Pulse-duration memory effect and deformable charge-density waves

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When repeated identical voltage pulses are applied to a material with a sliding charge-density wave, the current response synchronizes with the *end* of the applied pulse. We discuss how this observation can be understood using the model of classical deformable charge-density waves. The effect provides insight into unusual behavior that may be typical of many multivariate dynamical systems.

I. INTRODUCTION

Fleming and Schneemeyer[1] have recently reported that the charge-density wave (CDW) in blue bronze ($K_{0.3}MoO_3$) at 77 K exhibits a "pulse-duration memory effect." This nomenclature is used to describe a transient oscillation in response to repeated square-wave driving pulses which has a phase that is synchronized with the *end* of the pulse. For instance, in the voltage-driven configuration, after an initial transient, the current is always increasing as the pulse ends, even at the cost of distinctly aperiodic oscillations in the middle of the pulse. Thus, the CDW appears to "remember" the duration of the preceding pulse and adjust its response to be in synchrony.

In this paper we interpret this curious phenomenon in terms of the Fukuyama-Lee-Rice classical model of deformable CDW's.[2] We first discuss a numerical simulation of a one-dimensional version of the model that shows features very similar to those seen experimentally. Then we isolate the features of the model that cause the effect to occur. We propose that the "memory" is a signature of a process that will occur in many multivariate dynamical systems. The crucial features needed are a large number of metastable states and a negative feedback mechanism, features that often occur in nature.

When the system is subjected to identical repeated pulses, it attempts to reach a state for which further pulses induce no further changes (a "fixed point"). For an extended system, an infinite number of such states can exist. The system samples its available configuration space until it finds a fixed point, which may be atypical of the vast majority of configurations. In fact, it turns out that the fixed points selected out by letting the system evolve under the influence of repeated identical pulses are in some sense the *least stable*. This is because the system stops as soon as it reaches any fixed point, and the fixed point reached first tends to be only barely stable. The pulse-duration memory effect is the system's signature of being on the edge of its region of stability.[3,4]

In order to investigate this hypothesis further, we study multivariate discrete time mappings, some of which correspond to a particular limit of the differential equations used in our simulation. These mappings yield behavior that is consistent with our explanation of the pulse-duration memory effect.

The paper is organized as follows. In Sec. II we describe the pulse-duration memory effect in more detail, stressing the aspects that imply that simple models involving uncoupled degrees of freedom cannot explain the observations. In this section we also discuss our numerical simulations of about 50 coupled differential equations and compare the results to experiment. In Sec. III we discuss the results using the discrete time mappings with thousands of degrees of freedom, and in Sec. IV we summarize and make suggestions for future work.

II. BACKGROUND AND THEORETICAL CONSIDERATIONS

In order to understand the mechanisms underlying the pulse-duration memory effect, it is useful to recall the "mode-locking" and "interference" features in sliding CDW's pinned by disorder and subject to a voltage drive periodic in time.[5–7] Because the local pinning potential is invariant under displacement of the CDW by an integral number of wavelengths, the *local* motion of the CDW is periodic with the washboard frequency $\omega_0 = Q_z v$ (here, $\lambda = 2\pi/Q_z$ is the CDW wavelength in the sliding direction and v is the average CDW velocity). In the presence of a biased sinusoidal drive $V(t) = V_0 + V_1 \sin(\omega t)$, strong interference features (peaks in dV_0/dI_0, the derivative dc resistance) occur when the CDW current (proportional to v) is such that the washboard frequency is a harmonic or subharmonic of the external drive frequency: $\omega_0 = (p/q)\omega$. If the experimental conditions are arranged so that the CDW can undergo significant relaxation while the total applied voltage is below threshold, then true locking of the CDW velocity to the external drive occurs (experimentally, this appears as a plateau in dV_0/dI_0 at a value equal to the low-field resistance of the sample), both for harmonics and for low-order subharmonics.

A successful model of CDW treats it as an elastic object pinned by a random distribution of impurities.[2,8] A simplified one-dimensional model[9,10] has been shown to capture many of the qualitative features of the full model, and has the equation of motion

$$\dot{\phi}_i = -\frac{\delta H}{\delta \phi_i} , \tag{1a}$$

with

$$H = \sum_i \left[\frac{1}{2}\frac{(\phi_{i+1}-\phi_i)^2}{(r_{i+1}-r_i)^2} - V_i \cos(\phi_i - \beta_i) + F(t)\phi_i \right] , \tag{1b}$$

where the β_i are random variables uniformly distributed in $(0, 2\pi)$, and $F(t)$ is the driving field. The impurity positions r_i are random, and we shall renormalize length scales so that their average separation is unity. The pinning strengths V_i can also be chosen to be randomly distributed, although for our purposes this is not necessary. A crucial feature of this model is the existence of a very large number of metastable pinned states (growing exponentially with the size of the system). In contrast, a model with a single degree of freedom has a *unique* ground state up to translations by one wavelength (assuming a single-valued potential).

We have considered the behavior of the model when a sequence of repetitive square-wave pulses of duration $t_{\rm on}$ is applied (as in Fig. 1) with the time $t_{\rm off}$ between the pulses long enough that the system relaxes to a stationary (metastable) state in between each pulse. In an earlier paper,[11] we showed that under these conditions the plethora of metastable states leads to both harmonic *and* subharmonic mode locking.[12] Under the same conditions a single degree of freedom model gives rise only to trivial harmonic locking.

Each pinned state α can be represented by a set of variables $\{\phi_i^\alpha\}$, $i=1,\ldots,N$, where ϕ_i^α is the phase of the CDW oscillation at site i in the αth configuration and N is the number of degrees of freedom in the system. The effect of a single voltage pulse is just a mapping $\{\phi_i^\alpha\}\to\{\phi_i^\beta\}$ from state α to state β; in general, these states will be different. The transformation T such that $T\{\phi_i^{n-1}\}=\{\phi_i^n\}$ describes the change in configuration caused by application of the nth pulse. If the system evolves to a fixed point, such that $T^q\{\phi_i^\alpha\}=\{\phi_i^\alpha+2\pi p\}$ for integral p and q, the whole system moves by p wavelengths but is in the same state α, and the total charge transported per unit volume in q pulses is thus $p\rho_c$, where ρ_c is the collective charge density of the condensate.

Because the total number of states is approximately e^N, the naive expectation is that exponentially many pulses must be applied before a configuration repeats. Remarkably, it was found that after a short transient the system settles down into typically quite short cycles (of length $q \lesssim N$), with the same states (displaced by p wavelengths) appearing after every qth pulse. This situation corresponds to a p/q mode-locked step, since the current density (averaged over a cycle) is $(\rho_c/Q_z)(p/q)(2\pi/T)$, where $T=t_{\rm on}+t_{\rm off}$ is the repeat period of the pulse sequence.

One can attempt to describe the behavior in terms of a mapping involving a very few "effective" degrees of freedom.[13] A more complicated, but essentially trivial, possibility is that the system behaves as a set of *independent* nonlinear elements, each of which can be described by a simple map; the complexity of the behavior is just the result of averaging over the independent elements.

Neither of these hypotheses is consistent with the observation of the pulse-duration memory effect. Rather, the model is a representative of a general class of dynamical systems in which the coupled nonlinearity and disorder does not permit a reduction in the number of effective degrees of freedom, but whose properties are macroscopically regular and which exhibit features such as mode locking similar to those of low-dimensional maps. These systems have large numbers of fixed points, and one must look at the properties of the particular fixed point that is reached.

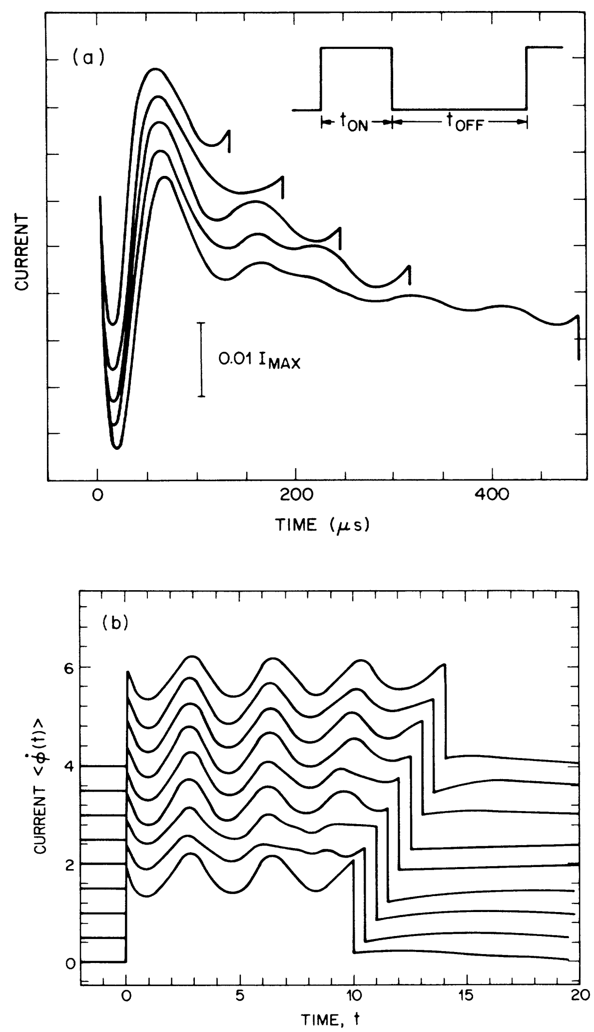

FIG. 1. (a) Current oscillations in response to a square-wave driving field of about $10E_T$ (inset) in $K_{0.3}MoO_3$ at 45 K (Ref. 1). The data were obtained in a current-driven configuration and have been inverted; however, the current oscillations are also clearly observed in a voltage-driven configuration (Ref. 18). (b) Current oscillations observed in numerical simulations of Eqs. (1) for a one-dimensional system with 50 degrees of freedom. (For details, see text.) In both experiment and model the pulse length is varied for fixed pulse height; different curves have their vertical axis offset for clarity.

We now discuss the experimental observations of the pulse-duration memory effect in more detail. When a square-wave voltage pulse from zero to above threshold is applied to a CDW, one typically observes a transient oscillation[14] at a frequency approximately equal to the washboard frequency ω_0, which is proportional to the (time- and space-averaged) CDW velocity. These oscillations could be interpreted as the sum of contributions of many independent degrees of freedom, each in a periodic potential with a random amplitude. Because each oscillator is at equilibrium when the pulse is applied (point A in Fig. 2), the current oscillations in $\dot{\phi}_i$ will all begin with the same phase. The oscillations dephase as time progresses, so that the oscillations in the *average* current $\langle \dot{\phi}(t) \rangle = N^{-1} \sum_i \dot{\phi}_i(t)$ gradually decay. However, notice in Fig. 1 that at the very end of the pulse there is a sharp upward cusp in the current. Experimentally, this cusp occurs over a broad range of pulse durations $t_{\rm on}$, so that the CDW appears to be adjusting itself to the pulse duration so that its velocity is always rising as the pulse ends.[1] The system has "remembered" the length of the preceding pulses, so that for subsequent pulses it "knows" precisely when to expect the pulse to end; this phenomenon has been termed the "pulse-duration memory effect."[1]

If a single, or independent, oscillator description were invoked, this rise in $\dot{\phi}$ just before $t_{\rm on}$ would correspond to $\phi(t_{\rm on})$ being at point B' in Fig. 2 rather than at a general point B determined by the length and height of the pulse and the strength of the potential. Just as the field is turned off, the particle lies at a maximum of the pinning potential, so the particle can roll down to either of the neighboring wells (i.e., C or C'). For a single oscillator, such an endpoint B' is only reached for special values of the pulse height or length. [This fact is easily seen because the motion is described by a deterministic differential equation and initial conditions $\dot{\phi}(t=0)$ and $\ddot{\phi}(t=0)$, which are both zero under these conditions.] However, both in CDW experiments, and in numerical simulations of Eqs. (1) [shown in Fig. 1(b), and to be discussed in more detail later], the upwards cusp always appears at the end of the pulse, for large ranges of $t_{\rm on}$ and pulse height F_0. This synchronization is a cooperative effect that cannot result from an assembly of independent oscillators.

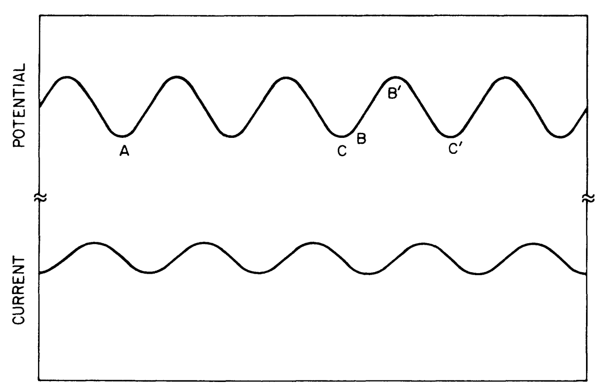

FIG. 2. Current oscillations I (lower trace) for overdamped motion of a single particle moving in a sinusoidal potential V (upper trace).

Although the independent oscillator description is clearly oversimplified, Fig. 2 already illustrates two key points. First, the existence of a cusp in the current at the end of the pulse *requires* a disproportionate number of the degrees of freedom ϕ_i to be at a maximum B' of their local pinning potential, precisely at the time the field is turned off.[15] Second, the potential maxima divide the basins of attraction of the local minima.

The equations of motion (1) are that of oscillators coupled by an elastic term that provides negative feedback. We now imagine starting from an arbitrary pinned configuration $\{\phi_i^0\}$ and applying a sequence of widely spaced, identical field pulses. In the first few pulses, the ϕ_i for which the pinning V_i is small will tend to move faster than the average, and to move ahead of the more strongly pinned sites, until their relative displacement from the strongly pinned regions is balanced by the elastic coupling. Thus the local velocities will become more homogeneous, and the initial transient-current oscillations will become well defined.

If the pulses were infinitely long, the CDW would eventually reach a *unique* sliding equilibrium configuration;[10] the finite pulse length prevents this evolution from being completely realized because the CDW relaxes to a static metastable state at the end of each pulse. Thus the local velocities will not become completely homogenized (as they do in dynamic equilibrium); even in a mode-locked configuration there is a distribution of local velocities, though it is much narrower than that obtained during the first few pulses.

Once the width of the velocity distribution Δv becomes $\lesssim 2\pi/t_{\rm on}$, memory of the overall shape of the configuration preceding the pulse will be retained. If, in addition, the average velocity v is such that $vt_{\rm on}$ (mod2π) is close to zero, it is likely that the system will relax to an identical configuration to that preceding the pulse. Because the equation of motion is deterministic, once a repeating state occurs, it will repeat indefinitely, and the system is (in this case harmonically) mode locked.

Note that we can imagine carefully preparing a state which has a very narrow distribution of velocities for a given value of the applied field; thus there are many *different* states which all mode-lock in *identical* driving fields.

However, starting from an arbitrary initial condition, the system will mode-lock first in the state when a large number of degrees of freedom are found (at time $t_{\rm on}$) at the boundary of the basin of attraction of the local minima. This is precisely the feature required to explain the rising cusp in the average current (Fig. 1) at the end of the pulse. Most of the locked states will not have this property, but we argue that it will be a *generic* feature observed for those locked states reached from an *arbitrary* initial configuration. This can be pictured easily in a general configuration space, where the locked states (for a particular value of the pulse height F and duration $t_{\rm on}$) are attracting points. Because their global configurations are quite similar, they will cluster together, and provide a wide basin of attraction. As the configuration evolves toward the cluster, it will be captured by states on the cluster *boundary*. These states are likely to be only marginally stable to variations in F and t_{on}; the more typical (and more stable) locked states in the interior of the cluster will rarely be reached.

An important point is that the tendency to end up in a marginally stable state is not restricted only to systems with many degrees of freedom; it is a property of having many inequivalent metastable states and a feedback process that causes evolution to a set of fixed points. Admittedly, having many metastable states is a property usually associated with many-particle systems. However, a system consisting of one particle in a sinusoidal potential connected by a weak spring to the point $x=0$ can also display a pulse-duration memory effect.[16] To see this, consider the dissipative equation of motion

$$\dot{x} = -V\sin x - kx + F(t) \ , \tag{2}$$

where $F(t)$ describes force pulses, as before. Consider the limit $k \ll 1$. When $F=0$, $\dot{x}=0$ for a discrete set of x's spaced by approximately 2π in the range $-V/k$ to V/k. When a pulse of height $F_0 > V$ and duration $t_0 > \pi/(F_0^2 - V^2)^{1/2} + O(k)$ is applied, the particle moves to another well. For the second pulse, x has increased, and the ball will not move as far, since kx is larger. As $k \to 0$, the spring force may be approximated as kx_n, where x_n is the position at the start of the nth pulse, and the equation of motion can be integrated analytically. (The approximation holds because the particle moves a distance of order unity during each pulse, though after many pulses x_n can be large.) Even without doing the integration, one can compare the motion in the mth pulse and $(m-1)$st pulse, where m is the first pulse where the configuration repeats (i.e., the particle remains in the same well). The $(m-1)$st pulse causes the particle to move by 2π, and the mth pulse causes no net motion (after allowing for relaxation after the field is turned off).

We consider the quantities x'_{m-1} and x'_m, the values of x modulo 2π just as the $(m-1)$st and mth pulses are turned off. To compare x'_{m-1} and x'_m, we first note that the position at the start of the mth pulse $x_m = x_{m-1} + 2\pi + O(k/V)$. So suppose we know $x_{m-1}(t)$, the position as a function of time during the $(m-1)$st pulse. Let $u = x_m(t) - x_{m-1}(t) - 2\pi$. Since $|\sin x| \le |x|$,

$$\left|\frac{du}{dt}\right| \le 2V|u| + k|u| + 2\pi k \ ,$$

and

$$|u(t_{on})| \le A\frac{k}{V}e^{(2V+k)t_{on}} \ ,$$

where A is a constant. Since V and t_{on} are fixed, as $k \to 0$, $u_m(t_{on}) \to 0$. (This bound can be improved considerably by accounting for the shape of the sine function.) Hence, $x'_{m-1} - x'_m = Bk$, where B depends on V and t_{on}, which are both fixed and of order unity. We will assume V is large and that F and t_{on} are chosen so that the particle originally moves only one well to the right per pulse. Thus, the particle will move over one period of the potential until the spring is stretched enough. After the pulse is turned off, the particle moves to the right if $x(t_{on}) > x_0$ and to the left if $x(t_{on}) < x_0$, where it is easily shown that $\pi < x_0 < 3\pi/2$. Now we know that $x'_{m-1} \ge x_0$ and $x'_m \le x_0$, so both x'_{m-1} and x'_m must be in the range $(x_0 - Bk, x_0 + Bk)$. Hence $x'_m \to x_0$ as $k \to 0$. The velocity is thus always increasing at the end of the pulse, and if the spring distortion is not too large, the particle ends very near a well top.

It is nontrivial to show that many nonlinearly interacting degrees of freedom also display the same behavior. We have investigated this question numerically, both by examining Eqs. (1) and by looking at discretized mappings, which are discussed in the next section.

The results of numerical simulations of Eqs. (1) are shown in Fig. 1, for the case of pulsed forcing in a system of size $N=50$. In order to reduce the size of the computation, we have worked in the intermediate-pinning limit $(V=1)$; although most experimental systems are believed to be described by the weak-pinning limit $(V \ll 1)$, we do not believe that this will qualitatively affect our conclusions.[17] Starting from an initial configuration (chosen to be $\{\phi\}=0$, which is far from the final locked stationary states), the equations of motion (1) were solved for a sequence of identical pulses of height $F=2$ and length t_{on}. The time t_{off} in between the pulses was long enough that the system relaxed to an equilibrium metastable state. In Fig. 1(b) we plot the average current $\langle\dot{\phi}(t)\rangle$ obtained when the system reaches its repeating, locked configuration (with a value of p/q shown in the figure). For the case when the locking is at a subharmonic $(q>1)$, we have averaged the current over the q pulses in the cycle; in such a case, the current response of different pulses in the cycle is not identical, which is evidence for finite-size effects. As the pulse length changes, the number of well-defined oscillations increases with t_{on}, but the rising behavior at the end of the pulse is quite evident in almost all cases. The occasional failure to observe a clear cusp, as well as the difference in shapes of the different curves close to the pulse end, is, we believe, a result of finite-size limitations. The arguments given above should apply only to the case of a very large number of degrees of freedom, and when the locked configuration is approached infinitesimally slowly. Overall, the numerical results are very similar to the experimental data on $K_{0.3}MoO_3$, also shown in Fig. 1.[18]

The existence of the "pulse-duration memory" effect within the model provides strong evidence that the coupling between oscillators [provided by the elastic term in Eq. (1b)] qualitatively affects the behavior of the system. Even though the model is extremely simplified, direct solution of the equation of motion is tedious. We also expect that the picture we have described has considerably more generality than the application to CDW motion. We now show that the principal features can be captured by a simple discrete map for a multivariate system, which we describe in the next section.

III. DISCRETE TIME MAPPINGS

We now illustrate the principles described above by constructing a simple dynamical system, which is a set of variables $\{x_j\}$ and a mapping F: $\{x_j^n\} \to \{x_j^{n+1}\}$.

The main features required are, first, a system with many different states, including a large number which are invariant under application of the mapping, or fixed points. These fixed points lie in a fairly compact region in the configuration space. Second, there must be a source of fairly weak negative feedback so that repeated application of the mapping causes the system to evolve towards the region of fixed points without overshooting. Finally, the system should start out in a state that is *not* in the region of fixed points, so that properties that arise during the relaxation towards the fixed-point region can be studied.

The mapping we use is most easily visualized as consisting of two steps, one analogous to the change in configuration caused by application of a voltage, and the second analogous to the relaxation to a metastable state that occurs when the field is off. The defining equations are

$$y_j^n = t(k(x_{j+1}^n - 2x_j^n + x_{j-1}^n) + F - d_j) , \tag{3a}$$

$$x_j^{n+1} = \mathrm{int}(y_j^n + \tfrac{1}{2}) , \tag{3b}$$

with periodic boundary conditions $x_1 = x_{N+1}$. The variables d_j are reminiscent of the pinning potential in Eqs. (1), and they are taken to be random numbers uniformly distributed between 0 and 1. The mapping thus depends on the three parameters t, k, and F, with the notation chosen to suggest analogy with the coupled differential equations (1).

This mapping is extremely similar to those that can be derived directly from the differential equations (1) in the limit $F \gg V \gg k$. Heuristically, the map can be understood as follows. The first step mimics the behavior of the system during the pulses, when the applied force dominates the motion, since $F \gg V \gg k$. To a first approximation, the position just as the pulse ends y_j is just $x_j + Ft_{\mathrm{on}}$, with the springs and potential providing small perturbations. However, because the system is randomly pinned, there is a tendency for different parts of the chain to move at different velocities, an effect which is accounted for by the random distribution of d_j's. As the pulses are applied, the springs stretch in a manner that causes the velocity to become more uniform. Since each pulse is short, the change in the spring forces during a given pulse is small, even though large excursions are possible if many pulses are applied. Thus, it is adequate to take the spring force during the entire pulse to be its value at the moment when the force is turned on. The pinning potential is ignored during the pulse because it is not a source of feedback that causes the system to approach a mode-locked configuration, and we are not concerned with the velocity during the pulse.

The second step mimics the effects of turning off the field, when the balls tend to fall down into the closest potential minimum. Although large enough distortions are built up such that well after the pulse is turned off, the balls are not actually at potential minima, this step captures the qualitative feature that the system finds a nearby metastable state.

However, we would like to stress that our arguments do not depend on the details of the equations of motion (1), but rather on the presence of many metastable states and weak feedback. Therefore, it is perhaps more useful to view the mapping as an example of a dynamical system with these features rather than as a model of charge density waves.

The system has a countably infinite number of configurations $\{x_j\}$ ("metastable states"); each x_j can be any integer. The invariant configurations are those for which all the $y_j^n - x_j^n$ lie within the interval $(m - \frac{1}{2}, m + \frac{1}{2})$ for some integer m. For $k = 0$, the random d_j cause the different x_j to move different distances in each cycle, and the system does not evolve towards a fixed point (though it could start out in a configuration that is invariant under application of the mapping). For $k > 0$, as the map is iterated the "feedback" term acts to counteract the random d_j's and causes the system to evolve towards the region of fixed points.

We will always choose $F > 1$, so the external driving force is greater than 0 at all points. How fast the system approaches the region of fixed points depends on the strength of the feedback k. If k is large, then large changes in the configuration occur in each iteration. For large enough k, the system overshoots its optimum configuration and various instabilities occur, leading to oscillatory motion of some degrees of freedom (e.g., $x_j^{n+1} < x_j^n$ for some j). As k is decreased, all the x_j are nondecreasing, which is the situation observed for the differential equations (1). Thus, there are additional instabilities in the mapping that are not present in the coupled differential equations. For moderate k ($\gtrsim 0.1$), the feedback is strong enough so that fairly large steps are taken toward the region of fixed points. As k is reduced further, the steps become smaller, and one expects the system to end up closer and closer to the boundary of the region of fixed points.

Whether or not degrees of freedom are piling up at the boundaries of the basins of attraction can be studied by examining the y_j's; the boundaries lie where the fractional part of the y's are exactly $\frac{1}{2}$ [see Eq. (3b)]. Therefore, we expect that a plot of the number of y's with a fractional part in a given range should show a peak near $\frac{1}{2}$.

As expected, the tendency to pile up is greater for small k.[19] Figure 3 shows the results of a calculation performed

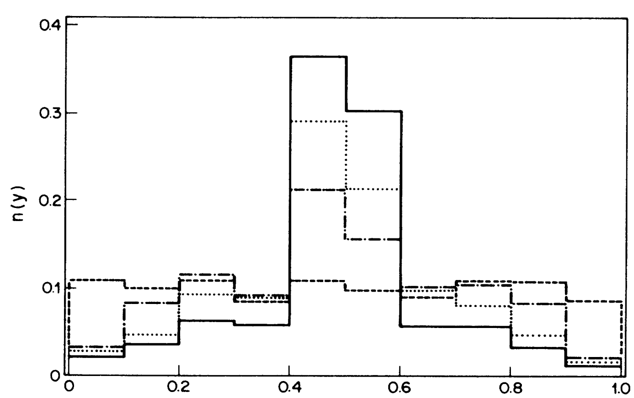

FIG. 3. Histogram of fractional part of y_j [Eqs. (3)] after 1 (dashed line), 1000 (dotted-dashed line), 2000 (dotted line), and 5000 (solid line) iterations of the map with 1000 degrees of freedom. The boundary of the basin of attraction is at $y = \frac{1}{2}$.

with the mapping defined by Eqs. (3) with the parameters $t=1$, $F=5$, and $k=0.0001$. Initially, the distribution of y_j is smooth, but as the system evolves a pronounced sharp peak grows at $y=\frac{1}{2}$. This mapping also yields both harmonic and subharmonic locking, although the subharmonic locking is quite weak with this particular map.[20] The piling up occurs well before locking, for even after 5000 iterations the system is still evolving for the conditions shown.

The piling-up process occurs locally, and one can again illustrate the basic mechanism using a mapping with a single degree of freedom and a feedback term, which can be obtained by keeping x_1 and x_3 fixed, $x_1=0$ and $x_3=0$, and allowing $x_2 \equiv x$ to evolve under the mapping

$$y(n)=[-kx(n)+F]+x(n) , \tag{4a}$$

$$x(n+1)=\text{int}[y(n)+\tfrac{1}{2}] . \tag{4b}$$

Assume that $x(0)\leq 0$, and $F>\frac{1}{2}$, so $x(1)>x(0)$, and that k is small. It will be true that $x(n+1)>x(n)$ so long as $F-kx(n)>\frac{1}{2}$. Thus, the system reaches its fixed point the first time that $y(n^*)-x(n^*)<\frac{1}{2}$. Now,

$$x(n^*)-x(n^*-1)=\text{int}[F-kx(n^*)+\tfrac{1}{2}]\leq F+1 .$$

Since, by hypothesis,

$$z(n^*-1)\equiv y(n^*-1)-x(n^*-1)>\tfrac{1}{2} ,$$

and

$$z(n^*)\equiv y(n^*)-x(n^*)<\tfrac{1}{2} ,$$

one must have

$$z(n^*)\geq z(n^*-1)-k(F+1) ,$$

so

$$z(n^*)\geq \tfrac{1}{2}-k(F+1) .$$

Since F is of order unity, as $k\to 0$, $z(n^*)\to\frac{1}{2}$, as claimed.

The nontrivial point is that increasing the number of degrees of freedom does not destroy the tendency to pile up. Thus, one can envision a process where longer and longer length scales have homogenized their velocities, with piling up at boundaries occurring at each stage. One might hope to describe this process with a mode-coupling theory.

IV. CONCLUSIONS

We have demonstrated that the phenomenon of "pulse-duration memory" can be reproduced by numerical simulations of a one-dimensional model of CDW motion, which has been found to be successful in explaining many other features of CDW dynamics. While we believe that this gives further evidence in favor of a many-degree-of-freedom interpretation of CDW dynamics, that evidence was already strong.[21] More important was to show that such behavior is likely to be characteristic of a variety of dynamical systems, where there are many possible attractors clustering in phase space, and the *generic* properties are determined by attractors on the boundary of the cluster. These states are the *least* stable, and one manifestation of this low stability is the pulse memory, shown in Fig. 1.[1]

These ideas were generalized by studying a two-stage multivariate map for many degrees of freedom. By studying the distribution of variables at the intermediate stage of the mapping, we were able to show that the generic behavior was for the system to evolve toward a fixed point characterized by the local degrees of freedom piling up at the least stable point in their local potential.

Because there are generic properties of the locked states, even in a random system (although randomness is not, in fact, necessary as an ingredient; similar behavior is found for the Frenkel-Kontorova model[3]) one may hope to provide a complete characterization of the "devil's staircase" observed[11] for the locking behavior of Eq. (1) and similar models.

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[1]R. M. Fleming and L. F. Schneemeyer, Phys. Rev. B **33**, 2930 (1986). See also S. E. Brown, G. Grüner, and L. Mihaly, in Ref. 7.

[2]H. Fukuyama and P. A. Lee, Phys. Rev. B **17**, 535 (1978); P. A. Lee and T. M. Rice, *ibid.* **19**, 3970 (1979).

[3]This generic behavior has been dubbed "phase organization;" C. Tang *et al.*, Phys. Rev. Lett. **58**, 1161 (1987).

[4]The idea that the dynamic behavior of a complex system can be governed by the minimally stable states first appeared in studies of dendritic pattern formation as the "marginal stability hypothesis;" see E. Ben-Jacob, N. Goldenfeld, B. G. Kotliar, and J. S. Langer, Phys. Rev. Lett. **53**, 2110 (1984).

[5]G. Grüner, A. Zawadowski, and P. M. Chaikin, Phys. Rev. Lett. **46**, 511 (1981).

[6]J. Richard, P. Monceau, and M. Renard, Phys. Rev. B **25**, 948 (1982).

[7]S. E. Brown, G. Mozurkewich, and G. Grüner, Phys. Rev. Lett. **52**, 2277 (1984); R. P. Hall and A. Zettl, Phys. Rev. B **30**, 2279 (1984); M. F. Hundley and A. Zettl, *ibid.* **33**, 2883 (1986); S. E. Brown, G. Grüner, and L. Mihaly, Solid State Commun. **57**, 165 (1986); S. Bhattacharya, J. P. Stokes, and M. J. Higgins (unpublished).

[8]For theoretical papers exploring the behavior of this model, see, e.g., L. Sneddon, M. C. Cross, and D. S. Fisher, Phys. Rev. Lett. **49**, 292 (1982); L. Sneddon, Phys. Rev. B **29**, 719 (1984); **29**, 725 (1984).

[9]N. Teranishi and R. Kubo, J. Phys. Soc. Jpn. **47**, 720 (1979); L. Pietronero and S. Strässler, Phys. Rev. B **28**, 5863 (1983).

[10]P. B. Littlewood, Phys. Rev. B **33**, 6694 (1986); P. B. Littlewood, Physica **23D**, 45 (1986), and references therein.

[11]S. N. Coppersmith and P. B. Littlewood, Phys. Rev. Lett. **57**, 1927 (1986).

[12]Subharmonic and harmonic mode locking has been seen in models similar to that of Eqs. (1), but with a sinusoidal driving field. P. F. Tua and J. Ruvalds, Solid State Commun. **54**, 471 (1985); M. Inui and S. Doniach (unpublished).

[13]The experiments of Brown, Mozurkewich, and Grüner, Ref. 7, have been interpreted in terms of a circle map for a single degree of freedom; see, e.g., P. Bak, in *Charge Density Waves in Solids,* Vol. 217 of *Lecture Notes in Physics,* edited by G. Hutiray and J. Sólyom (Springer, Berlin, 1985); M. H. Jensen, P. Bak, and T. Bohr, Phys. Rev. A **30**, 1960 (1984); **30**, 1970 (1984); see also R. E. Thorne, J. R. Tucker, John Bardeen, S. E. Brown, and G. Grüner, Phys. Rev. B **33**, 7342 (1986), who have claimed that a single-degree-of-freedom model with a nonsinusoidal pinning potential can quantitatively explain their experimental observations.

[14]R. M. Fleming, L. F. Schneemeyer, and R. J. Cava, Phys. Rev. B **31**, 1181 (1985).

[15]In the case of weak pinning ($V \ll 1$), this statement must be interpreted as referring to effective degrees of freedom in a renormalized potential.

[16]The spring breaks discrete translational symmetry, distinguishing this system from that described by a pendulum equation.

[17]For further discussion of this point, and for details of the numerical procedures, see Ref. 10.

[18]R. M. Fleming and L. F. Schneemeyer (unpublished). These experiments, and those of Ref. 1, do not show mode locking of the dc current, which may be due to current inhomogeneities in the large samples used. However, numerically it is seen that the "pulse-memory effect" occurs in the approach toward the fixed point, and before the development of the repeating locked configurations; S. N. Coppersmith (unpublished).

[19]For CDW systems the feedback is effectively small because the relevant distortions occur over very long length scales.

[20]The truncation performed in Eq. (3b) is similar to mappings derived from differential equations using a pinning potential with cusps at the minimum and the maximum. A slightly more complicated version of Eq. (2b) allows us to represent a potential with a smooth maximum, and restores the high-order subharmonic locking features. The piling up of the intermediate variables y_j at the boundary of the basin of attraction is preserved; see S. N. Coppersmith (unpublished).

[21]See discussion in Refs. 8–11, for example.