

Electron-phonon interaction, ultrasonic attenuation, and Eliashberg function $\alpha^2 F(\omega)$ in impure metals

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A relation between the ultrasonic attenuation coefficient and the Eliashberg function $\alpha^2 F$ is derived, which holds up to leading order in the impurity density. This shows that all electron-phonon interaction models which yield Pippard's result for the sound attenuation are equivalent with respect to calculation of electronic properties as well. The low-frequency behavior of $\alpha^2 F(\omega)$ is calculated for a jellium model, and several mutually contradicting results in the literature are discussed.

I. MOTIVATION

In recent years, interest has grown in a number of topics, for which availability of a reliable electron-phonon interaction model in the presence of disorder is of vital importance. Among these are dirty superconductors and sound attenuation and polaronic effects in impure metals, to mention only a few. Unfortunately, the problem of electron-phonon interaction in impure systems is not straightforward, mainly because uncritical application of the standard clean-limit concepts yields incorrect results. In the standard model,¹ screening leads to a short-ranged effective interaction between longitudinal phonons and electron-density fluctuations. Since disorder leads to diffusive behavior of the electronic density, one might naively expect an increase of the effective interaction due to disorder. That this is actually not the case follows from Pippard's famous result² for the longitudinal (L) ultrasonic attenuation coefficient $\alpha_L(q)$: with a decrease in the product of the phonon wave number q and the electronic mean free path l , α_L decreases monotonically, and for $ql \ll 1$ α_L is smaller than the clean-limit result by a factor of ql . For transverse (T) phonons, the result is more complicated, and shows that disorder *does* increase α_T in certain regimes. The physical reason for the behavior of α_L is the system's tendency to maintain approximate local charge neutrality, as has been discussed by Pippard² and more thoroughly by Holstein.³ A microscopic derivation of Pippard's result was obtained by Tsuneto,⁴ who used a unitary transformation to a frame of reference which moves with the ions. This transformation takes care of the approximate charge neutrality mentioned above. The unitary transformation has been further exploited by Schmid,⁵ who derived a model to replace the standard model in impure systems. This Tsuneto-Schmid (TS) model essentially replaces the standard model's coupling of the phonon to the electronic density by a coupling of the phonon to the electronic stress tensor, and it is very easy to use within the usual many-body formalism. Despite this advantage, some authors felt uneasy about the transformation to the moving frame of reference. Hence Eisenriegler and Grünwald and Schainberg,⁶ considered a model where the electrons are viewed from the laboratory frame of reference, and

the long-range Coulomb interactions are considered explicitly (EGS model). This model requires explicit consideration of collisions of electrons with moving impurities. From this there arises a host of new diagrams. Taking these properly into account, the authors of Ref. 6 recovered Pippard's result. Finally, the same result has been obtained⁷ within the short-range Fröhlich model with additional consideration of the mentioned inelastic processes. So for the sound attenuation, the standard Pippard result has been corroborated by three different microscopic calculations, of which the one using the TS model is by far the simplest.

In contrast to that, the electronic self-energy, and the related Eliashberg function, have been a source of confusion for some time. Bergmann⁸ was the first to argue that the frequency dependence of the Eliashberg function $\alpha^2 F(\omega)$ would be changed at small ω due to disorder. From phase-space arguments, he concluded $\alpha^2 F(\omega \rightarrow 0) \propto \omega$, where both longitudinal and transverse phonons contribute to the increase over the usual quadratic behavior. This contradicted earlier arguments by Ginsberg,⁹ who argued that an overall increase of the coupling parameter $\lambda = \int d\omega \alpha^2 F(\omega) / \omega$ was due to a negative longitudinal contribution overcompensated by a positive transverse one. More recently, Poon and Geballe¹⁰ obtained an expression for $\alpha^2 F$ which is identical with that of Bergmann.⁸ This result also got strong empirical support, since most experiments find a linear low-frequency behavior of $\alpha^2 F$.¹¹

On a microscopic level, Schmid⁵ obtained $\alpha^2 F(\omega \rightarrow 0) \propto \omega^3$ within the TS model. On the other hand, calculations¹² within the EGS model found a linear (and even a small constant) contribution from longitudinal phonons alone. Later Keck and Schmid⁷ reexamined the TS model. They confirmed Schmid's ω^3 result for frequencies $\omega \ll c/l$, where c is the sound velocity. For $c/l \ll \omega \ll \omega_D$, they found a linear contribution to $\alpha^2 F$ from the transverse phonons, the prefactor of which is somewhat reduced by the longitudinal ones. ($\omega_D = cq_D$ is the Debye frequency.) This was in accord with the arguments of Ref. 9, and in contradiction to the results of Refs. 8 and 12.

This contradiction has very recently been put to rest by Reizer and Sergeyev.¹³ They show that careful calcula-

tions within both the TS and the EGS model yield identical results, namely the ω^3 law of Keck and Schmid. They also point out explicitly the errors of Ref. 12. However, some problems still remain. In the first place, Ref. 13 is very technical, and does not show what is physically wrong with the semiphenomenological arguments of Refs. 8 and 10. In the second place, the ω^3 law is still not the correct asymptotic low-frequency result, as we will see below. Finally, one would like to understand why the (wrong) result of Refs. 8 and 10 agrees with experiment, while the one of Ref. 5, 7, and 13 does not. The latter point is of particular importance, since the observed absence of the ω^3 law did much to discredit the TS model in the experimental community.

In the present paper it is shown by a simple argument that *all* models yielding the Pippard result for the sound attenuation also give the same answer for $\alpha^2 F$, namely that of Keck and Schmid, if the latter is corrected for an effect at very small frequencies. Our argument is closely related to a less explicit one given before in Ref. 7. To this end, we derive in Sec. II a relation between $\alpha^2 F$, and the sound-attenuation coefficient. If damping of the phonon propagator entering $\alpha^2 F$ is neglected (as has been done in all previous work), this turns out to be Allen's relation,¹⁴ which holds in disordered as well as in clean systems. Insertion of Pippard's formula yields the result of Keck and Schmid for $\alpha^2 F$. All other conjectures for $\alpha^2 F$ violate Pippard's result (and all known experimental facts) for the sound attenuation. However, if the correct phonon propagator (with damping taken into account) is used to calculate $\alpha^2 F$, the result of Keck and Schmid is qualitatively changed at small frequencies. This is important for comparison with experiments, and invalidates the experimental argument against the TS model mentioned before. In Sec. III we discuss the result, and comment on the experimental situation.

$$\Pi_b(\mathbf{q}, i\Omega) = \int d\mathbf{x}_1 d\mathbf{x}_2 e^{i\mathbf{q}\cdot(\mathbf{x}_1 - \mathbf{x}_2)} T \sum_{i\omega} \int d\mathbf{x}'_1 d\mathbf{x}'_2 d\mathbf{x}'_3 \langle G(\mathbf{x}'_1, \mathbf{x}'_2, i\omega - i\Omega) G(\mathbf{x}'_2, \mathbf{x}'_3, i\omega) g_b(\mathbf{x}'_2, \mathbf{x}_2, \mathbf{x}'_2) g_b(\mathbf{x}'_1, \mathbf{x}_1, \mathbf{x}'_1) \rangle_{\text{av}}. \quad (2.4)$$

Here T denotes the temperature, $i\Omega$ and $i\omega$ are Matsubara frequencies, and $\langle \dots \rangle_{\text{av}}$ denotes the ensemble average over the disorder. Notice that in general g may not be taken out of the average. As in the clean case, we can replace the self-consistent electron Green's functions G by bare (with respect to the phonons) ones, $G^{(0)}$.¹ We now make use of the exact-eigenstates formalism.^{16,7} Thereby one assumes that the electronic part of the Hamiltonian has been diagonalized with eigenfunctions $\psi_n(\mathbf{x})$, and eigenenergies E_n . $G^{(0)}$ is diagonal in this rep-

$$R_b(\mathbf{x}, E, E') = \left\langle \sum_{n,m} \delta(E - E_n) \delta(E' - E_m) \int d\mathbf{x}_1 d\mathbf{x}'_1 g_b(\mathbf{x}'_1, \mathbf{x}, \mathbf{x}_1) \psi_n^*(\mathbf{x}_1) \psi_m(\mathbf{x}'_1) \int d\mathbf{x}_2 d\mathbf{x}'_2 g_b(\mathbf{x}_2, 0, \mathbf{x}'_2) \psi_m^*(\mathbf{x}'_2) \psi_n(\mathbf{x}_2) \right\rangle_{\text{av}}. \quad (2.6)$$

It is easy to show¹⁷ that for small $E - E'$, the function R_b depends on $E - E'$ only, and is the dissipative part of a corresponding causal function Φ_b , which allows for the spectral representation

II. THEORY

A. A general model

We consider electrons of effective mass m in a disordered medium, whose interaction with phonons is described by the following Hamiltonian:

$$H_{e\text{-ph}} = \sum_{\mathbf{k}, \mathbf{k}', \mathbf{q}} \sum_b g_b(\mathbf{k}, \mathbf{k}', \mathbf{q}) c_{\mathbf{k}}^\dagger c_{\mathbf{k}'} \phi_b(\mathbf{q}). \quad (2.1)$$

Here the c^\dagger, c are electron creation and annihilation operators. The phonon-field operator is defined as

$$\phi_b(\mathbf{q}) = i[\omega_b(\mathbf{q})/2]^{1/2} (a_{\mathbf{q}b} + a_{-\mathbf{q}b}^\dagger). \quad (2.2)$$

Here $\omega_b(\mathbf{q})$ is the dispersion relation for polarization branch b [$b=L$ (longitudinal), $b=T$ (transverse)], and the a^\dagger, a are phonon creation and annihilation operators. We choose units such that $\hbar=1$. g is the vertex for scattering an electron from state \mathbf{k} to state \mathbf{k}' by means of a phonon of wave vector \mathbf{q} . We assume that the vertex has been corrected for static screening, so we do not have to consider the Coulomb interaction explicitly.¹⁵ We do not require, however, momentum to be conserved in the process. If it is, as in the TS model,⁵ we have $g(\mathbf{k}, \mathbf{k}', \mathbf{q}) \propto \delta_{\mathbf{k}' - \mathbf{k}, \mathbf{q}}$. If it is not, as in the EGS model,¹² g depends explicitly on the disorder which absorbs the "missing" momentum. We will also need the real-space representation of g , defined by

$$g_b(\mathbf{x}_1, \mathbf{x}_2, \mathbf{x}_3) = \sum_{\mathbf{k}, \mathbf{k}', \mathbf{q}} e^{i\mathbf{k}'\cdot\mathbf{x}_1 - i\mathbf{q}\cdot\mathbf{x}_2 - i\mathbf{k}\cdot\mathbf{x}_3} g_b(\mathbf{k}, \mathbf{k}', \mathbf{q}). \quad (2.3)$$

B. The phonon self-energy

We now calculate the phonon self-energy Π . According to the usual rules, we have

resentation: $G_{nm}^{(0)} \simeq \delta_{nm} G_n^{(0)}$. Rewriting Eq. (2.4) in the exact-eigenstate basis, we obtain

$$\Pi_b(\mathbf{q}, i\Omega) = \int dE dE' \frac{f(E') - f(E)}{i\Omega + E' - E} \times \int d\mathbf{x} d\mathbf{y} e^{i\mathbf{q}\cdot(\mathbf{x} - \mathbf{y})} \mathbf{R}_b(\mathbf{x} - \mathbf{y}, E, E'). \quad (2.5)$$

Here f is the Fermi distribution function, and R is defined as

$$\Phi_b(\mathbf{q}, z) = \int d\omega R_b(\mathbf{q}, \omega) / (\omega - z), \quad (2.7)$$

for any complex frequency z . With the help of these relations, we obtain for the imaginary part of the retarded

phonon self-energy at zero temperature

$$\text{Im}\Pi_b(\mathbf{q}, i\Omega \rightarrow \Omega + i0) = \Omega R_b(\mathbf{q}, \Omega) . \quad (2.8)$$

The sound-attenuation coefficient is defined as

$$\alpha_b(\mathbf{q}) = [\omega_b(\mathbf{q})/c_b] \times \text{Im}\Pi_b(\mathbf{q}, i\Omega \rightarrow \omega_b(\mathbf{q}) + i0) .$$

For $\omega_b(\mathbf{q})\tau \ll 1$, (i.e., $q l \ll v_F/c_b$ with Fermi velocity $v_F = k_F/m$), we can put the frequency argument of R_b equal to zero, and obtain

$$\alpha_b(\mathbf{q}) = \frac{1}{c_b} \omega_b^2(\mathbf{q}) R_b(\mathbf{q}, 0) . \quad (2.9)$$

Within the TS model, R_b is an electronic stress correlation function. It can be evaluated either by solving a kinetic equation, or by standard diagrammatic techniques. For jellium, the result in leading order in the disorder is⁵ the Pippard formula, viz.,

$$R_b(q, 0) = d_b^2 (3m^2/\pi^2) (1/q^2 l) f_b(q l) , \quad (2.10a)$$

where

$$\begin{aligned} \Sigma_n(i\omega) = -T \sum_{i\omega'} \sum_b \int d\mathbf{x}_1 d\mathbf{x}_2 D_b(\mathbf{x}_1, \mathbf{x}_2, i\omega - i\omega') \sum_m G_m^{(0)}(i\omega') \int d\mathbf{x}'_1 d\mathbf{x}'_2 d\mathbf{x}'_2' g_b(\mathbf{x}'_1, \mathbf{x}'_1) \psi_n^*(\mathbf{x}'_1) \psi_m(\mathbf{x}'_1) \\ \times g_b(\mathbf{x}'_2, \mathbf{x}'_2, \mathbf{x}'_2') \psi_m^*(\mathbf{x}'_2') \psi_n(\mathbf{x}'_2) . \end{aligned} \quad (2.11)$$

We now carry out the ensemble average according to Ref. 17. As in all previous work,^{7,12} we average separately over the phonon propagator. This means that the result will be exact only to leading order in the impurity density. Within the same accuracy, we can calculate the electronic inelastic lifetime from the ensemble-averaged self-energy rather than from the Green's function. For the former we obtain

$$\Sigma_E(i\omega) = - \sum_{\mathbf{q}, b} \int \frac{dy}{\pi N_F} [f(y) D_b(\mathbf{q}, i\omega - y) R_b(\mathbf{q}, E - y) + n(y) D_b''(\mathbf{q}, y) \Phi_b(\mathbf{q}, E - y - i\omega)] , \quad (2.12)$$

where D'' is shorthand for the imaginary part of the retarded phonon propagator, and n denotes the Bose distribution function. The imaginary part of the retarded self-energy $\Sigma(E, \omega) = \Sigma_E(i\omega \rightarrow \omega + i0)$, is

$$\begin{aligned} \text{Im}\Sigma(E, \omega) = \int \frac{dy}{\pi} [f(y + \omega) + n(y)] \\ \times \sum_{\mathbf{q}, b} D_b''(\mathbf{q}, y) \frac{1}{N_F} R_b(\mathbf{q}, E - \omega - y) . \end{aligned} \quad (2.13)$$

From this we read off the Eliashberg function

$$\alpha^2 F(\omega) = \sum_b \alpha^2 F_b(\omega) = \frac{1}{2\pi^2 N_F} \sum_{\mathbf{q}, b} D_b''(\mathbf{q}, \omega) R_b(\mathbf{q}, \omega) . \quad (2.14)$$

For $\omega\tau \ll 1$, we can again neglect the frequency argument of R_b , and with Eq. (2.9) we have the following relation between $\alpha^2 F$ and the sound-attenuation coefficient:

$$\alpha^2 F(\omega) = \frac{1}{2\pi^2 N_F} \sum_{\mathbf{q}, b} \alpha_b(\mathbf{q}) \frac{c_b}{\omega_b^2(\mathbf{q})} D_b''(\mathbf{q}, \omega) . \quad (2.15)$$

$$f_L(x) = \frac{1}{3} \frac{x^2 \arctan x}{x - \arctan x} - 1 , \quad (2.10b)$$

$$f_T(x) = \frac{1}{2x^3} [2x^3 + 3x - 3(x^2 + 1)\arctan x] , \quad (2.10c)$$

and

$$d_L = (c_T/c_L) d_T = k_F^2 / 3m \rho_{\text{ion}}^{1/2} c_L ,$$

where ρ_{ion} denotes the ion mass density. Within the EGS and Fröhlich models, R_b is considerably more difficult to evaluate, since g_b itself depends on the disorder. The equivalent of evaluating R_b for these models has been carried out by Eisenriegler,⁶ and by Keck and Schmid,⁷ respectively. The result was again the Pippard formula.

C. The electron self-energy

Now we calculate the electron self-energy Σ by the same method. Again we may neglect the self-energy of the electron Green's function.¹ In the exact-eigenstate representation, we find

III. DISCUSSION

Equation (2.15) is our central result. It expresses $\alpha^2 F$ in terms of the sound attenuation coefficient, and the phonon propagator. It is an exact identity for clean systems, and for impure systems up to leading order in the impurity density. To leading order in the disorder, the sound attenuation is known, and given by the Pippard formula, Eqs. (2.9) and (2.10). This proves that the sound attenuation determines the Eliashberg function. In the following, we discuss this result.

A. Undamped phonons

The phonon propagator in Eq. (2.15) contains, of course, the self-energy calculated in Sec. II B. Nevertheless, we will consider undamped phonons for a while in order to make contact with previous results. So we put

$$D_b''(\mathbf{q}, \omega) \simeq \pi \omega_b(\mathbf{q}) [\delta(\omega - \omega_b(\mathbf{q})) - \delta(\omega + \omega_b(\mathbf{q}))] , \quad (3.1)$$

and obtain

$$\alpha^2 F(\omega) = \frac{1}{2\pi^2 N_F} \frac{1}{\omega} \sum_{\mathbf{q}, b} \alpha_b(\mathbf{q}) c_b \delta(\omega - \omega_b(\mathbf{q})) . \quad (3.2)$$

This is the relation Allen¹⁴ derived for clean systems, and it is interesting to see that it holds in the presence of disorder as well. This relation has a number of immediate consequences. If we use the Pippard formula, Eqs. (2.9) and (2.10), in Eq. (3.2), we obtain for the low-frequency Eliashberg function the result of Keck and Schmid,⁷ namely $\alpha^2 F(\omega \rightarrow 0) \propto \omega^3$. From Eq. (2.15) it follows that this is the result for every model which yields the Pippard result for $\alpha(q)$ (if damping of the phonon propagator in $\alpha^2 F$ is neglected).

Pippard's work, as well as all three microscopic models mentioned in the Introduction, apply for a crystalline system (or jellium) with impurities. On the other hand, the authors of Refs. 8 and 10 considered amorphous metals. Nevertheless, the general calculation of Sec. II applies to this case equally well, and Eq. (2.15) holds. Consequently, the result^{8,10} $\alpha^2 F(\omega \rightarrow 0) \propto \omega$ implies $\alpha(q \rightarrow 0) = \text{const}$, where the constant is proportional to the resistivity ρ . That is, there is no propagating sound mode at small frequencies. This has been realized in Ref. 10, where Allen's relation has been tacitly assumed to hold for amorphous materials as well. While the experimental situation is not entirely conclusive, there seems to be little support for this conjecture. In the local limit, $ql \ll 1$, very general arguments based on screening show that the electronic damping mechanism is viscous damping, which always goes like ω^2 . This is certainly true in liquid metals.¹⁸ In metallic glasses, and amorphous metallic NiP, experiments show a linear behavior,¹⁹ which is probably due to nonelectronic relaxation processes. Even in that case there is, to the author's knowledge, no evidence for overdamping at small frequencies. It should also be mentioned that the result

$\alpha(q \rightarrow 0) = \text{const} \propto \rho$ in Refs. 8 and 10 stems technically from a coupling to the electronic density, which is incorrect in the case of jellium with impurities.

B. Damped phonons

We now consider the correct phonon propagator in Eq. (2.15). Accordingly, instead of the approximation (3.1), we have

$$D_b''(\mathbf{q}, \omega) = \frac{2\omega\omega_b^2(\mathbf{q})2\gamma_b(\mathbf{q}, \omega)}{[\omega^2 - \Omega_b^2(\mathbf{q})]^2 + 4\omega^2\gamma_b^2(\mathbf{q}, \omega)}. \quad (3.3a)$$

Here $\Omega_b(\mathbf{q})$ is the dressed phonon frequency, and

$$\gamma_b(\mathbf{q}, \omega) = \frac{1}{2}\omega_b^2(\mathbf{q})R_b(\mathbf{q}, \omega) \quad (3.3b)$$

is the damping. For $\omega\tau \ll 1$, we again can neglect the ω dependence, and have

$$\gamma_b(\mathbf{q}, \omega) \simeq \gamma_b(\mathbf{q}) \equiv (c_b/2)\alpha_b(\mathbf{q}). \quad (3.3c)$$

With this phonon spectrum, the discussion of the integral in Eq. (2.15) proceeds analogously to the clean case.¹ For frequencies $\omega \gg \max[\gamma(\mathbf{q})] = \omega_D\eta$, where $\eta = k_F^4/9\rho_{\text{ion}}c$, Eq. (3.3a) reduce to the bare spectrum, Eq. (3.1), and we find

$$\alpha^2 F(\omega) = \frac{3m}{4\pi^3 k_F l} \sum_b d_b^2 \frac{\omega}{c_b} f_b(\omega l/c_b), \quad (\omega \gg \omega_D\eta). \quad (3.4)$$

Here we have neglected the renormalization of the phonon frequency, and assumed Debye dispersion. In typical metals, $\eta \simeq 0.01$. We now have to distinguish between the cases $\omega l/c \gtrsim 1$. From Eq. (2.10), we obtain

$$\alpha^2 F(\omega) = h_L \frac{q_D/k_F}{5\pi} \frac{8}{\hat{\rho}} (1 + \frac{3}{2}c_L^5/c_T^5)(\omega/\omega_D)^3, \quad (\eta \ll \omega/\omega_D \ll \hat{\rho}k_F/q_D), \quad (3.5a)$$

$$\alpha^2 F(\omega) = h_L \left[(\omega/\omega_D)^2 + \frac{k_F/q_D}{\pi} [12c_L^3/c_T^3 - (8 - \pi^2/2)] \hat{\rho}\omega/\omega_D \right], \quad (\hat{\rho}k_F/q_D \ll \omega/\omega_D \ll \hat{\rho}2\varepsilon_F/\omega_D). \quad (3.5b)$$

Here we have introduced the dimensionless resistivity $\hat{\rho} = 1/k_F l$, $h_L = d_L^2 N_F q_D^2 / 4k_F^2$, and both expressions are valid to first order in the respective expansion parameters. Equation (3.5) is the result obtained before by Keck and Schmid,⁷ who assumed, however, that the ω^3 law holds down to arbitrarily small frequencies. We see that this is actually not the case. Rather there is a window of frequencies where $\alpha^2 F \propto \omega^3$ only if the resistivity is large enough, viz., $\hat{\rho} \gg \eta \simeq 0.01$. With typical numbers, that means $\rho \gg 10 \mu\Omega \text{ cm}$. We will come back to that.

The true asymptotic behavior, $\omega \ll \omega_D\eta$, is dominated by the phonon damping, and we can replace the phonon spectrum by

$$D_b''(\mathbf{q}, \omega) \simeq \omega 2\gamma_b(\mathbf{q}, \omega) / \omega_b^2(\mathbf{q}). \quad (3.6)$$

This leads to

$$\alpha^2 F(\omega) = h_L^2 \frac{4k_F^3}{q_D^3} (\omega/\varepsilon_F) \left[1 - \hat{\rho} \frac{9}{\pi^2} \frac{4k_F}{q_D} F + O(\hat{\rho}^2) \right], \quad (\omega/\omega_D \ll \eta), \quad (3.7a)$$

where

$$F = \int_0^\infty dx [\pi^2/36 - f_L^2(x)/x^2 - (c_L/c_T)^4 f_T^2(x)/x^2]. \quad (3.7b)$$

This result shows that the true asymptotic behavior of $\alpha^2 F$ at low frequencies is linear, and is qualitatively the same for clean and disordered systems. The resulting inelastic lifetime is of the same structure and order of magnitude as the one arising from electron-electron interaction in the clean limit. For the physical interpretation of this result, we refer the reader to Refs. 1 and 20.

C. Conclusion

We summarize the low-frequency behavior of the Eliashberg function. For $\omega/\omega_D \ll \eta \simeq 0.01$, $\alpha^2 F(\omega)$ is linear with a very small slope, Eq. (3.7a). The slope is disorder dependent, but the exponent is not. If the resistivity is small, $\rho \lesssim \eta e^2 3\pi^2/k_F \simeq 10 \mu\Omega \text{ cm}$, this crosses over at $\omega/\omega_D \simeq \eta$ to another linear law, Eq. (3.5b), the slope of which depends linearly on the resistivity. In the clean limit, it gives way to the ω^2 behavior of free electrons. For larger resistivities $\rho \gg \eta e^2 3\pi^2/k_F \simeq 10 \mu\Omega \text{ cm}$, there is a window of frequencies $\eta \ll \omega/\omega_D \ll \rho k_F^2/3\pi^2 e^2 q_D$, where $\alpha^2 F(\omega) \propto \omega^3$, Eq. (3.5a). For $\omega/\omega_D \gg \rho k_F^2/3\pi^2 e^2 q_D$, we have again the linear behavior, Eq. (3.5b). All these results follow from Pippard's law for the sound attenuation, if one neglects nontrivial phonon dispersion, and possible phonon softening in amorphous materials. To obtain Pippard's formula, one has neglected Brillouin-zone and band-structure effects. Though neither of these approximations should be of qualitative importance at small frequencies, they clearly render impossible quantitative comparison with experiments. We therefore confine ourselves to a brief qualitative discussion of the experimental situation.

Experiments on quench-condensed simple metals show a linear behavior of $\alpha^2 F(\omega)$.¹¹ Unfortunately, for all experiments quoted in Ref. 11, the resistivity of the samples is not known. For the measurements on tin by Knorr and Barth,²¹ however, we can estimate the resistivity by comparison with the work of Bergmann,²² who prepared tin samples in a very similar way, and obtained comparable transition temperatures. In that case, the maximum resistivity was 10 to 20 $\mu\Omega \text{ cm}$. It is certainly realistic to assume similar values for all other quench-condensed microcrystalline samples. For these values of ρ , the ω^3 dependence of $\alpha^2 F$ is absent. The true asymptotic behavior, Eq. (3.7a), is too small to be seen in these experiments. Therefore, $\alpha^2 F$ at small ω is essentially linear, Eq. (3.5b), shifted to high ω by an amount $\Delta\omega \simeq 0.01\omega_D \simeq 0.1$

meV, which is again smaller than present experimental accuracy. We conclude that in this case, theory and experiment agree qualitatively. The ω^3 law is no argument against the theory, simply because it is absent up to much larger resistivities. This brings us back to the behavior proposed in Refs. 8 and 10. As we have argued above, these results miss the correct low-frequency behavior, and contain an unphysical contribution from longitudinal phonons. For $\rho \lesssim 10 \mu\Omega \text{ cm}$, however, the former is too small to be measured, and the latter changes only the prefactor of the linear behavior. We conclude that in this regime, the formulas given in Refs. 8 and 10 give a qualitatively correct picture, even though their derivations give rise to criticism.

For the amorphous transition metals Mo and Nb with resistivities of 65 to 200 $\mu\Omega \text{ cm}$, Kimhi and Geballe²³ measured $\alpha^2 F(\omega \rightarrow 0) \propto \omega^n$, where n is definitely larger than one. Even though our jellium considerations are clearly inadequate for transition metals, it is tempting to interpret this effect as the ω^3 law, which should be well pronounced at these resistivities. If this is actually the case, the nonlinear behavior should disappear at smaller resistivities. It would be interesting to check this experimentally. On the other hand, in amorphous simple metals with comparable resistivities, $\alpha^2 F$ is found to be linear.¹¹ This difference between simple and transition metals cannot be understood within existing theory (ironically, it is the simple metals which deviate qualitatively from the jellium results). Clearly, more theoretical work is needed to understand amorphous materials.

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¹See, e.g., A. A. Abrikosov, L. P. Gor'kov, and I. E. Dzyaloshinskii, *Methods of Quantum Field Theory in Statistical Physics* (Prentice Hall, Englewood Cliffs, NJ, 1969).

²A. B. Pippard, *Philos. Mag.* **46**, 1104 (1955).

³T. Holstein, *Phys. Rev.* **113**, 479 (1959).

⁴T. Tsuneto, *Phys. Rev.* **121**, 402 (1961).

⁵A. Schmid, *Z. Phys.* **259**, 421 (1973).

⁶E. Eisenriegler, *Z. Phys.* **258**, 185 (1973); G. Grünwald and K. Scharnberg, *ibid.* **268**, 197 (1974); *Z. Phys. B* **20**, 61 (1975).

⁷B. Keck and A. Schmid, *J. Low Temp. Phys.* **24**, 611 (1976).

⁸G. Bergmann, *Phys. Lett.* **29A**, 492 (1969); *Phys. Rev. B* **3**, 3797 (1971).

⁹D. M. Ginsberg, *Phys. Rev.* **138**, A1409 (1965).

¹⁰S. J. Poon and T. H. Geballe, *Phys. Rev. B* **18**, 233 (1978).

¹¹For a review, see, G. Bergmann, *Phys. Rep.* **27**, 159 (1976).

¹²H. Takayama, *Z. Phys.* **263**, 329 (1973); S. G. Lisitin, *Fiz. Nizk. Temp.* **1**, 1516 (1975) [*Sov. J. Low. Temp. Phys.* **1**, 728

(1975)].

¹³M. Yu Reizer and A. V. Sergeev, *Zh. Eksp. Theor. Fiz.* **90**, 1056 (1986) [*Sov. Phys.—JETP* **63**, 616 (1986)].

¹⁴P. B. Allen, *Phys. Rev. B* **6**, 2577 (1972); *Solid State Commun.* **14**, 937 (1974).

¹⁵This assumption does not mean a loss of generality. It follows from Ref. 5, and has been checked explicitly in Ref. 12, that for coupling to long wavelength phonons, one does not have to take into account dynamical screening. Consequently, disorder changes the screening only via its effects on static correlation functions. These effects are of quantitative nature only [P. G. De Gennes, *J. Phys. Radium (Paris)* **23**, 630 (1962)], and can be absorbed in g .

¹⁶P. W. Anderson, *J. Phys. Chem. Solid* **11**, 26 (1959).

¹⁷E. Abrahams, P. W. Anderson, P. A. Lee, and T. V. Ramakrishnan, *Phys. Rev. B* **24**, 6783 (1981); D. Belitz, *J. Phys. F* **15**, 2315 (1985).

¹⁸R. T. Beyer and E. M. Ring, in *Liquid Metals*, edited by S. Z.

- Beer (Dekker, New York, 1972).
- ¹⁹H. Araki, G. Park, A. Hikata, and C. Elbaum, *Phys. Rev. B* **21**, 4470 (1980); P. Doussineau, A. Levelut, and G. Bellessa, in *Proceedings of the International Conference on Lattice Dynamics, Paris, France, 1977*, edited by M. Balkanski (Flammarion, Paris, 1978).
- ²⁰A. Schmid, in *Localization, Interaction, and Transport Phenomena in Impure Metals*, edited by B. Kramer, G. Bergmann, and Y. Bruynseraede (Springer, Berlin, 1985).
- ²¹K. Knorr and N. Barth, *J. Low. Temp. Phys.* **4**, 469 (1971).
- ²²G. Bergmann, *Z. Phys.* **228**, 25 (1969).
- ²³D. B. Kimhi and T. H. Geballe, *Phys. Rev. Lett.* **45**, 1039 (1980).