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Zener transitions in dissipative driven systems

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The effect of inelastic interactions on Zener transitions is studied in the context of externally driven systems (flux-driven normal metallic rings, current-biased tunnel junctions, etc.). We employ a time-dependent Hamiltonian and use a two-level approximation to study these systems. We consider a coupling (not necessarily weak) of the system to a phonon bath and allow for a large class of phonon spectra. The coupling tends to reduce the Zener tunneling. The Zener transitions are enhanced at finite temperatures. We also consider the effect of a phenomenological relaxation time τ on the behavior of the system, and express the (nonlinear) resistance R in terms of τ and the other important physical parameters. When the bias goes to zero, R vanishes.

I. INTRODUCTION

Recent progress in the fabrication of submicrometer systems has made it possible to study physical phenomena on mesoscopic length scales, i.e., the intermediate regime between macroscopic and microscopic length scales. From the theoretical viewpoint this is of interest because one is dealing with systems which, although consisting of many degrees of freedom, are not sufficiently large for the thermodynamic limit to be achieved: various types of fluctuations (e.g., deviations from ensemble averages) may be of importance. In many instances one has to deal with such systems coupled to an external driving source (e.g., a current source). This poses the question of how to describe an open system in a nonequilibrium situation. To make things even more complicated—but also more interesting—we note that a complete description of the underlying physics should account for inelastic, or dissipative, processes which are present due to the coupling of the system to certain external modes ("heat bath"). Such dissipative processes can be separated out (at least in principle) when dealing with reversible, microscopic systems, but should be taken into account when larger, mesoscopic or macroscopic, systems are considered. From the experimental and technological viewpoint the fabrication and study of submicrometer systems pose difficult challenges, but may be rewarding as well; one might hope that applications to logic and memory circuits as well as new sensitive measurement devices could emerge from studies in these directions.

The coupling to the external source induces transitions among the relevant energy levels. As was pointed out recently by Landauer,[1] the energy pumped into the system in this way should not be interpreted as dissipation, since it is associated with a reversible process and is, at least in principle, retrievable. The coupling of the system to an external bath affects the transition rates but, even more importantly, introduces the notion of irreversibility, which is required in order to account for dissipation. We shall study here the dependence of the dissipation on the parameters of the system, including the driving source, the strength of the coupling to the reservoir, and the temperature.

More specifically, in this paper we consider some general features of a dissipative Zener tunneling. This should apply to systems whose energy (as a function of some external parameter) resembles that of an electron in a periodic potential. The uncoupled system is treated here as a two-level system (similarly to what has been done by Landauer and Büttiker[2] in the context of thermal transitions). This evidently facilitates an analytic study of the problem. Our main purpose here is to elucidate some principal aspects of the physics involved, including how dissipation arises and its dependence on various important parameters of the problem. It should be emphasized that in order to make accurate quantitative predictions one should consider more detailed models, and in particular one should account for the system's many internal degrees of freedom.[3]

We find that the coupling of the driven system to a heat bath tends to suppress the transition rates. How strong this suppression is depends on some details of the excitation spectrum of the bath and on the system-reservoir coupling. Increasing the temperature enhances the transition rates. Our results suggest that Ohmic dissipation does not vanish in the zero-temperature limit. This dissipation, induced by Zener tunneling (as opposed to thermally induced dissipation), vanishes in the limit of zero external bias.

Some results of the present analysis can be found in previous works. Microscopic models for dissipation which employ the idea of coupling the system to many

harmonic degrees of freedom[4] have recently gained renewed popularity in the study of macroscopic quantum tunneling and macroscopic quantum coherence. In this context two-level systems, coupled to a heat reservoir, have also been studied.[5] In certain molecular and biological[5–8] systems one is interested in a situation in which the system is coupled to an additional, macroscopic coordinate. This coordinate, if treated classically, can be parametrized as function of time, yielding an effective, time-dependent Hamiltonian of an externally driven system. Although the works referred to here[9] bear some formal similarity to our problem, we remark that (a) the formulation of our problem contains important differences (e.g., in the way time dependence enters into the Hamiltonian); (b) in some respects our treatment is more general (we consider, for example, arbitrary spectral densities of the reservoir); and (c) the types of systems we specifically study and the questions we ask (e.g., concerning the onset of dissipation and its dependence on various parameters) have not yet been treated systematically.

At this point we note that our mechanism is different from that considered by Landauer and Büttiker,[2] who studied thermally activated transitions in the presence of an external source. Such a source induces continuous variation of a certain external parameter of the system (in their specific example, that was the magnetic flux confined within a metallic ring). The delay in the response of the system to the change in its external conditions leads eventually to dissipation of energy. (Another mechanism, suggested by Büttiker,[10] is the gradual destruction of the phase of the wave function; see also Gefen and Schön.[11]) In the above-mentioned analysis of Landauer and Büttiker, time was treated as an external parameter, and transition rates were calculated by employing the static Hamiltonian that corresponds to the instantaneous value of the external parameter. *In this respect* their mechanism describes an adiabatic limit, and is expected to prevail in the high-temperature, weak-external-bias limit. When the rate of change of the external parameter is large (i.e., when the system is coupled to a strong driving source), dissipative Zener-Landau transitions (i.e., Zener-Landau transitions affected by the coupling to the reservoir) may take over. These transitions will survive in the zero-temperature limit, where the Landauer-Büttiker mechanism gives no dissipation.

Perhaps the simplest example of the family of systems we are interested in is the the charged quantum rotator in a magnetic field, subject to an external potential[12] (e.g., gravitational field); see Fig. 1. The Hamiltonian of the system is

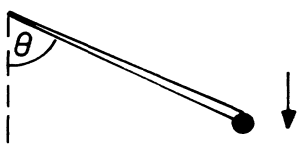

FIG. 1. Quantum rotator in an external field.

$$\mathcal{H}_{\rm rot}=\frac{1}{2I}\left[-i\hbar\frac{\partial}{\partial\theta}-\frac{e}{c}A\right]^2-V_0\cos\theta\ , \tag{1.1}$$

where I is the moment of inertia, e is the electron charge, c is the speed of light, and A is the magnitude of the vector potential, assumed to be aligned in the $\hat{\theta}$ direction. If A is independent of θ one can perform a gauge transformation on the Hamiltonian. The resulting Schrödinger equation is, in fact, the Mathieu equation with boundary conditions which depend on the value of A (see Appendix A).[13] The low-lying energy levels are plotted schematically in Fig. 2 as function of the parameter A. One can show (see Appendix A) that the energy spectrum is a periodic function of ϕ, with a period $\phi_0=hc/e$. Here ϕ is the magnetic flux confined within a circle of radius r (see Fig. 1), where r is the radius of the trajectory of the rotator. When ϕ varies adiabatically $\phi=\alpha t$, the energy is a periodic function of time.

Interestingly enough, there are several other systems which are physically quite different from the quantum rotator but whose Hamiltonian is formally similar to that of Eq. (1.1). Those systems have been discussed extensively in recent publications; we shall therefore mention them only briefly and refer the reader to other papers with more exhaustive lists of references.

An electron in a one-dimensional normal ring.[14,15] At sufficiently low temperatures the electronic eigenstates around the ring are well defined. Due to the Aharonov-Bohm effect, the quasimomentum of each eigenstate is shifted by the flux ϕ confined within the ring; the energy levels are periodic functions of ϕ, reminiscent of the flux quantization that occurs in superconducting rings.

An electron in a one-dimensional potential with an external field.[16] One may consider a random one-dimensional system connected to a voltage source. When

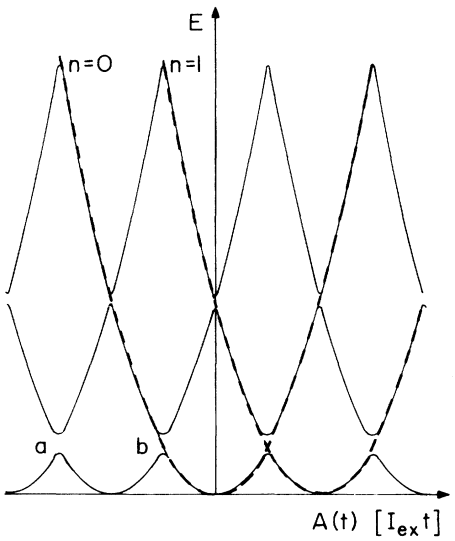

FIG. 2. Low-lying energy levels of the quantum rotator (schematic) as function of the vector potential. The same energy structure is obtained for the Hamiltonian of a normal tunnel junction (see text).

the system is sufficiently short, inelastic effects will take place mainly in the electrodes. This problem can be related to that of electrons in a one-dimensional ring, with the external bias being represented by a magnetic field which increases (decreases) linearly in time. The Hamiltonian of the system is similar to that of Eq. (1.1), with the appropriate boundary conditions. The cosinusoidal potential energy term is replaced by a random potential. Treating time as a parameter, one can show, along the lines presented in Appendix A, that the energy spectrum oscillates with t. The "energy bands" structure, however, will be different from that shown in Fig. 2. The band width will fluctuate about a certain typical value (the average level spacing) and will *not* increase systematically with E. The gap between two consecutive bands (for a fixed value of t) is also a random function and may not decrease with E [see Fig. 2 of Ref. 16(c)].

Superconducting quantum-interference device[17] *(a superconducting ring containing a single Josephson junction).* The minimum energy state is a periodic function of the externally applied magnetic field. One usually includes in the Hamiltonian of such a system a self-inductance term which makes the formalism of this problem somewhat different from the cases mentioned above.

Small current-driven Josephson junctions (JJ's). Neglecting coupling to a heat bath, this system is described by the Hamiltonian[18–22]

$$\mathcal{H}_{\rm JJ}=\frac{(q\hat{n}-I_{\rm ex}t)^2}{2C}+\mathcal{H}_{\rm tun}\ , \tag{1.2}$$

where the operator $\hat{n}$ measures the net number of elementary charges (Cooper pairs, $q=2e$) that tunneled from, say, right to left. $I_{\rm ex}t$ is the total charge that has been injected into the junction by the external current source $I_{\rm ex}$ continuously. C is the self-capacitance of the junction. The tunneling part of the Hamiltonian does not commute with the charging part. This can be seen easily from the fact that $\mathcal{H}_{\rm tun}$ may connect different $\hat{n}$ states. Previous works assumed $\mathcal{H}_{\rm tun}=E_J(1-\cos\hat{\theta})$ (E_J being the Josephson energy). This has been obtained from a degenerate perturbation theory treatment in the limit of large C and may be modified for small junctions.

It is also evident that the dependence of the spectrum of $\mathcal{H}_{\rm JJ}$ on the *parameter* t is periodic in t (with a period of $q/I_{\rm ex}$). One way to see this is to consider first the spectrum of the charging part of $\mathcal{H}_{\rm JJ}$, which, as function of t, defines a set of mutually intersecting parabolas. These parabolas are indexed by n. The main effect of the addition of small $\mathcal{H}_{\rm tun}$ in to remove the degeneracies at the intersection points. As an example, $\mathcal{H}_{\rm tun}$ removes the degeneracy at the intersection of the parabolas 0 and 1. We now obtain at this point a linear combination of the states corresponding to $n=0$ and $n=1$. One then obtains a structure similar to the one shown in Fig. 2. The time evolution of a system that follows adiabatically the lower band consists of transitions from parabola 0 to 1, then from 1 to 2, etc. These correspond to transferring pairs across the junction, a process which takes place in relatively short bursts.

Small current-driven normal tunnel junctions. In earlier work[21] we predicted a similar effect for small normal junctions with a characteristic frequency of $I_{\rm ex}/e$. In this case one also obtains time-dependent energy levels. A more subtle picture arises when one includes dissipative effects,[23] as well as the internal degrees of freedom of the junction.[3] Following our predictions, other works, using different approaches,[24,25] have derived similar effects.

In the following analysis we shall make reference mostly to small tunnel junctions. The reader should bear in mind, however, that our analysis, with some modifications, may be relevant to all the other systems listed above, as well as to charge-density-wave transport and Bloch oscillations in superlattices.

The outline of this paper is as follows. In Sec. II we introduce our model, which includes coupling to an external heat bath. General expressions for the transition rates at zero and finite temperatures will be derived, and the dependence on the bath's phonon spectrum will be indicated. In Sec. III we consider the behavior of our externally driven two-level system when a phenomenological relaxation time is introduced. In particular, we calculate the dependence of the dissipation of our model system on the current and temperature. The paper contains two appendices; they are included not because they contain new physics, but rather for the sake of completeness of the presentation. In Appendix A the transitions between the two levels are calculated (in the absence of a driving source). In Appendix B we sketch the derivation of the well-known Zener-Landau tunneling in a two-level system (without dissipation).

II. DEFINITION OF THE MODEL AND CALCULATIONS OF TRANSITION RATES

We are interested in interband transitions (Fig. 2) that occur due to the external driving source. We employ a two-band approximation; that is, for the purpose of calculating the rate of transitions through a gap, we consider only the energy bands just below and above this gap. Moreover, we note that the main contribution to the transitions comes from those values of the external parameter (A, $I_{\rm ex}t$, etc.) for which the interband gaps are narrowest; we shall therefore consider a single narrow gap (e.g., points a,b in Fig. 2). The dependence of the energy levels on the external parameter (t) in the vicinity of such a narrow gap is approximated by a simple analytic expression [see Eq. (2.2) below]. The effect of possible interference between consecutive transitions (e.g., transitions that occur at a and b) is ignored. This is justified when phase smearing of the electronic wave function due to inelastic events is strong enough to destroy possible correlations among different transitions. These interference effects will be treated in future work.

Following the above discussion we shall consider a system whose Hamiltonian is

$$\mathcal{H}=\mathcal{H}_S+\mathcal{H}_b+\mathcal{H}_c\ . \tag{2.1}$$

(Hereafter we put $\hbar=1$.) The Hamiltonian of the system when decoupled from the bath is (see Fig. 3)

$$\mathcal{H}_S=\alpha t S_z+\Delta S_x\ . \tag{2.2}$$

Here S_z and S_x are Pauli's spin-$\frac{1}{2}$ operators,

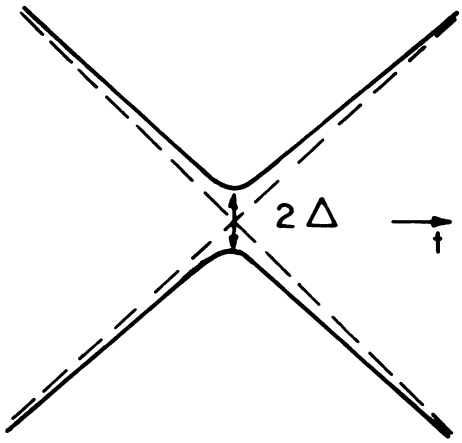

FIG. 3. Energy levels of the decoupled system as function of time (schematic).

$$S_x=\begin{bmatrix}0 & 1\\ 1 & 0\end{bmatrix},\quad S_y=\begin{bmatrix}0 & -i\\ i & 0\end{bmatrix},\quad S_z=\begin{bmatrix}1 & 0\\ 0 & -1\end{bmatrix}.$$

Note that this Hamiltonian is time dependent. Near the minimum gap the energy levels are parabolic. The detailed dependence of the spectrum at $|t|\to\infty$ (when the gap is large) has little effect on the transition rates; the specific form of Eq. (2.2) is chosen for reasons of analytic convenience. The Hamiltonian of the thermal bath, represented by harmonic degrees of freedom, is

$$\mathcal{H}_b=\sum_\alpha \omega_\alpha(a_\alpha^\dagger a_\alpha+\tfrac{1}{2}) , \tag{2.3}$$

whereas the system-bath coupling is given by

$$\mathcal{H}_c=\sum_\alpha A_\alpha S_z(a_\alpha+a_\alpha^\dagger) . \tag{2.4}$$

Throughout this paper, unless otherwise stated, we set $\hbar=1$. The calculation of the transition rates for $\mathcal{H}_c=0$ is sketched in Appendix B. The transition probability (from $t\to-\infty$ to $t\to+\infty$) in the adiabatic limit is

$$P=e^{-\pi\Delta^2/\alpha} . \tag{2.5}$$

Qualitatively, the source-driven transitions can be understood from the following argument. Let us assume that at time t the quantum state of the system, ψ_S, corresponds to the system being in the lower eigenstate of $\mathcal{H}_S$, i.e., $\psi_S=|-\rangle_t$ (see Appendix B). This state, of course, is orthogonal to $|+\rangle_t$ (($|+\rangle_t$ and $|-\rangle_t$ are the eigenstates of $\mathcal{H}_S$, where t is taken as a parameter). At a later time, $t+dt$, the eigenstates of $\mathcal{H}_S(t+dt)$ are $|+\rangle_{t+dt}$ and $|-\rangle_{t+dt}$. These states are mutually orthogonal, but in general ${}_t\langle -|+\rangle_{t+dt}\neq 0$, and therefore at $t+dt$ there is a nonvanishing probability of finding the system in the upper eigenstate. For $\Delta=0$, ${}_t\langle -|+\rangle_{t+dt}=0$, and no Zener transitions take place.

Let us consider now the two-level system coupled to the phonon bath. Within our treatment we wish to study an arbitrary coupling strength and to include the explicit time dependence of the Hamiltonian as well. To facilitate this we shall consider the term ΔS_x as a perturbation. For $\Delta=0$ the eigenstates of the Hamiltonian $\mathcal{H}(\Delta=0)$ are also eigenstates of the operator S_z, and no Zener transitions between the states $|\uparrow\rangle$ and $|\downarrow\rangle$ are possible (even in the presence of coupling to the bath). In order to account for such transitions we have to consider a nonvanishing ΔS_x term.

Employing the unitary transformation[26]

$$\bar{\mathcal{H}}(\Delta=0)=e^{\tilde{T}}\mathcal{H}(\Delta=0)e^{-\tilde{T}} \tag{2.6}$$

with

$$\tilde{T}\equiv S_x\sum_\alpha \frac{A_\alpha}{\omega_\alpha}(a_\alpha^\dagger-a_\alpha) ,$$

we obtain a diagonalized Hamiltonian

$$\bar{\mathcal{H}}(\Delta=0)=\alpha t S_z-S_z\sum_\alpha\frac{A_\alpha^2}{\omega_\alpha}+\sum_\alpha\omega_\alpha(a_\alpha^\dagger a_\alpha+\tfrac{1}{2}) \tag{2.7}$$

(t is still regarded as a parameter here). The eigenstates of $\bar{\mathcal{H}}(\Delta=0)$ are

$$\bar{\psi}_0(\{n_\alpha\},S_z)\equiv\bar{\psi}_0=\left[\prod_\alpha|n_\alpha\rangle\right]\otimes|S_z\rangle , \tag{2.8}$$

where $|S_x\rangle$ is either $|\uparrow\rangle$ or $|\downarrow\rangle$ (see Appendix B for notation) and $\prod_\alpha|n_\alpha\rangle$ is an external product over the harmonic modes of the bath; $|n_\alpha\rangle$ stands for n "phonons" in the α mode. The subscript of $\bar{\psi}$ (and subsequently ψ) is intended to emphasize that the eigenstates are calculated for $\Delta=0$. The corresponding eigenstate of $\mathcal{H}(\Delta=0)$ is obtained by applying the inverse transformation:

$$\psi_0(\{n_\alpha\},S_z)\equiv\psi_0=e^{-\tilde{T}}\bar{\psi}_0=e^{-\tilde{T}}\left[\prod_\alpha|n_\alpha\rangle\right]\otimes|S_z\rangle . \tag{2.9}$$

The time dependence of the amplitude of $\psi(\{n\beta\},S_z)$—an amplitude denoted by $b(\{n_\beta\},S_z)$—is given by [see Eq. (B9)]

$$\dot{b}(\{n_\alpha\},S_z)=\sum_{\{m_\alpha\},\tilde{S}_z}{}'\frac{b(\{m_\alpha\},\tilde{S}_z)}{E(\{n_\alpha\},S_z)-E(\{m_\alpha\},\tilde{S}_z)}\exp\left[i\int_{-\infty}^{t}dt'[E(\{n_\alpha\},S_z)-E(\{m_\alpha\},\tilde{S}_z)]\right]\left[\frac{\partial\mathcal{H}}{\partial t}\right]_{\psi(\{n_\alpha\},S_z),\psi(\{m_\alpha\},\tilde{S}_z)} \tag{2.10}$$

The summation is over all the initial states (denoted by the spin and phonon indices) that are different from the final state. The energies $E(\{n_\alpha\},S_z)$ and $E(\{m_\alpha\},\tilde{S}_z)$ are those of the final and initial state, respectively. We use the relation[27]

$$\left\langle \psi(\{n_\alpha\},S_z)\left|\frac{\partial}{\partial t}\right|\psi(\{m_\alpha\},\tilde{S}_z)\right\rangle = -\frac{\left[\frac{\partial \mathcal{H}}{\partial t}\right]_{\psi(\{n_\alpha\},S_z),\psi(\{m_\alpha\},\tilde{S}_z)}}{E(\{n_\alpha\},S_z)-E(\{m_\alpha\},\tilde{S}_z)} . \quad (2.11)$$

The states $\psi(\{n_\alpha\},\tilde{S}_z)$ are derived from the eigenstates of the unperturbed Hamiltonian, $\psi_0(\{n_\alpha\},\tilde{S}_z)$, using first-order perturbation theory. One may write to first order in Δ

$$\psi(\{n_\alpha\},\tilde{S}_z)=\psi_0(\{n_\alpha\},\tilde{S}_z) + \sum_{\{m_\alpha\},S_z'}{}' c(\{n_\alpha\},\tilde{S}_z;\{m_\alpha\},S_z') \times\psi_0(\{m_\alpha\},S_z') , \quad (2.12)$$

where the coefficients c are given by

$$c(\{n_\alpha\},\tilde{S}_z;\{m_\alpha\},S_z')=\frac{(\Delta S_x)_{\psi_0(\{m_\alpha\},S_z'),\psi_0(\{n_\alpha\},\tilde{S}_z)}}{E_0(\{n_\alpha\},\tilde{S}_t)-E_0(\{m_\alpha\},S_z')} . \quad (2.13)$$

[The prime over the summation sign in Eq. (2.12) is introduced here in order to emphasize that the set of indices $\{m_\alpha\},S_z'$ should not include $\{n_\alpha\},\tilde{S}_z$. The energies E_0 are those of the unperturbed ($\Delta=0$).]

We shall consider a situation in which the system is initially ($t\to-\infty$) in the lower "band" ($|\tilde{S}_t\rangle=|\uparrow\rangle$). We assume that the bath is initially in equilibrium; the probability $g(n_\alpha)$ of finding n_α "phonons" in the α mode is given by

$$g(n_\alpha)=(1-\Gamma_\alpha)\Gamma_\alpha^{n_\alpha} , \quad (2.14)$$

where

$$\Gamma_\alpha=e^{-\hbar\omega_\alpha/k_BT} ; \quad (2.15)$$

here T is the temperature.

In the limit of low temperatures the excitation energy of the bath is small; the energy differences in the denominators of Eqs. (2.11) and (2.13) may then be approximated by the difference between the energy of the "upper band" (i.e., the energy of the uncoupled system in the state $|S_x\rangle=|+\rangle$; see Appendix B for notation) and the energy of the "lower band" (i.e., the energy associated with the state $|\tilde{S}_z\rangle=|-\rangle$) at a given time. The "phonon" corrections to the energies are neglected at this stage. Thus, our treatment excludes resonant phonon absorption which may be relevant at higher temperatures. It should be noted that even within these approximations, which imply

$$E_0(\{n_\alpha\},\tilde{S}_z)\approx E_0(\tilde{S}_z) , \quad (2.16)$$

the energies are still functions of time.

If we apply the above approximation [Eq. (2.16)] to the energy difference that appears in the phase factor of Eq. (2.10), we immediately see that the expression for the transition probability decouples into a product of two terms. These terms are as follows.

(a) The standard Zener expression for a two-level system:

$$Z_0(S_z=\downarrow;\tilde{S}_z=\uparrow)\approx\int_{-\infty}^{+\infty}dt\frac{\left[\langle\downarrow|+\langle\uparrow|\frac{\langle\uparrow|\Delta S_x|\downarrow\rangle}{E_0(S_z=\downarrow)-E_0(\tilde{S}_z=\uparrow)}\right]\left[\frac{\partial\mathcal{H}_s}{\partial t}\right]\left[|\uparrow\rangle+\frac{\langle\downarrow|\Delta S_x|\uparrow\rangle}{E_0(\tilde{S}_z=\uparrow)-E_0(S_z=\downarrow)}|\downarrow\rangle\right]}{E_0(S_z=\downarrow)-E_0(\tilde{S}_z=\uparrow)} \times\exp\left[i\int_{-\infty}^{t}dt'[E_0(S_z=\uparrow)-E_0(\tilde{S}_z=\downarrow)]\right] . \quad (2.17)$$

The calculation of the bare Zener transition (in the absence of coupling to a phonon bath) is discussed in Appendix B. The corresponding contribution to the transition probability is $|Z_0|^2$.

(b) The second factor represents the overlap between the initial and the final phonon state. This term arises from the factor

$$\langle\psi_0\{n_\alpha\},\downarrow|S_x|\psi_0(\{m_\alpha\},\uparrow\rangle+\text{c.c.} , \quad (2.18)$$

which consists of first-order corrections to the wave function [see Eq. (2.13)].

Before calculating the term (2.18), let us evaluate the matrix element that connects an initial state with n "phonons" in the α mode to a final state with m phonons in that mode. Employing the relation (2.9) [with the unitary transformation $\bar{T}$ defined by Eq. (2.6)] this factor is

$$\langle m_\alpha,\downarrow|e^{\bar{T}}S_xe^{-\bar{T}}|n_\alpha,\uparrow\rangle = \left\langle m_\alpha,\downarrow\left|\exp\left[2\frac{A_a}{\omega_\alpha}(a_\alpha^\dagger-a_\alpha)\right]\right|n_\alpha,\uparrow\right\rangle . \quad (2.19)$$

To evaluate the contribution to the total transition *probability* we consider the initial and final states with the occupation numbers $\{n_\alpha\}$ and $\{m_\alpha\}$, respectively. Employing the relations (2.14) and (2.15) we can account for the probability of finding the system with initial occupation numbers $\{n_\alpha\}$. We then obtain a factor that multiplies $|Z_0|^2$:

$$Y=\prod_{\alpha}(1-\Gamma_{\alpha})\sum_{n_{\alpha}=0}^{\infty}\sum_{m_{\alpha}=0}^{\infty}\Gamma_{\alpha}^{n_{\alpha}}\left|\left\langle m_{\alpha}\left|\exp\left[2\frac{A_{\alpha}}{\omega_{\alpha}}(a_{\alpha}^{\dagger}-a_{\alpha})\right]\right|n_{\alpha}\right\rangle\right|^{2} . \tag{2.20}$$

The tacit assumption in deriving Eq. (2.20) is that the various transition channels (from $|\uparrow\rangle$ to $|\downarrow\rangle$) that involve changing the phonon occupation numbers add incoherently (i.e., one should add probabilities rather than amplitudes).

In order to make further progress we call the Weyl's identities:[28]

$$e^{u_{\alpha}v_{\alpha}/2}e^{va_{\alpha}^{\dagger}}e^{ua_{\alpha}}=e^{u_{\alpha}a_{\alpha}+v_{\alpha}a_{\alpha}^{\dagger}}=e^{-u_{\alpha}v_{\alpha}/2}e^{u_{\alpha}a_{\alpha}}e^{v_{\alpha}a_{\alpha}^{\dagger}} , \tag{2.21}$$

and the identity

$$\exp(u_{\alpha}a_{\alpha}+v_{\alpha}a_{\alpha}^{\dagger})\exp(N_{\alpha}\ln w_{\alpha})=\exp(N_{\alpha}\ln w_{\alpha})\exp(u_{\alpha}w_{\alpha}a_{\alpha}+v_{\alpha}a_{\alpha}^{\dagger}/w_{\alpha}) . \tag{2.22}$$

Here $u_{\alpha},v_{\alpha},w_{a}$ are c-numbers. The operators $a_{\alpha},a_{\alpha}^{\dagger}$ satisfy the commutation relation

$$[a_{\alpha},a_{\beta}^{\dagger}]=\delta_{\alpha,\beta} . \tag{2.23}$$

The number operator N_{α} is defined by

$$N_{\alpha}=a_{\alpha}^{\dagger}a_{\alpha} . \tag{2.24}$$

Employing Eqs. (2.21) and (2.22) with

$$u_{\alpha}=-\frac{2A_{\alpha}}{\omega_{\alpha}} \tag{2.25a}$$

and

$$v_{\alpha}=\frac{2A_{\alpha}}{\omega_{\alpha}} , \tag{2.25b}$$

we obtain from Eq. (2.20)

$$Y=\prod_{\alpha}(1-\Gamma_{\alpha})\sum_{n_{\alpha}=0}^{\infty}\prod_{m_{\alpha}=0}^{\infty}\langle m_{\alpha}|\exp(u_{\alpha}a_{\alpha}+v_{\alpha}a_{\alpha}^{\dagger})\exp(\tfrac{1}{2}N_{\alpha}\ln\Gamma_{\alpha})|n_{\alpha}\rangle\langle n_{\alpha}|\exp(\tfrac{1}{2}N_{\alpha}\ln\Gamma_{\alpha})\exp(u_{\alpha}a_{\alpha}^{\dagger}+va_{\alpha})|m_{\alpha}\rangle$$

$$=\prod_{\alpha}(1-\Gamma_{\alpha})\mathrm{Tr}[\exp(u_{\alpha}a_{\alpha}+v_{\alpha}a_{\alpha}^{\dagger})\exp(N_{\alpha}\ln\Gamma_{\alpha})\exp(u_{\alpha}a_{\alpha}^{\dagger}+va_{\alpha})] . \tag{2.26}$$

Using the relation[29]

$$(1-w_{\alpha})\mathrm{Tr}[\exp(N_{\alpha}\ln w_{\alpha})\exp(u_{\alpha}a_{\alpha})\exp(v_{\alpha}a_{\alpha}^{\dagger})]=\exp\left[\frac{u_{\alpha}v_{\alpha}}{1-w_{\alpha}}\right] , \tag{2.27}$$

we obtain after some algebra

$$Y=1 ;$$

i.e., within the above approximate scheme the coupling to the phonon reservoir does *not* modify the bare Zener tunneling. In fact, this result is not surprising. The factor Y [see Eq. (2.20)] that multiples the bare Zener transition probability accounts for all possible channels of phonon exchange, $\{n_{\alpha}\}\rightarrow\{m_{\alpha}\}$. Conservation of probability requires that the sum over all channels of the unitary operator that appears in the matrix element of Eq. (2.20) is equal to 1.

In order to evaluate the effect of the inelastic channels on the bare Zener transition we need to employ a more refined approximation. As before, we shall approximate the energy differences that appear in the denominators of Eqs. (2.10) and (2.13) by the energy difference between the $|S_z\rangle=|+\rangle$ and $|S_z\rangle=|-\rangle$ levels, neglecting phonon corrections. However, the phase factor in Eq. (2.10) will be treated more carefully, and the "phonon" contributions to the energy difference will be taken into account. These corrections to the phase factor affect a rapidly oscillating term and may therefore be quite important, as opposed to the energy denominators in Eq. (2.10). The latter vary relatively slowly in time and contain rather small "phonon" corrections.

The energy difference in the phase factor of Eq. (2.10) reads

$$E_0(S_z=\downarrow)-E_0(S_z=\uparrow)+\sum_{\alpha}\hbar\omega(n_{\alpha}-m_{\alpha}) . \tag{2.28}$$

The calculation of the bare Zener tunneling is not decoupled now from the term that contains the phonon contribution. We continue our calculation along the lines of the analysis of Appendix B. The result of the integration over time (in the phase factor), Eq. (B12) is now replaced by

$$\frac{i\Delta^{2}}{\alpha}\left[\frac{\alpha t}{\Delta}\left[\frac{\alpha^{2}t^{2}}{\Delta^{2}}+1\right]^{1/2}+\sinh^{-1}\left[\frac{\alpha t}{\Delta}\right]\right]+\sum_{\substack{\text{phonon}\\ \text{modes }\beta}}it\hbar\omega_{\beta}(m_{\beta}-n_{\beta}) . \tag{2.29}$$

The integration over time is performed similarly to the evaluation of (B9), using a contour in the complex plane. For a given channel of phonon exchange (i.e., for a certain choice of $\{m_\beta\},\{n_\beta\}$) we obtain for $\Delta \ll a$ the analogue of Eq. (B13):

$$\frac{\pi}{2}\exp\left[-\frac{\pi}{2}\frac{\Delta^2}{\alpha}\right]\exp\left[-\frac{\Delta}{\alpha}\sum_\beta \hbar\omega_\beta(m_\beta-n_\beta)\right] . \quad (2.30)$$

The above expression (2.30) is the contribution to the transition amplitude arising from a particular inelastic channel.

We next have to sum over all possible channels (i.e., all $\{m_\beta\}$ and $\{n_\beta\}$). Assuming that different transition channels are incoherent, we have to add the transition *probabilities* of all channels. Equations (2.20) and (2.26) are now replaced by

$$\tilde{Y}=\frac{\pi^2}{4}\exp\left[-\frac{\pi\Delta^2}{\alpha}\right]\left\{\prod_\alpha (1-\Gamma_\alpha)\mathrm{Tr}[\exp(-N_\alpha \ln\chi_\alpha)\exp(v_\alpha a_\alpha^\dagger + u_\alpha a_\alpha)\exp(N_\alpha \ln\chi_\alpha\Gamma_\alpha)\exp(N_\alpha\ln\chi_\alpha) \times \exp(u_\alpha a_\alpha^\dagger + v_\alpha a_\alpha)\exp(-N_\alpha\ln\chi_\alpha)]\right\} . \quad (2.31)$$

The parameters Γ_α, u_α, and v_α are defined by Eqs. (2.15) and (2.25). The operators a_α, $a_\alpha^\dagger$, and N_α are related by Eqs. (2.23) and (2.24). The quantity χ_α is given by

$$\chi_\alpha \equiv \exp\left[\frac{\Delta}{\alpha}\hbar\omega_\alpha\right] . \quad (2.32)$$

Employing the identities (2.21), (2.22), and (2.27), we calculate the four terms that appear on the right-hand side of Eq. (2.31). After some algebra we find

$$\tilde{Y}=\frac{\pi^2}{4}\exp\left[-\frac{\pi\Delta^2}{\alpha}\right] \times \exp\left[\sum_\alpha \frac{4A_\alpha^2}{\omega_\alpha^2}\frac{\exp\left[\frac{2\Delta}{\alpha}\omega_\alpha\right]\exp\left[-\frac{\hbar\omega_\alpha}{k_BT}\right]+\exp\left[\frac{-2\Delta}{\alpha}\omega_\alpha\right]-1-\exp\left[-\frac{\hbar\omega_\alpha}{k_BT}\right]}{1-\exp\left[-\frac{\hbar\omega_\alpha}{k_BT}\right]}\right] \quad (2.33)$$

The above general expression enables us to study how various factors such as temperature, coupling to the phonon bath, the excitation spectrum of the bath, the external bias, etc., affect the Zener tunneling.

We should recall that our calculation consists of corrections to the adiabatic approximation. It should be valid as long as the Zener tunneling is small, i.e., $\tilde{Y}$ [Eq. (2.33)] is small. The factor

$$B=\frac{\pi^2}{4}\exp\left[-\frac{\pi\Delta^2}{\alpha}\right] , \quad (2.34)$$

which is approximately equal to the bare Zener probability [cf. Eq. (B13)], is multiplied by inelastic corrections. We shall therefore focus on the effect of the inelastic corrections arising from this term.

At zero temperature ($T=0$) the coupling to the phonon reservoir attenuates the transition rate. This can be qualitatively understood along the lines of the discussion following Eq. (2.5). The Zener transition is associated with the overlap among states at time t and states at time $t+dt$. When the two-level system is coupled to the phonon bath we have to consider a larger, multidimensional Hilbert space. As a direct consequence the overlap among the t and $t+dt$ states is reduced; hence the attenuation in the Zener transition.

An important factor in determining the effect of the reservoir is its spectral density. We have

$$\tilde{J}(\omega)=\sum_\alpha \frac{A_\alpha^2}{\omega_\alpha^2}\delta(\omega-\omega_\alpha) . \quad (2.35)$$

Note the difference between our definition and the one used, e.g., by Caldeira and Leggett.[4(c)] Their condition on the spectral density to exhibit dissipation in the classical limit is, in our notation,

$$\tilde{J}(\omega)=\eta . \quad (2.36)$$

The correction to the bare Zener tunneling $\tilde{Y}/B$ is shown in Fig. 4 versus the normalized temperature for various values of η. More generally we consider

$$\tilde{J}(\omega)\sim\omega^q . \quad (2.37)$$

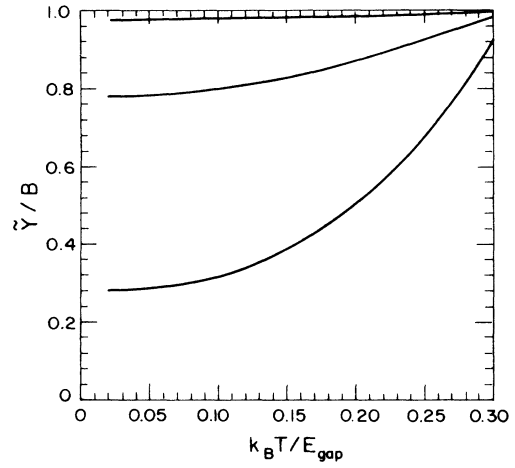

FIG. 4. Correction to the bare Zener probability $\tilde{Y}/B$ vs the normalized temperature $k_BT/E_{\rm gap}=k_BT/2\Delta$ for various values of dissipation, η. The current is chosen such that $\Delta^2/\alpha=5$.

The correction factor that multiplies B [cf. Eq. (2.34)] in the first term on the right-hand side of Eq. (2.33) is $(T=0)$

$$\exp\left\{\sum_\alpha \frac{4A_\alpha^2}{\omega_\alpha^2}\left[\exp\left[-\frac{2\Delta}{\alpha}\omega_\alpha\right]-1\right]\right\}. \tag{2.38}$$

The sum in the above exponential may be separated into the contribution of the low-frequency $[(2\Delta/\alpha)\omega_\alpha \ll 1]$ modes and that of the higher frequency. The former is proportional to ω^{q+1}. Note that for $q<-2$ the sum in (2.38) is dominated by the infrared divergence (unless there is a lower cutoff on the excitation frequencies of the reservoir). This result suggests that the Zener tunneling not only may be attenuated but, for spectral densities with a sufficiently strong infrared divergence, could be suppressed.

Equation (2.33) implies that as temperature is increased, the Zener tunneling rate is enhanced. Qualitatively, this is due to the availability of more phonons, which assist the Zener tunneling. The quantitative aspects of the temperature dependence are related to the nature of the reservoir and its coupling to the system. For example, at low temperatures, the contribution of the low frequencies (which is important for large negative q) in the exponential of Eq. (2.33), assumes the form

$$\text{const}\times\left[\int d\omega\left[-\frac{2\Delta}{\alpha}\omega^{1+q}\right] + \int d\omega\frac{4\Delta^2}{\alpha^2\hbar}\omega^{1+q}k_BT\right]. \tag{2.39}$$

In that limit, the temperature correction to $\tilde{Y}$ are exponential in k_BT.

III. INELASTIC INTERACTIONS AND DISSIPATION

In the previous section we have derived an expression for a single Zener transition in the presence of coupling to a heat bath. We shall now try to study how such transitions can lead to dissipation and evaluate the dependence of the dissipation on the parameters of the system. One of the basic assumptions made in the calculation above was that there was a characteristic phase-smearing time which is small enough that quantum correlations (interference) are destroyed between two consecutive transitions. An earlier work[1] considered the other extreme limit, where no phase-smearing mechanism is present. It has been argued that under these circumstances transitions to higher-lying "bands" can be undone; the Zener tunneling is reversible and no energy is dissipated.

The present case is different. We consider an irreversible process and evaluate the dissipation associate with it. Our treatment in this section is semiphenomenological. We *assume* that the Zener transitions to the higher band are quantum-mechanically uncorrelated. We also introduce a phenomenological relaxation time τ. Within the two periodic bands approximation (cf. Fig. 5) the system may undergo transitions from the lower energy band to the upper band, with the transition probability given by Eq. (2.33). It takes the system a time τ on the average to relax back to the lower band via an inelastic interaction. In order to evaluate the energy dissipated by the system in this excitation-relaxation process we consider now a relaxation time τ which is shorter than the period of energy oscillations (e.g., $\tau \ll e/I_{\rm ex}$ for the current-biased normal tunnel junction; see Fig. 5). We assume that a Zener transition (which occurs with a probability $\tilde{Y}$) takes place at the narrow gap and that the system relaxes to the lower energy band a time τ later (τ larger than the tunneling time). The system goes through $f\alpha$ narrow gaps per unit time (the constant f depends on the system considered). Therefore, there are $\tilde{Y}f\alpha$ Zener transitions per unit time. Hence, the dissipated power P is given by

$$P=\tilde{Y}f\alpha\Delta E\ . \tag{3.1}$$

In our case [cf. Eq. (B1)],

$$\Delta E=2(\alpha^2\tau^2+\Delta^2)^{1/2}\ , \tag{3.2}$$

and, for a sufficiently large τ, $\Delta E=2\alpha\tau$. In the case of a normal tunnel junction, the parameter α is proportional

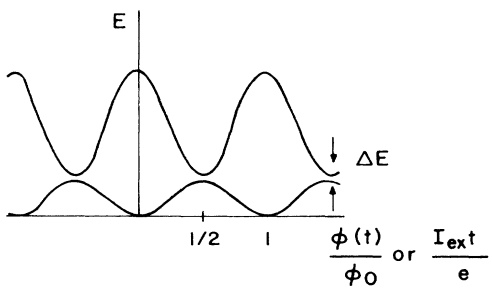

FIG. 5. Two-band picture: a two-level system whose energy is a periodic function of an external parameter.

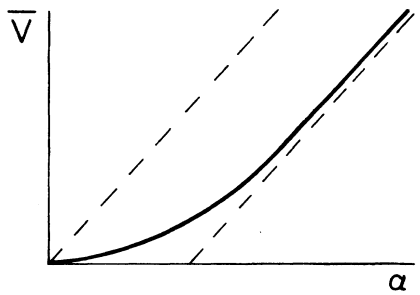

FIG. 6. Voltage-current characteristic for following the two-band picture. The relaxation time τ is assumed to be independent of both current and energy.

to the external driving current. Thus, one may define the resistance R of a junction. From Eq. (3.2) this is proportional to

$$R \sim \tilde{Y}\frac{\Delta E}{\alpha} \sim \bar{V}\alpha \ , \tag{3.3}$$

where $\bar{V}$ is the dc voltage.

In order to study the dependence of $\bar{V}$ on the external current we replace $\tilde{Y}$ by the bare Zener probability, discussed in Appendix B. The strongly nonlinear dependence of the voltage on the current is shown schematically in Fig. 6. The reason that the dissipated power *increases* with the relaxation time τ (assumed to be independent of ΔE and α) is that for larger τ (such that τ is still shorter than the period of the oscillations) more energy is pumped into the system before a relaxation event takes place.

A more realistic model should include the effect of the inelastic interactions and temperature on the transition probability $\tilde{Y}$. In Sec. II we have shown that increasing the temperature enhances the Zener tunneling [see Eqs. (2.33) or (2.39): in the latter exponential dependence on $k_B T$ is derived]. Higher temperatures usually mean shorter relaxation times; this tends to reduce the "energy term" $[\tau^2+\Delta^2/\alpha^2]^{1/2}$. Thus, as the temperature is varied the energy term may compete with the transition-rate term. The overall effect of varying $k_B T$ depends on details, such as the dependence of τ on $k_B T$, the phonon spectrum, etc.

IV. FINAL REMARKS

In this paper we have addressed the issue of the effect of the environment on externally driven systems which can be described by an effective time-dependent Hamiltonian. These systems, including current-biased tunnel junctions and flux-driven normal rings, exhibit oscillations (of energy, voltage, etc.) versus time. The external bias induces Zener transitions which depend on various parameters of the system, including the magnitude of the bias, the coupling to the environment, and temperature. Our analysis is intended to give only qualitative indications on how the Zener tunneling rate and the oscillations are affected by these parameters. To allow for a detailed comparison with experiment, extension of the present analysis is required. In particular, one would like to go beyond the two-level picture and the adiabatic approximations that were applied here. We nevertheless hope that the qualitative picture that emerges from our analysis will motivate comparison with experiment.[29] In particular, we have shown that the Zener tunneling is an increasing function of the external bias and temperature; we also studied how the resistance varies with these parameters and found that it goes to zero in the limit of zero bias.

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APPENDIX A: ENVIRONMENT-INDUCED TRANSITIONS

In this appendix we calculate transition rates between states of the electronic system ("spin states") due to the coupling to an external heat bath. Our result is basically equivalent to a "golden rule" expression for the transition rates. However, we feel that it may be useful to sketch, in some detail, the derivation of such transition rates in our case.

For the sake of clarity we confine our discussion here to a specific model of a single charged particle (whose charge is e) of mass m, which is confined to move along a one-dimensional ring of radius $r=1$. The latter encloses a magnetic flux ϕ. The reader should bear in mind, though, that this problem is closely related to several other problems (as discussed in the Introduction), and in that respect the results presented here apply to other systems which exhibit a similar energy-level structure.

The Hamiltonian of the system has the form

$$\mathcal{H}^{(A)}=\frac{1}{2I}\left[P_\theta-\frac{e}{c}A\right]^2+V(\theta) \ , \tag{A1}$$

where A is the magnitude of the vector potential, chosen to be tangential to the ring and independent of θ; I is the moment of inertia of the particle; and P_θ is the momentum canonically conjugate to θ. The potential $V(\theta)$ is, obviously, 2π periodic. Let us choose, for simplicity,

$$V(\theta)=-V_0\cos(\theta) \ . \tag{A2}$$

By a gauge transformation[12] one can easily show that the eigenvalue problem

$$\mathcal{H}^{(A)}=|\psi_n\rangle=E_n|\psi_n\rangle \ , \tag{A3}$$

with the eigenfunctions satisfying the boundary conditions

$$|\psi_n(\theta)\rangle=|\psi_n(\theta+2\pi)\rangle \ , \tag{A4}$$

is equivalent to

$$\mathcal{H}^{(A=0)}|\phi_n\rangle = E_n|\phi_n\rangle \ , \tag{A5}$$

with the boundary conditions

$$|\phi_n(\theta)\rangle = |\phi_n(\theta+2\pi)\rangle e^{-i2\pi\Phi/\Phi_0} \ . \tag{A6}$$

Here $\Phi = \oint A\,dl$ (the contour is along the ring, $\Phi_0 = hc/e$). The two sets of eigenfunctions are related to each other by

$$|\psi_n(\theta)\rangle = |\phi_n(\theta)\rangle e^{-i\theta\Phi/\Phi_0} \ , \tag{A7}$$

which is compatible with Eqs. (A4) and (A6). In particular, in the absence of an external electric field ($V_0=0$) the eigenfunctions are

$$|\psi_n\rangle = \frac{1}{\sqrt{2\pi}} \exp(in\theta) \ , \tag{A8}$$

with the corresponding eigenvalues

$$E_n = \frac{\hbar^2}{2I}\left[n - \frac{\Phi}{\Phi_0}\right]^2 \ . \tag{A9}$$

In the following discussion we shall use the "nearly-free-electron" approximation. By that we mean that the potential along the one-dimensional ring is assumed to be small. Thus, expressions (A8) and (A9) will be used as approximations for the actual eigenfunctions and eigenvalues, except near the values of the flux, where degeneracy occurs. The analysis in the vicinity of these values necessitates certain modifications (see below).

The "band" structure that arises from this model is shown schematically in Fig. 2. The gap between the first and second "band" is equal to $2V_0$, and the energy near the narrow gap is given by

$$E^{\pm} = \frac{\hbar^2}{8I} \pm V_0 \pm \frac{\hbar^4}{8I^2V_0}\epsilon^2 \ , \tag{A10}$$

for $\epsilon \equiv \Phi/\Phi_0 - \frac{1}{2} \ll 1$. Here $+$ $(-)$ stands for the second (first) band. The wave functions corresponding to $E^{(+)}$ and $E^{(-)}$ are

$$|\psi^{(+)}\rangle \simeq \frac{1}{\sqrt{\pi}} e^{-i\theta/2} \cos\frac{\theta}{2} \tag{A11a}$$

and

$$|\psi^{(+)}\rangle \simeq \frac{i}{\sqrt{\pi}} e^{-i\theta/2} \sin\frac{\theta}{2} \ , \tag{A11b}$$

respectively. In the gauged representation, where the boundary conditions on the eigenfunctions depend on the flux, we have ($\Phi/\Phi_0 \simeq \frac{1}{2}$)

$$|\phi^{(+)}\rangle \simeq \frac{1}{\sqrt{\pi}} \cos\frac{\theta}{2} \tag{A12a}$$

and

$$|\phi^{(-)}\rangle \simeq \frac{i}{\sqrt{\pi}} \sin\frac{\theta}{2} \ . \tag{A12b}$$

The gaps among the higher levels decrease with the level index n faster than exponentially.[13] However, for more complex periodic potentials the higher gaps may not be vanishingly small due to the existence of higher harmonic components of the potential. Thus, for example, the tenth gap may be mainly dominated by the amplitude of the $\cos(10\theta)$ component of the potential.

In the following discussion we shall concentrate on the transitions between the two low-lying levels, but the results are easily generalized to transitions between any pair of levels. We shall consider here only environmentally induced transitions, and take the enclosed flux to be a parameter; i.e., we neglect the explicit time dependence of Φ.

Following previously developed models,[4(c)] we describe the environment by a reservoir of harmonic oscillators coupled to the electronic system. The total Hamiltonian is then given by

$$\begin{aligned}\mathcal{H}_T = \mathcal{H}^{(A=0)} &+ \sum_\alpha C_\alpha(1)x_\alpha \sin\theta + \sum_\alpha C_\alpha(2)y_\alpha \cos\theta \\ &+ \sum_\alpha \left[\frac{P_\alpha^2(1)}{2m_\alpha(1)} + \tfrac{1}{2}m_\alpha(1)\omega_\alpha^2(1)x_\alpha^2\right] \\ &+ \sum_\alpha \left[\frac{P_\alpha^2(2)}{2m_\alpha(2)} + \tfrac{1}{2}m_\alpha(2)\omega_\alpha^2(2)y_\alpha^2\right] \ .\end{aligned} \tag{A13}$$

The reservoir consists of two baths of oscillators ($\{x_\alpha\}$ and $\{y_\alpha\}$), denoted by 1 and 2, which are coupled to $\sin\theta$ and $\cos\theta$, respectively. Note that the coupling part of the Hamiltonian preserves the invariance under $\theta \to \theta + 2n\pi$. By imposing some constraints on the microscopic parameters of the theory one can obtain the desired behavior of the system in the classical limit. Following Caldeira and Leggett[4(c)] we shall consider here a quantum-mechanical Hamiltonian which describes standard dissipation in the classical limit. This is satisfied provided that

$$\sum_i \frac{\pi}{2} \sum_\alpha \left[\frac{\partial F_\alpha(i)}{\partial\theta}\right]^2 \frac{1}{m_\alpha(i)\omega_\alpha(i)} \delta(\omega - \omega_\alpha(i)) = \eta\omega \ , \tag{A14}$$

where $i=1,2$, $F_\alpha(1) = C_\alpha(1)\sin\theta$, and $F_\alpha(2) = C_\alpha(2)\cos\theta$. Choosing $C_\alpha(1) = C_\alpha(2) = C_\alpha$, $m_\alpha(1) = m_\alpha(2) = m_\alpha$, and $\omega_\alpha(1) = \omega_\alpha(2) = \omega_\alpha$, we obtain the constraint

$$\frac{\pi}{2} \sum_\alpha \frac{C_\alpha^2}{m_\alpha\omega_\alpha} \delta(\omega - \omega_\alpha) = \eta\omega \ . \tag{A15}$$

In order to make further progress we assume that the coupling of the electronic system to the environment is weak in the sense that it is still meaningful to ask questions about transition rates among electronic states (and how these are affected by the environment). The transition rates will be calculated within first-order perturbation theory, using the influence functional approach of Feynman and Vernon.[4(a)]

The transition probability from the state $|\phi_m\rangle$ to $|\phi_n\rangle$ is given by

$$P(\phi_m \to \phi_n) = \int\int\int\int d\theta_i d\theta_f d\tilde{\theta}_i d\tilde{\theta}_f \phi_n^*(\theta_f)\phi_n(\tilde{\theta}_f) J(\theta_f,\tilde{\theta}_f,t;\theta_i,\tilde{\theta}_i,0)\phi_m(\theta_i)\phi_m^*(\tilde{\theta}_i) , \tag{A16}$$

where

$$J(\theta_f,\tilde{\theta}_f,t;\theta_i,\tilde{\theta}_i,0) = \int\int \mathcal{D}\theta\,\mathcal{D}\tilde{\theta} \exp\left[\frac{1}{\hbar}S[\theta]\right] \exp\left[-\frac{1}{\hbar}S[\tilde{\theta}]\right] \mathcal{F}[\theta,\tilde{\theta}] . \tag{A17}$$

Here $S[\theta]$ is the action of the uncoupled system and $\mathcal{F}[\theta,\tilde{\theta}]$ is the influence functional. We shall compute $\mathcal{F}[\theta,\tilde{\theta}]$ for only one oscillator, coupled, for example, to $\sin\theta$, at zero temperature. It is straightforward to extend the above calculation and include the other modes, the coupling to $\cos\theta$, and finite-temperature effects. This will be carried out when necessary.

Within the above-mentioned approximation we can write

$$\mathcal{F}[\theta,\tilde{\theta}] = \exp\left[-\frac{1}{\hbar}\int_0^t\int_0^\tau dz\,ds \frac{C_\alpha^2}{2m_\alpha\omega_\alpha}[\sin\theta(z)-\sin\tilde{\theta}(z)][e^{-i\omega_\alpha(z-s)}\sin\theta(s) - e^{i\omega_\alpha(z-s)}\sin\tilde{\theta}(s)]\right] . \tag{A18}$$

Now, assuming that C_α is a weak perturbation, we can expand the exponent in (A18) up to the first order in C_α^2, and the expression for J becomes

$$\begin{aligned} J = {} & k(\theta_f,t;\theta_i,0)k^*(\tilde{\theta}_f,t;\tilde{\theta}_i,0) \\ & - \frac{C_\alpha^2}{2m_\alpha\hbar\omega_\alpha}\int_0^t\int_0^\tau dz\,ds\, e^{-i\omega(z-s)}k^*(\tilde{\theta}_f,t;\tilde{\theta}_i,0)\int \mathcal{D}\theta \sin\theta(z)\sin\theta(s)\exp\left[\frac{i}{\hbar}S[\theta]\right] \\ & - \frac{C_\alpha^2}{2m_\alpha\hbar\omega_\alpha}\int_0^t\int_0^\tau dz\,ds\, e^{i\omega_\alpha(z-s)}k(\theta_f,t;\theta_i,0)\int \mathcal{D}\tilde{\theta} \sin\tilde{\theta}(z)\sin\tilde{\theta}(s)\exp\left[-\frac{i}{\hbar}S[\tilde{\theta}]\right] \\ & + \frac{C_\alpha^2}{2m_\alpha\hbar\omega_\alpha}\int_0^t\int_0^\tau dz\,ds\, e^{i\omega_\alpha(z-s)}\int \mathcal{D}\theta \sin\theta(z)\exp\left[-\frac{i}{\hbar}S[\theta]\right]\int \mathcal{D}\tilde{\theta}\sin\tilde{\theta}(s)\exp\left[-\frac{i}{\hbar}S[\tilde{\theta}]\right] \\ & + \frac{C_\alpha^2}{2m_\alpha\hbar\omega_\alpha}\int_0^t\int_0^\tau dz\,ds\, e^{-i\omega_\alpha(z-s)}\int \mathcal{D}\tilde{\theta}\sin\tilde{\theta}(z)\exp\left[-\frac{i}{\hbar}S[\tilde{\theta}]\right]\int \mathcal{D}\theta\sin\theta(s)\exp\left[\frac{i}{\hbar}S[\theta]\right] . \end{aligned} \tag{A19}$$

The k's are the Feynman propagators for the uncoupled system. The transition element can then be written as

$$\begin{aligned} P(\phi_m \to \phi_n) = {} & \delta_{mn} - \delta_{mn}\frac{C_\alpha^2}{m_\alpha\hbar\omega_\alpha}\mathrm{Re}\int_0^t\int_0^\tau dz\,ds\, e^{-\omega_\alpha(z-s)}\int\int d\theta_i d\theta_f \phi_n^*(\theta_f)\phi_m(\theta_i) \\ & \times \int \mathcal{D}\theta \sin\theta(z)\sin\theta(s)\exp\left[\frac{i}{\hbar}S[\theta]\right] \\ & + \frac{C_\alpha^2}{m_\alpha\hbar\omega_\alpha}\mathrm{Re}\int_0^t\int_0^\tau dz\,ds\, e^{-i\omega_\alpha(z-s)}\int d\theta\sin\theta\,\phi_n^*(\theta,s)\phi_m(\theta,s)\int d\tilde{\theta}\sin\tilde{\theta}\,\phi_n(\tilde{\theta},z)\phi_m^*(\tilde{\theta},z) . \end{aligned} \tag{A20}$$

This expression can be computed easily in two cases: either far from the narrow gaps (see Fig. 2) when the wave functions are approximately the free-particle ones; or near

$$\frac{\Phi}{\Phi_0} = \tfrac{1}{2} + n ,$$

where the wave functions are given by Eq. (A12).

Case 1: Far from the narrow gaps. In this case the wave functions are given by Eqs. (A7) and (A8), E_n is given by Eq. (A9). From Eq. (A20) we obtain ($m \neq n$)

$$P(\phi_m \to \phi_n) = \frac{c_\alpha^2 |J_{mn}|^2}{4\pi^2 m_\alpha\hbar\omega_\alpha}\mathrm{Re}\int_0^t\int_0^\tau dz\,ds\, e^{-i(\omega_\alpha-\omega_{mn})z} \times e^{i(\omega_\alpha-\omega_{mn})s} , \tag{A21}$$

with

$$|J_{mn}|^2 = \left|\int d\theta \sin\phi_n^*(\theta)\phi_m(\theta)\right|^2 . \tag{A22}$$

This yields

$$P(\phi_m \to \phi_n) = \frac{C_\alpha^2}{4\pi^2 m_\alpha\hbar\omega_\alpha}\frac{\sin^2[(\omega_\alpha+\omega_{mn})t/2]}{(\omega_\alpha+\omega_{mn})^2}\delta_{n,m\pm1} , \tag{A23}$$

where

$$\omega_{mn} = \frac{\hbar}{2I}\left[\left[n+m-\frac{2\Phi}{\Phi_0}\right](n-m)\right] . \tag{A24}$$

Taking the limit $t \to \infty$, we obtain

$$\frac{dP_{mn}}{dt}=\frac{C_\alpha^2}{8\pi m_\alpha\hbar\omega_\alpha}\delta(\omega_\alpha+\omega_{mn})\delta_{n,m\pm1}\ , \tag{A25}$$

which clearly shows that only transitions such that $E_n > E_m$ are allowed at $T=0$.

Our result can be generalized to a bath of uncoupled oscillators, initially at equilibrium at temperature T.[30] We obtain

$$\frac{dP_{mn}}{dt}=\sum_k \frac{C_\alpha^2}{8\pi m_\alpha\hbar\omega_\alpha} \times\left[\frac{\exp(\hbar\omega_\alpha/kT)}{\exp(\hbar\omega_\alpha/kT)-1}\delta(\omega_\alpha+\omega_{mn})+\frac{1}{\exp(\hbar\omega_\alpha/kT)-1}\delta(\omega_\alpha-\omega_{mn})\right] . \tag{A26}$$

The same expression is obtained by considering the other set of oscillators, namely those which have the $\cos\theta$ coupling. Incorporating the Caldeira-Leggett constraint, Eq. (A15), we finally obtain

$$\frac{dP_{mn}}{dt}=\begin{cases}\dfrac{\eta\omega_{mn}}{2\pi^2\hbar}\dfrac{1}{\exp\left[\dfrac{\hbar\omega_{mn}}{kT}\right]-1} & \text{if } \omega_{mn}>0,\\[2ex] \dfrac{\eta\,|\,\omega_{mn}\,|}{2\pi^2\hbar}\dfrac{\exp\left[\dfrac{\hbar\,|\,\omega_{mn}\,|}{kT}\right]}{\exp\left[\dfrac{\hbar\,|\,\omega_{mn}\,|}{kT}\right]-1} & \text{if } \omega_{mn}<0,\end{cases} \tag{A27}$$

where the selection rule $n-m=\pm1$ [for our specific form of potential, Eq. (A2)] is always implicit.

Case 2: Near the narrow gaps. The analysis of this case follows closely that of the previous one, the main difference being that the energies [Eq. (A10)] and the wave functions [Eq. (A12)] are different. The final result is similar to Eq. (A27) with ω_{mn} replaced by

$$\Delta\omega=2V_0+\frac{\hbar^4}{2I^2V_0}\epsilon^2 \tag{A28}$$

[cf. Eq. (A10)].

APPENDIX B: ZENER TRANSITIONS IN A TWO-LEVEL SYSTEM

We consider the Hamiltonian $\mathcal{H}_s$ [Eq. (2.2)]. Considering t as a parameter, the eigenvalues are

$$\lambda_\pm=\pm(\alpha^2t^2+\alpha^2+\Delta^2)^{1/2}\ . \tag{B1}$$

The corresponding eigenfunctions are

$$|+\rangle_t=\tilde{N}\begin{bmatrix}1\\ \dfrac{(\alpha^2t^2+\Delta^2)^{1/2}-\alpha t}{\Delta}\end{bmatrix}, \tag{B2}$$

$$|-\rangle_t=\tilde{N}\begin{bmatrix}\dfrac{\alpha t-(\alpha^2t^2+\Delta^2)^{1/2}}{\Delta}\\ 1\end{bmatrix}, \tag{B3}$$

with the normalization factor given by

$$\tilde{N}=\frac{\Delta}{\sqrt{2}[\alpha^2t^2+\Delta^2-\alpha t(\alpha^2t^2+\Delta^2)^{1/2}]^{1/2}}\ . \tag{B4}$$

Let us assume that the quantum state of the system at $t\to-\infty$, ψ_s, is given by

$$\psi_s(t\to-\infty)=|\uparrow\rangle=|-\rangle_{t\to-\infty}\ . \tag{B5}$$

Here we use the notation

$$|\downarrow\rangle\equiv\begin{bmatrix}0\\1\end{bmatrix},\quad |\uparrow\rangle=\begin{bmatrix}1\\0\end{bmatrix}. \tag{B6}$$

The state of the system at time t, expressed in terms of the above spinors, is

$$\psi_s(t)=\tilde{a}_\uparrow(t)\,|\uparrow\rangle+\tilde{a}_\downarrow(t)\,|\downarrow\rangle =a_+(t)\,|+\rangle_t+a_-(t)\,|-\rangle_t\ . \tag{B7}$$

We have $\tilde{a}_\downarrow(t\to-\infty)=a_-(t\to-\infty)=1$. Within the adiabatic approximation, one assumes that

$$a_-(t)\approx1;\quad |\,a_-(t)\,|\gg|\,a_+(t)\,|\ , \tag{B8}$$

for all values of t. We then have[23]

$$a_+(t)=\int_{-\infty}^{t}\frac{dt}{\lambda_+(t)-\lambda_-(t)}\left[\frac{\partial\mathcal{H}_s}{\partial t}\right]_{+-} \times\exp\left[i\int_{-\infty}^{t}dt'[\lambda_+(t')-\lambda_-(t')]\right], \tag{B9}$$

where

$$\left[\frac{\partial\mathcal{H}_s}{\partial t}\right]_{+-}\equiv{}_t\left\langle+\left|\frac{\partial\mathcal{H}_s}{\partial t}\right|-\right\rangle_t\ . \tag{B10}$$

In our case

$$\left[\frac{\partial\mathcal{H}_s}{\partial t}\right]_{+-}=\frac{-\Delta\alpha}{(\alpha^2t^2+\Delta^2)^{1/2}}\ . \tag{B11}$$

The expression in the exponential in Eq. (B9) is (up to a constant phase term)

$$\frac{i\Delta^2}{\alpha}\left[\frac{\alpha t}{\Delta}\left[\frac{\alpha^2t^2}{\Delta^2}+1\right]^{1/2}+\sinh^{-1}\left[\frac{\alpha t}{\Delta}\right]\right]. \tag{B12}$$

One can then calculate the integral in Eq. (B9) by, e.g., using steepest descent method in the complex plane.[31] For $t\to\infty$ we obtain for the total transition amplitude in the adiabatic limit

$$\mathcal{A}=e^{-\pi\Delta^2/2\alpha}\ . \tag{B13}$$

This expression [Eq. (B13)] is also compatible with the re-

sult obtained in the opposite limit, i.e., $a \to 0$. the approximations involved in the calculation are valid as long as the magnitude of this transition amplitude is much smaller than unity. The expression (B13) (obtained by integrating the transition rate for the energy levels shown in Fig. 3 between $t = -\infty$ and $t \to +\infty$) may be taken as a reasonable approximation for the transition amplitude "per period" for the oscillating energy bands (cf. Fig. 2).

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