sigma\rangle}b_z(\mathbf{q}) \ . \tag{B10}$$

Using $\langle\sigma_H\rangle/\langle\sigma\rangle=\omega_c^0\tau$, we finally obtain

$$\mathbf{e}=\mathbf{e}_t+\mathbf{e}_l=\left[\frac{\mathbf{q}\times(\mathbf{v}\times\mathbf{q})}{q^2}-\omega_c^0\tau\frac{[\mathbf{q}\cdot(\hat{\mathbf{z}}\times\mathbf{v})]\mathbf{q}}{q^2}\right]b_z(\mathbf{q}) \ . \tag{B11}$$

*Present address.
†Permanent address.
[1]For reviews on this development, we refer to B. L. Altshuler, A. G. Aronov, D. E. Khmel'nitskii, and A. I. Larkin, in *Quantum Theory of Solids,* edited by I. M. Lifshitz (MIR Publishers, Moscow, 1982). P. A. Lee and T. V. Ramakrishnan, Rev. Mod. Phys. **57**, 287 (1985).
[2]P. W. Anderson, Phys. Rev. **109**, 1492 (1958).
[3]D. E. Khmel'nitskii, Physica (Amsterdam) **126B**, 288 (1984).
[4]D. E. Khmel'nitskii and A. I. Larkin, Usp. Fiz. Nauk **136**, 533 (1982) [Sov. Phys.—Usp. **25**, 185 (1982)].
[5]For a description of the quasiclassical theory of weak localization we refer to the excellent review article by S. Chakravarty and A. Schmid, Phys. Rep. **140**, 193 (1986).
[6]B. L. Altshuler, A. G. Aronov, A. I. Larkin, and D. E. Khmel'nitskii, Zh. Eksp. Teor. Fiz. **81**, 768 (1981) [Sov. Phys.—JETP **54**, 411 (1981)]. For simplicity we neglect the effects of spin-orbit and spin-flip scattering.
[7]For a review on the experimental investigations, we refer to G. Bergmann, Phys. Rep. **107**, 1 (1984).
[8]It is worth emphasizing that the relation between the current and the electric field is local, at least insofar as the electric field varies smoothly on the scale of the mean free path.
[9]We note, that a nonuniform magnetic field may also be created by the domain structure in a ferromagnetic film.
[10]M. Tinkham, *Introduction to Superconductivity* (McGraw-Hill, New York, 1975).
[11]P. P. M. Meincke, Phys. Rev. Lett. **17**, 390 (1966).
[12]I. Giaever, Phys. Rev. Lett. **15**, 825 (1966).
[13]We note, that our definition of the particle-particle diffusion propagator C differs from the definitions used in Refs. 1 and 6 by the factor τ.
[14]M. Kretzschmar, Z. Phys. **185**, 84 (1965).
[15]S. Olariu and I. Iovitzu Popescu, Rev. Mod. Phys. **57**, 339 (1985).
[16]For example, I. S. Gradshteyn and I. M. Ryzhik, *Tables of Integrals, Series and Products* (Academic, New York, 1980).
[17]B. L. Altshuler, A. G. Aronov, and V. B. Spivak, Pisma Zh. Eksp. Teor. Fiz. **33**, 101 (1981) [JETP Lett. **33**, 94 (1981)].
[18]This estimate is in accordance with the standard arguments as used for the derivation of Eq. (1.2).
[19]B. L. Altshuler, Pis'ma Zh. Eksp. Teor. Fiz. **41**, 530 (1985) [Sov. Phys.—JETP Lett. **41**, 648 (1985)]; P. A. Lee and A. D. Stone, Phys. Rev. Lett. **55**, 1622 (1985); B. L. Altshuler and D. E. Khmel'nitskii, Pis'ma Zh. Eksp. Teor. Fiz. **42**, 291 (1985) [Sov. Phys.—JETP Lett. **42**, 359 (1985)].
[20]L. D. Landau and E. M. Lifshitz, *Electrodynamics of Continuous Media* (Pergamon, New York, 1960), p. 45.
[21]Neglecting the spreading of the magnetic field inside the superconductor and taking advantage of Eq. (1.7) and the expression for the Fourier component of the field inside the superconductor [P. G. de Gennes, *Superconductivity of Metals and Alloys* (Benjamin, New York, 1966)] we have $b(\mathbf{q},z)=B(1+\lambda_L^2 q^2)^{-1}e^{-|q|z}$, where λ_L is the London penetration depth.