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Theory of orientational relaxation in systems with continuous symmetry

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We study the response to a magnetic field variation of a system with continuous symmetry in the ordered phase. Nonlinear relaxation of a vector order parameter is expected to occur even for a very small field if this is switched on in a direction not close to the direction of the initial magnetization. The transient behavior is explained in terms of fluctuations associated with the decay from the initial state which, immediately after a field inversion, is unstable. We find a metastablelike behavior of the magnetization, anomalies in the fluctuations of the order parameter, and a large growth of the correlation length above the equilibrium values. We study the effects of both homogeneous and inhomogeneous fluctuations using two complementary theoretical approaches, i.e., the quasideterministic theory and the spherical limit.

I. INTRODUCTION

Highly nonlinear phenomena occur when a system with continuous symmetry is suddenly forced to be in an unstable state, for instance, by varying some external control parameter such as the temperature or an external field.[1] A typical nonlinear relaxation process which has been extensively studied, both experimentally and theoretically,[1–3] is the quench of a magnetic system below the critical point, where the growth of a new phase is observed. An analogous phenomenon is the nonlinear orientational response to a magnetic field switched on in a direction very different from that of the initial magnetization.

The aim of this paper is to study in detail the mechanism of the orientational relaxation process for a nonconserved vectorial order parameter with n components, in the case of a complete inversion of the direction of the external field. We treat both the case when only homogeneous spatial fluctuations are important and the case in which inhomogeneous fluctuations are equally relevant. The most relevant result we get is the interpretation of the main features of the relaxation as due to the vicinity of the unstable state that the system reaches immediately after a magnetic field inversion. The fluctuations associated with the decay from the initial unstable state characterize the temporal relaxation.

The physical phenomenon we discuss can be summarized considering the following four different temporal regimes in the transient behavior of the system.

(i) An early stage when the system, after the magnetic field variation, quickly reaches an unstable state.

(ii) Subsequently, the system remains close to the unstable state until thermal fluctuations cause the decay. This behavior shows up in a quasimetastable state of the macroscopic magnetization for times that depend logarithmically on the applied magnetic field and the strength of the thermal fluctuations. A concomitant anomaly develops in the fluctuations' correlation length which grows to values much larger than the initial and final steady-state ones. A further characterization of this regime can be given in terms of the statistical properties of the temporal averages of the order parameter which show an anomalous dispersion.

(iii) The magnetization inversion happens then very rapidly at a time which coincides with the maximum of the magnetization fluctuations. The vicinity of the instability manifests itself as anomalously large fluctuations of the component of the order parameter transverse with respect to the magnetization. This phenomenon is due to the large spread of the time it takes to the system to leave the unstable state.

(iv) Finally transverse fluctuations are reduced, the system feels the direction of the external field and the magnetization reaches the new equilibrium value.

From the theoretical point of view two special cases are of particular interest since they can be treated in an essentially exact fashion. The first one is the case of a two-component order parameter ($n=2$) with homogeneous fluctuations, that can be solved taking into account phase fluctuations by means of the quasideterministic approximation.[4,5] The other important case is the large-n limit (or spherical model), which gives a consistent description of all the phenomena related to the inhomogeneous fluctuations. In this case the modulus of the order parameter is a nonfluctuating variable, the process becomes a Gaussian process whose characterization is given by a single deterministic nonlinear integro-differential equation.[3,6–8] The first approach is typical of the transient phase fluctuations of a field in a laser; the second refers to the kinetics of phase transitions in magnetic systems.

We have studied the homogeneous case in terms of modulus and phase of a two-component order parameter. In such a case the relevant relaxation phenomenon occurs in the phase. Thermal noise is essential to initiate the relaxation after a field inversion since, as we already mentioned, the phase is initially in an unstable state. The decay is studied from two points of view, namely, the temporal evolution of the magnetization and its anomalous fluctuations and the passage time statistics.

An exact theory of the relaxation phenomena can also be worked out for large n both in the case of homogeneous and nonhomogeneous fluctuations. An analytic solution is easily derived in the homogeneous case, while in the nonhomogeneous case we get a piecewise analytical result. In the first case a comparison can be made with the results for $n=2$; in the second one with a numerical solution of the relevant dynamical equation. In the inhomogeneous case the time behavior of the relaxation function is quite close to that of the homogeneous system both in the initial and final regime. However, nonhomogeneous fluctuations are important for the intermediate growth and to trigger the final regime. One of the important features of the kinetics after a quench is the appearance of a scaling regime at intermediate times. We find that in the field inversion case, both in the decay and growth regime, magnetization fluctuations exhibit a scaling behavior with a spatial correlation which increases in time according to the Lifshitz-Cahn-Allen $t^{1/2}$ law.[9,10] The analysis of the structure factor also shows how the long-range correlation of transverse fluctuations, if present in the initial state, will be maintained after the field switch but with a relative weight which decreases in time.

The theory of the orientational relaxation is also able to evidentiate the ergodic behavior of the system, that can be summarized as follows. After the decay from the metastable system the phase space available to the systems is enlarged because of the possibility of very different orientations of the magnetization in different regions of the system. During the growth regime a local ordering develops via a reorientation of the local magnetization which produces local domains of growing size. In the final ergodic regime domains with orientations different from that of the applied magnetic field will disappear.

Section II introduces the theoretical model and the discussion of the homogeneous two-component system. In Sec. III we introduce the large-n limit and the analytic solution of the model in the homogeneous case and the numerical and approximate analytical solution for the nonhomogeneous system. Section IV is devoted to the conclusions.

II. THE MODEL: HOMOGENEOUS FLUCTUATIONS

We assume as evolution equations for our system the time-dependent Ginzburg-Landau model[11] for a nonconserved order parameter

$$\frac{\partial}{\partial t}\boldsymbol{\psi}(\mathbf{x},t)=-\frac{\delta H}{\delta \boldsymbol{\psi}}+\sqrt{\epsilon}\boldsymbol{\xi}(\mathbf{x},t) \ , \qquad (1)$$

where $\boldsymbol{\psi}(\mathbf{x},t)$ is the n-dimensional vector associated with the local magnetization and $\boldsymbol{\xi}(\mathbf{x},t)$ is the local thermal noise with the usual properties

$$\begin{aligned}&\langle \xi_i(\mathbf{x},t)\rangle=0 \ ,\\ &\langle \xi_i(\mathbf{x},t)\xi_j(\mathbf{x}',t')\rangle=\delta_{i,j}\delta(\mathbf{x}-\mathbf{x}')\delta(t-t') \ .\end{aligned} \qquad (2)$$

We assume Eq. (1) to be an Ito stochastic differential equation and, as is customary in the physics literature, we use the stochastic force $\boldsymbol{\xi}(\mathbf{x},t)$ instead of the multivariable Wiener process $dW(\mathbf{x},t)$, according to the relation $dW(\mathbf{x},t)=\xi(\mathbf{x},t)dt$. The effective free energy H is given by

$$H=\int_V dV\left[\frac{r_0}{2}\,|\boldsymbol{\psi}|^2+\frac{u}{4n}\,|\boldsymbol{\psi}|^4+\frac{D}{2}\,|\nabla\cdot\boldsymbol{\psi}|^2-\sqrt{n}\,\mathbf{h}\cdot\boldsymbol{\psi}\right] \qquad (3)$$

for a system of finite size $V=L^3$. In order to obtain the homogeneous limit we first Fourier transform Eq. (1),

$$\boldsymbol{\psi}(\mathbf{x},t)=\frac{1}{V}\sum_{\mathbf{q}} e^{i\mathbf{q}\cdot\mathbf{x}}\boldsymbol{\psi}_q(t), \quad q\equiv\left\{q_i=\frac{2\pi}{L}n_i, \quad i=1,2,3\right\} \qquad (4)$$

with n_i $(i=1,2,3)$ taking integer values. We then have

$$\begin{aligned}\frac{\partial}{\partial t}\boldsymbol{\psi}_q=&-(r_0+Dq^2)\boldsymbol{\psi}_q\\ &-\frac{u}{nV^2}\sum_{\mathbf{q}'\mathbf{q}''}(\boldsymbol{\psi}_{\mathbf{q}-\mathbf{q}'-q''}\cdot\boldsymbol{\psi}_{q'})\boldsymbol{\psi}_{\mathbf{q}''}\\ &+\sqrt{\epsilon}\boldsymbol{\xi}_q+\sqrt{n}\,\mathbf{h}V\delta_{\mathbf{q},0} \ .\end{aligned} \qquad (5)$$

The homogeneous limit is obtained selecting the $q=0$ mode in Eq. (5) and assuming it is macroscopic, $\boldsymbol{\psi}_0(t)=\sqrt{n}\,V\boldsymbol{\varphi}(t)$:

$$\frac{d}{dt}\boldsymbol{\varphi}(t)=-(r_0+u\,|\boldsymbol{\varphi}|^2)\boldsymbol{\varphi}+(\epsilon_0)^{1/2}\boldsymbol{\xi}(t)+\mathbf{h} \ , \qquad (6)$$

where $\boldsymbol{\xi}$ is a vectorial Gaussian white noise and $\epsilon_0=\epsilon/nV$.

Some of the most remarkable features of our solution can already be seen studying the homogeneous system with $n=2$. In this case an alternative picture is given by introducing the modulus R and the phase θ for the order parameter through

$$\varphi_1+i\varphi_2=Re^{i\theta} \ . \qquad (7)$$

We use a method[12] which takes advantage of the fact that when dealing with Ito equations, the noise is statistically independent from the field evaluated at the same time. Assuming the magnetic field in the direction 1, we have the Ito stochastic differential equations

$$\frac{d}{dt}R(t)=-(r_0+uR^2)R+\frac{\epsilon_0}{2R}+(\epsilon_0)^{1/2}\xi_R+h\cos\theta \ , \qquad (8)$$

$$\frac{d}{dt}\theta(t)=-\frac{h}{R}\sin\theta+\frac{(\epsilon_0)^{1/2}}{R}\xi_\theta \ , \qquad (9)$$

$$\xi_R=\xi_1\cos\theta+\xi_2\sin\theta, \quad \xi_\theta=-\xi_1\sin\theta+\xi_2\cos\theta \ , \qquad (10)$$

where $h=h_I>0$ and $h=-h_F<0$, respectively, before and after the inversion of the magnetic field at $t=0$. For $t<0$ the situation can be described as follows. From Eq. (8) we see that when $r_0<0$, for small enough ϵ_0 and h_I and very large $|r_0|$ and u, we can neglect the field and thermal-fluctuation effect on R so that we can assume $R\approx(|r_0|/u)^{1/2}$. On the other hand, the magnetic field and thermal fluctuations strongly affect the phase. Equation (9) shows that the phase undergoes a Brownian motion in a periodic potential $V(\theta)=-(h_I/R)\cos\theta$. The steady-state probability distribution

$$\mathbf{P}(\theta) \sim \exp\left[-\frac{1}{\epsilon_0}V(\theta)\right]$$

has a maximum, as expected, for $\theta=0$, which is also the equilibrium state for the process of Eq. (9). For large R, i.e., $R \gg \epsilon_0/h_I$, the phase has negligible fluctuations around is equilibrium value. Field inversion transforms the initial equilibrium state $\theta=0$ into an unstable one that the system will abandon because of thermal fluctuations. We will use the limit of very large $|r_0|$ and u with fixed ration $R^2 \sim (|r_0|/u)=1$ which is the so called hard-spin (HS) limit.[13] It allows a closer analogy with spin systems, and at the same time neglects the inessential fast initial dynamics of the modulus.

We will apply a quasideterministic theory to the transient behavior of fluctuations, exploiting the same approximation scheme already used to study the transient laser radiation statistics[4-5] and chemical explosions.[14] The idea is to derive an approximate solution of Eq. (9), i.e., a process which reduces for short times to the solution of the linear version of Eq. (9) and for long times to the solution of the deterministic limit of the same equation. In other words, for long times we neglect in Eq. (9) the noise obtaining the deterministic equation of motion. The main justification of such approximate solution is that noise is important to start the evolution and becomes negligible when the system leaves the unstable state. Fluctuations of the system when it comes close to the stable state can then be added as a perturbation according the results of Ref. 5. The approximate process we get is given by

$$\theta(t)=2\arctan[\tfrac{1}{2}\theta_0(t)e^{h_F t}]\ , \tag{11a}$$

$$\theta_0(t)=(\epsilon_0)^{1/2}\int_0^t dt' e^{-h_F t'}\xi_\theta(t')\ , \tag{11b}$$

and it can be understood as a mapping, defined by the solution of the deterministic version of Eq. (10), between the process $\theta(t)$ and a Gaussian variable $\theta_0(t)$ of zero average and variance given by

$$\langle \theta_0^2(t)\rangle = \sigma^2(t) = \frac{\epsilon_0}{2h_F}(1-e^{-2h_F t})\ . \tag{12}$$

The decay of an unstable state can be characterized in terms of the approximate solution of Eqs. (11) and (12) from two points of view that can be fully exploited in the case of orientational relaxation. First of all we study the transient statistics of the order parameter, which has an analogy with the experiments performed on laser transient radiation.[4] The decay of the unstable state shows up in the anomalous fluctuation phenomena, i.e., fluctuations in the transient much larger than at steady state. In our case this phenomenon is expected for the transverse components of the order parameter. A second point of view[15-17] is to consider the time the system needs to reach a given state as a stochastic variable, the passage time, and to study its statistics. We now discuss in some detail the main features of the decay according to these characteristics.

The longitudinal component of the order parameter, i.e., the macroscopic magnetization $m(t)$, is

$$m(t)=\langle\cos\theta\rangle = \left\langle \frac{1-\frac{1}{4}e^{2h_F t}\theta_0^2}{1+\frac{1}{4}e^{2h_F t}\theta_0^2}\right\rangle \tag{13}$$

and can be evaluated, using the properties of $\theta_0(t)$, as

$$m(t)=2H(z)-1,\quad H(z)=\sqrt{\pi}z e^{z^2}\mathrm{erfc}(z)\ ,$$
$$z=\left[\frac{2}{\sigma^2(t)}\right]^{1/2}\exp(-h_F t)\ , \tag{14}$$

where erfc is the complementary error function.[18] The magnetization is a function of the scaling variable z which vanishes for long times. Using a series development in z for $m(t)$ we can evaluate the instant t_i when it vanishes. To lowest order in z

$$t_i=\frac{1}{2h_F}\ln\left[\frac{16\pi h_F}{\epsilon_0}\right]\ . \tag{15}$$

The overall behavior of the magnetization is shown in Fig. 1.

Anomalous fluctuations are evidentiated in the transverse component of the order parameter. From Eq. (11a) we obtain

$$\varphi_2(t)=\sin\theta=\frac{e^{h_F t}\theta_0}{1+\frac{1}{4}e^{2h_F t}\theta_0^2}\ . \tag{16}$$

The average value is zero as expected from the symmetry of the problem, and the variance is given by

$$\langle\varphi_2^2(t)\rangle = 2H(z)+4z^2[H(z)-1] \tag{17}$$

with the same symbols used in Eq. (14). Again we obtain a macroscopic maximum value in the variance at times of the order of t_i when $\langle\sin^2\theta\rangle \approx 1$. The macroscopic variance is actually a natural consequence of the fact that different stochastic trajectories leave the unstable state at quite different times. The result of Eq. (17) is shown in Fig. 2.

We now consider[17] the time the systems needs to reach a given angle θ, if it is initially $\theta=0$, as a stochastic variable that can be obtained from Eq. (16). Recalling the

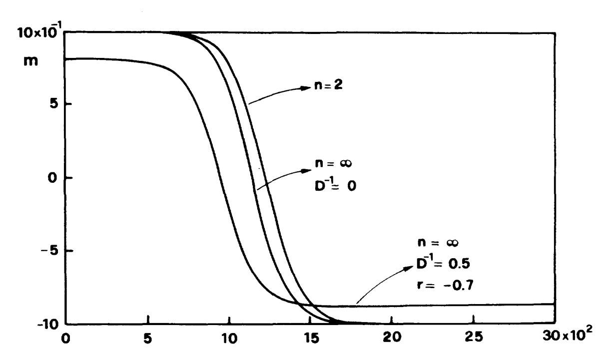

FIG. 1. The magnetization $m(t)$ for homogeneous and inhomogeneous fluctuations for $n=2$ and $n=\infty$. The values of the parameters are $h_I=h_F=4.2\times10^{-3}$; $\epsilon=2$; $V=10^7$; $u=1$ in the HS limit.

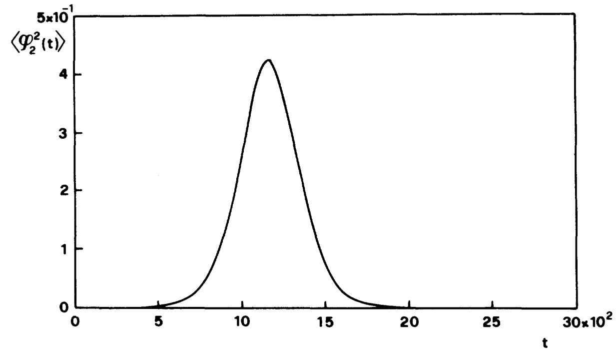

FIG. 2. The anomalous fluctuations of the transverse field for $n=2$. The parameters are the same as in Fig. 1.

properties of $\theta_0(t)$ we can write $\theta_0=\sigma(t)g$, where g is a time-independent Gaussian variable with zero mean and unitary variance and $\sigma(t)$ is given by Eq. (12). From Eqs. (16) and (11a) we obtain the passage as a function of θ and g

$$t_p=\frac{1}{2h_F}\ln\left[\left[\tan^2\frac{\theta}{2}\right]\frac{8h_F}{\epsilon_0 g^2}+1\right] . \tag{18}$$

This quantity, which we call passage time, is not the usual first passage time since it also allows multiple crossings of a given reference value θ. Moreover, in the quasideterministic part of the motion the two quantities should be essentially the same. The statistical properties of t_p can be obtained from the generating function

$$W(\lambda)=\langle e^{-\lambda 2h_F t_p}\rangle , \tag{19}$$

where the average is over the distribution of θ_0. The result is

$$W(\lambda)=\left[\frac{h_F}{\pi\epsilon_0}\right]^{1/2} 2\tan\frac{\theta}{2}\Gamma\left[\frac{1+\lambda}{2}\right] \times U\left[\frac{1+\lambda}{2},\frac{3}{2},\frac{4h_F}{\epsilon_0}\tan^2\frac{\theta}{2}\right] , \tag{20}$$

where Γ and U are the gamma and the confluent hypergeometric functions.[18] For the physical interesting case of a macroscopic angle and a magnetic field larger than the effective noise ϵ_0, i.e., for

$$\tan^2\frac{\theta}{2} >> \frac{\epsilon_0}{4h_F} ,$$

we obtain, using the asymptotic expansion of the function U to the leading order

$$W(\lambda)\approx\frac{1}{\sqrt{\pi}}\Gamma\left[\frac{1+\lambda}{2}\right]\left[\frac{4h_F}{\epsilon_0}\tan^2\frac{\theta}{2}\right]^{-\lambda/2} \tag{21}$$

and for the average passage time[14,15]

$$\langle t_p\rangle=\frac{1}{2h_F}\left\{\ln\left[\frac{16h_F}{\epsilon_0}\right]+\ln\left[\tfrac{1}{4}\tan^2\left[\frac{\theta}{2}\right]\right]-\psi(\tfrac{1}{2})\right\} , \tag{22}$$

where ψ is the digamma function.[18] The passage time as given in Eq. (22) is a sum of three terms, the first one being the time the system spends close to the unstable state and the second one the time the system needs to reach a given orientation. This second contribution, under the conditions mentioned above, is much smaller than the first one. It is worth noting to note that the average passage time is just the time at which the magnetization vanishes and the transverse anomalous fluctuations reach the maximum value [see Eq. (15)]. It is also interesting to note the logarithmic dependence both on thermal noise strength and external magnetic field. Thermal noise is essential to move the system away from the initial state while a vanishingly small magnetic field implies a very long time for orientational relaxation to occur.

III. THE SPHERICAL LIMIT

We now approach the study of the relaxation after a field inversion in the limit of a large number of components of the order parameter.[3,6–8] We assume again the external field in the 1 direction and therefore a macroscopic longitudinal component of the magnetization in that direction

$$\psi_i(\mathbf{x},t)=\sqrt{n}\,m(t)\delta_{i,1}+\delta\psi_i(\mathbf{x},t) . \tag{23}$$

The main point is that in such a limit the nonlinear term of Eq. (1) becomes a time-dependent nonfluctuating quantity

$$\lim_{n\to\infty}\frac{1}{n}|\boldsymbol{\psi}|^2=\lim_{n\to\infty}\frac{1}{n}\sum_{i=1}^{n}\langle\psi_i(\mathbf{x},t)\psi_i(\mathbf{x},t)\rangle =m^2(t)+\langle\delta\psi_i^2(\mathbf{x},t)\rangle=C(t) . \tag{24}$$

As a consequence, we have a linear stochastic differential equation with an effective time-dependent restoring force. In Fourier space

$$\frac{\partial}{\partial t}\delta\boldsymbol{\psi}_q(t)=-[r_0+Dq^2+uC(t)]\delta\boldsymbol{\psi}_{\mathbf{q}}+\sqrt{\epsilon}\xi_{\mathbf{q}}+\sqrt{n}\,\mathbf{h}V\delta_{\mathbf{q},0} . \tag{25}$$

The solution of this equation is conveniently expressed[6–8] in terms of the nonlinear relaxation function $y(t)$,

$$y(t)=\exp\left[2\int_0^t dt'[r_0+uC(t')]\right] , \tag{26}$$

as

$$\delta\boldsymbol{\psi}_q(t)=\frac{e^{-Dq^2t}}{\sqrt{y(t)}}\left[\delta\boldsymbol{\psi}_q(0)+\int_0^t dt' e^{Dq^2t'}\sqrt{y(t')}\sqrt{\epsilon}\xi_q(t')\right] . \tag{27}$$

From Eq. (25) we get the self-consistency condition which leads to the evolution equation for y,

$$\frac{1}{2}\frac{d}{dt}y(t)=(r+um^2)y+\frac{u\epsilon}{V}\int_0^t dt' y(t') +u\epsilon\int_0^t dt' f(t-t')y(t')+\frac{u}{V}\sum_{\mathbf{q}}S_q(0)e^{-2Dq^2t} . \tag{28}$$

The average magnetization is given by

$$m(t)=\frac{1}{\sqrt{y(t)}}\left[m(0)+h\int_0^t dt'\sqrt{y(t')}\right]. \quad (29)$$

As before $h=h_I>0$ for $t>0$ and $h=-h_F<0$ for $t>0$. The usual definition for the transverse structure factor holds,

$$\langle\delta\psi_{q,i}(t)\delta\psi_{q',i}(t)\rangle=V\delta_{\mathbf{q}+\mathbf{q}',0}S_q(t)\quad(i\neq 1), \quad (30)$$

and $S_q(0)$ is its value for $t=0$. The last term of Eq. (28) contains the memory effect due to the local fluctuations

$$f(t)=\frac{1}{V}\sum_{\mathbf{q}}{}'\left[e^{-2Dq^2t}-\frac{\delta(t)}{Dq^2}\right]. \quad (31)$$

The primed sum is restricted to values of q different from zero. The subtraction appearing in the preceding equation is actually present only for space dimensionality larger than 2, and is related to the renormalization of the parameter r which is given by

$$r=r_0+\frac{u\epsilon}{2V}\sum_{\mathbf{q}}{}'\frac{1}{Dq^2}. \quad (32)$$

and the sum is limited by an upper momentum cutoff. It is worth to recall that for negative r_0 the renormalized parameter r can be positive for high enough temperature and negative for small enough ones. The critical point is thus defined by $r=0$. The steady-state solution of Eq. (28) is

$$y\sim e^{\alpha t},\quad S_q(0)=\frac{\epsilon}{2Dq^2+\alpha},\quad m(0)=\frac{2h}{\alpha}, \quad (33)$$

where α is the solution of the equation

$$r-\frac{\alpha}{2}+\frac{4uh^2}{\alpha^2}-\frac{u\epsilon\alpha}{V}\sum_{\mathbf{q}}{}'\frac{1}{2Dq^2(2Dq^2+\alpha)}+\frac{u\epsilon}{\alpha V}=0. \quad (34)$$

We will always use a steady state as initial situation and, consequently, we will use α_I and α_F when referring, respectively, to the initial situation with $h=h_I$ or the final one with $h=-h_F$.

A. Homogeneous fluctuations

We shall first treat the homogeneous fluctuation limit (infinite D) which can be solved analytically and compared both with the $n=2$ case of Sec. II and the inhomogeneous case. Hence the equation for $y(t)$ reads as

$$\frac{1}{2}\frac{dy}{dt}=(r+um^2)y+\frac{u\epsilon}{V}\left[\frac{1}{\alpha_I}+\int_0^t dt'y(t')\right]. \quad (35)$$

From now on we will use the HS limit, in which Eq. (35) can be reduced to

$$y(\tau)=\mu^2\left[\frac{h_I\alpha_F}{h_F\alpha_I}-\tfrac{1}{2}\int_0^\tau d\tau'\sqrt{y(\tau')}\right]^2+\nu\left[\frac{\alpha_F}{\alpha_I}+\int_0^\tau d\tau' y(\tau')\right], \quad (36)$$

where we defined the scaled time $\tau=\alpha_F t$ and the constants

$$\mu=\frac{2h_F}{(|r|/u)^{1/2}\alpha_F},$$

$$\nu=(u\epsilon/|r|V\alpha_F),$$

that verify the relation $\mu^2+\nu=1$, derived from Eq. (34), which gives

$$\alpha_F\approx\frac{2h_F}{(|r|/u)^{1/2}}.$$

Equation (36) can be readily solved

$$y(\tau)=\left[\frac{1-\nu}{1-(\nu/2)}e^{-(1-\nu)(\tau/2)}+\frac{\nu}{2[1-(\nu/2)]}e^{\tau/2}\right]^2, \quad (37)$$

$$m(\tau)=\left[\frac{|r|}{u}\right]^{1/2}\mu\frac{1-\nu/2e^{[1-(\nu/2)]\tau}}{1-\nu+(\nu/2)e^{[1-(\nu/2)]\tau}}, \quad (38)$$

for $h_F=h_I$ and therefore $\alpha_F=\alpha_I$. The behavior of the magnetization, shown in Fig. 1, has a close analogy with the behavior of the two components model discussed in Sec. II. However, a qualitative difference is evident as far as higher moments are concerned and is a consequence of the fact that in such a limit the process is Gaussian at any time. The inversion time for the magnetization is easily estimated from Eq. (38) in the case $\nu\ll 1$ when $\alpha_F\approx 2h_F/(|r|/u)^{1/2}$,

$$\tau_i\approx\ln\left[\frac{4Vh_F}{\epsilon}\left[\frac{|r|}{u}\right]^{1/2}\right], \quad (39)$$

and it is very close to the one given by Eq. (15). It is also important to note the behavior of the function $y(t)$ in the early and final stages of the relaxation (see Fig. 3). It is very close, respectively, to a decreasing or an increasing exponential with a time constant inversely proportional to h_F. As it is clear from Eq. (38) the magnetization inversion occurs only because of the presence of thermal fluctuations which are able to move the system away from the unstable state. In the homogeneous limit these fluctuations are vanishingly small for large volume.

B. Inhomogeneous fluctuations

The situation is quite different in the inhomogeneous case; in fact, thermal fluctuations are present even in the infinite volume limit. In such a case Eq. (28) reads, in the HS limit,

$$y(\tau)=\mu^2\left[\frac{h_I\alpha_F}{h_F\alpha_I}-\tfrac{1}{2}\int_0^\tau d\tau'\sqrt{y(\tau')}\right]^{1/2}+\nu\left[\frac{\alpha_F}{\alpha_I}+\int_0^\tau d\tau' y(\tau')\right]+\xi^{1/2}\left[\left[\frac{\alpha_I}{\alpha_F}\right]^{1/2}\bar{F}\left[\frac{\alpha_I}{\alpha_F}\tau\right]+\int_0^\tau d\tau'\bar{f}(\tau-\tau')y(\tau')\right] \quad (40)$$

with

$$\zeta^{1/2}=\alpha_F^{1/2}\left[\frac{4\pi}{\epsilon}\frac{|r|}{u}\right]^{-1}(2D)^{-3/2},\quad 1-\mu^2+\zeta^{1/2}-\nu=0\ ,$$

$$\bar{F}(\tau)=\frac{1}{\sqrt{\pi\tau}}[1-\sqrt{\pi\tau}e^{\tau}\mathrm{erfc}(\sqrt{\tau})]\ , \tag{40a}$$

$$\bar{f}(\tau)=\frac{2}{\pi}\int_0^\infty dx\,x^2\left[e^{-x^2\tau}-\frac{2}{x^2}\delta(\tau)\right]\ ,$$

and thermal fluctuations appear through the last term on the right-hand side (rhs). This term has a memory effect which, for small enough ϵ, is small for small times, becomes important at intermediate times and again irrelevant for long times.

The numerical solution of Eq. (40) is plotted in Fig. 3. We distinguish three time regimes. In the first one, thermal fluctuations are irrelevant and the system behaves as in the homogeneous case. Then in analogy with the case $n=2$, the system spends some time in the vicinity of the unstable point before leaving it definitively. The subsequent relaxation from the unstable state and reorientation along the new magnetic field can be described by the approximate equation we obtain in the following way in the case $\nu=0$ and for $h_I=h_F$. We make a perturbative expansion in $\zeta^{1/2}$ which gives asymptotically for long times $y(\tau)\approx(\zeta^{1/2}/2\sqrt{\pi})\tau^{-3/2}$. Since the nonlinear memory term in Eq. (40) is only important for intermediate times, we substitute it with the previous asymptotic results. Use of the auxiliary variable

$$z(\tau)=1-\tfrac{1}{2}\int_0^\tau d\tau'\sqrt{y(\tau')} \tag{41}$$

then leads to

$$4\left[\frac{dz}{dt}\right]^2=z^2+\frac{\zeta^{1/2}}{2\sqrt{\pi}}\tau^{-3/2} \tag{42}$$

valid at intermediate times. The connection between the early stage behavior and the intermediate one gives an estimate of the separation time between the two regimes. Setting equal the two terms of the rhs of Eq. (42) with $\alpha_F\approx 2h_F/(|r|/u)^{1/2}$ yields approximately

$$\tau_1\approx\ln\left[\frac{2\sqrt{\pi}}{\zeta^{1/2}}\right]\approx\ln\left[\frac{8\pi^{3/2}}{\epsilon(2h_F)^{1/2}}\left[\frac{|r|}{u}\right]^{5/4}(2D)^{3/2}\right]. \tag{43}$$

At such time the function $z(\tau)$ reaches extremely small values, of the order of $\zeta^{1/4}$, then z varies as $z(\tau)\sim\zeta^{1/4}[\tau_1^{-3/4}+4(\tau_1^{1/4}-\tau^{1/4})]$ and becomes negative at the inversion time τ_i which is thus very close to τ_1, i.e., $\tau_i\approx\tau_1+1$. At this point $y(t)$ has a minimum of the order $4e^{-\tau_1}$. Then $y(t)$ eventually reaches the asymptotic regime obtained by matching with the preceeding value at $y(t)\sim 4e^{\tau-2\tau_i}$. This is shown in Fig. 3 where one clearly sees the initial decreasing exponential regime $y\sim e^{-\tau}$, the very short scaling intermediate regime and the final one $y\sim e^{\tau}$. It is clear from Eq. (43) that the inversion time varies logarithmically with the small external magnetic field. Such a variation is shown in Fig. 4 to hold on a range of values of h_F large enough to exclude the power-law dependence as hypothesized by Mazenko and Zannetti.[2]

Once the equation for the nonlinear relaxation function is solved it is easy to calculate all the relevant physical quantities. As far as the structure factor is concerned, it is given in terms of $y(t)$ by

$$S_q(\tau)=\frac{\epsilon}{\alpha_F}\left[\frac{\alpha_F e^{-(2Dq^2/\alpha_F)\tau}}{2Dq^2+\alpha_I}+\int_0^\tau d\tau' e^{-(2Dq^2/\alpha_F)(\tau-\tau')}\frac{y(\tau')}{y(\tau)}\right]. \tag{44}$$

Regarding the scaling behavior of this function we must note that it is verified in the intermediate time regime in which the system has already lost memory of the initial correlation and does not yet feel the presence of the new magnetic field. This is the regime in which the nonlinear relaxation function $y(t)$ has a power-law behavior. The extension of the scaling regime appears to be quite narrow, corresponding to times in which y is not represented by an exponential curve, as shown in Fig. 3. This scaling

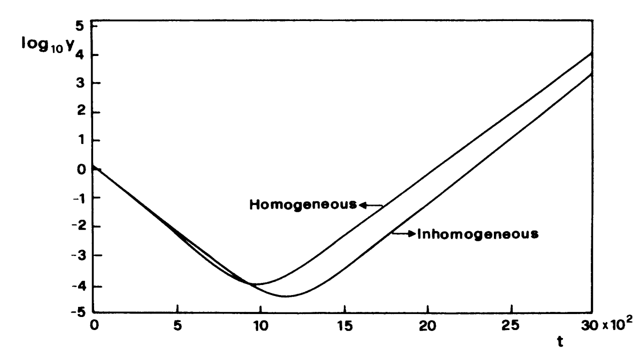

FIG. 3. The relaxation function $y(t)$ in the spherical limit for homogeneous and inhomogeneous fluctuations. Parameters as in Fig. 1.

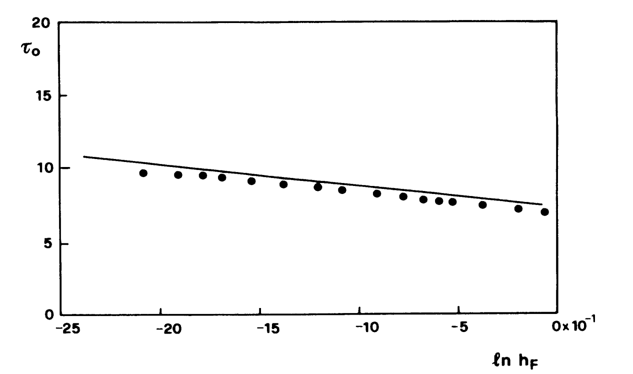

FIG. 4. Scaled time for the vanishing of the magnetization as a function of the magnetic field. Parameters same as in Fig. 1. The dots refer to the numerical solution, the line to Eq. (43).

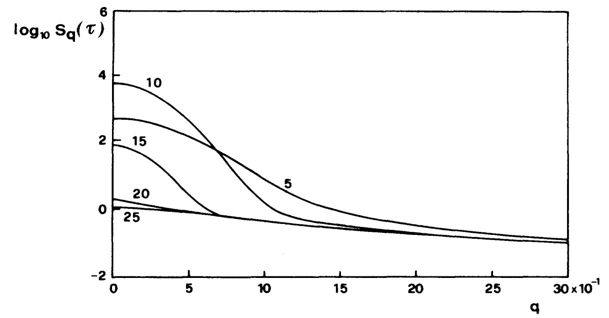

FIG. 5. The structure factor $S_q(\tau)$ for $h_I = h_F = 8.4\times 10^{-2}$ for the scaled times $\tau=5$, 10, 15, 20, and 25.

regime could be evidentiated starting with an initial state with very short correlation length (strong initial magnetic field). This kind of scaling behavior has been widely studied in Ref. 2 and our results are in complete agreement with theirs. The global evolution of the structure factor is shown in Fig. 5 for equal values of the magnetic field in initial and the final state. Figure 6 shows $S_q(\tau)$ for a weak initial magnetic field corresponding to a long correlation length. It evidentiates, especially when comparing with the situation of Fig. 5, the fact that the memory of the initial correlation survives for long times after the external field variation.

The global behavior of the structure factor can be described in terms of the time-dependent correlation length defined as

$$l^2(\tau) = \frac{1}{6}\frac{\int_0^\infty dx\, x^2 S(x,t)}{\int_0^\infty dx S(x,t)} = -\frac{1}{6S_0(t)}\sum_{i=1}^{3}\frac{\partial^2}{\partial q_i^2}S_q(t)\Bigg|_{q=0} . \qquad (45)$$

Using this definition and the equation of motion (26), with the initial condition $S_q(0)=\epsilon/(2Dq^2+\alpha_I)$, one gets the expression of $l^2(\tau)$ as a function of $y(\tau)$,

$$\frac{l^2(\tau)-l_I^2}{l_I^2} = \frac{\tau + \int_0^\tau d\tau'(\tau-\tau'-1)y(\tau')}{1+\int_0^\tau d\tau' y(\tau')} , \qquad (46)$$

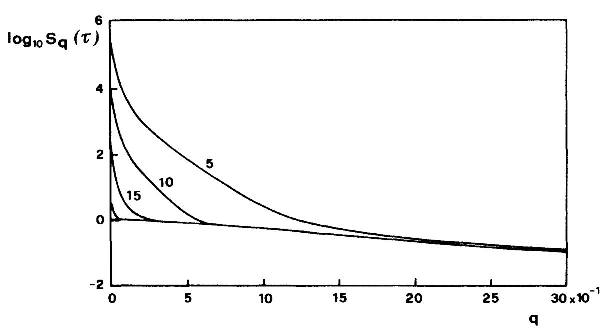

FIG. 6. The same as in Fig. 5 for $h_I = 8.3\times 10^{-6}$.

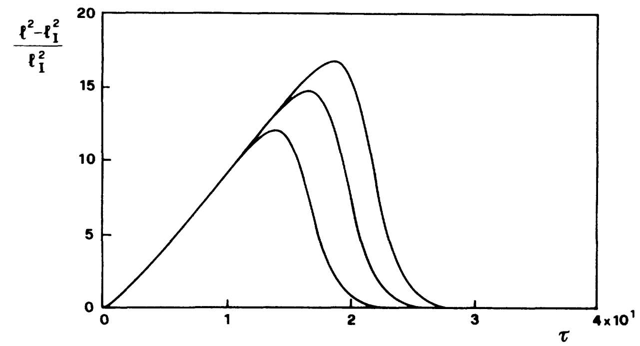

FIG. 7. The percentage increase of the squared correlation length as a function of the scaled time for decreasing magnetic fields $h_I = h_F = 8.4\times 10^{-1}$, 8.4×10^{-2}, and 8.4×10^{-3}.

where now $\tau=\alpha_I t$, and $l_I^2 = 2D/\alpha_I$ is the initial correlation length. At times $\tau \ll 1$ we get $l^2 - l_I^2/l_I^2 \sim \tau^2$, while for $1 \ll \tau < \tau_i$, $l^2 - l_I^2/l_I^2 \sim \tau$. This is the scaling regime with the typical square root of time law, characteristic of a system close to an unstable state. Neglecting the very small contribution of the short intermediate region and using the asymptotic form of y gives for $\tau > \tau_i$

$$\frac{l^2(\tau)-l_I^2}{l_I^2} \sim \frac{\tau}{1+2e^{\tau-2\tau_i}} . \qquad (47)$$

which shows the same behavior until it reaches a maximum at $\tau \sim 2\tau_i$ and then rapidly decays to zero. This behavior is verified numerically as shown in Fig. 7. It is a manifestation of the reorientation of the magnetization along the new direction of the magnetic field since the new maximum of the local potential is felt by the system. The anomalous increase of the correlation length is a qualitative feature associated with the decay from the unstable state.

It is worthy to emphasize that the time at which the correlation length has a maximum, i.e., $t \sim 2\tau_i/\alpha_I$ is dominated by the spatial correlation length associated to the initial state, i.e., $l_I = (2D/\alpha_I)^{1/2}$. The latter quantity can be very large if the system is initially in the presence of a vanishingly small magnetic field. This fact clearly illustrates the importance of long-range field configurations in the statistical properties of the system during the transient. The weight of these configurations is obviously emphasized in a quantity like the correlation length. From this point of view the possibility of having a correlation length which is still growing for times greater than the inversion time is not unexpected.

IV. CONCLUSIONS

Orientational relaxation is a conceptually simple phenomenon in which many topical aspects of nonlinear relaxation can be evidentiated. First of all we found all the phenomenology of the decay from an unstable state. This aspect has been analyzed in full detail in the case in which only homogeneous fluctuations are taken into account. At first glance such an approximation could be considered sufficient to describe the general case of the orientational decay. In fact, the system with inhomogeneous fluctuations can be considered as an ensemble of independent subsystems whose typical size is the final fluctuation correlation length $l_F=(2D/\alpha_F)^{1/2}$. Actually one observes that the inhomogeneous inversion time [Eq. (43)] is nearly equal to the homogeneous one for a system of volume $V=l_F^3$. Such a picture actually works as long as the average magnetization inversion time is considered. However, the fluctuation correlation length is a quantity which changes significantly during the transient, as we have seen in the discussion of inhomogeneous fluctuations in the large-n limit. Because of this circumstance the previous picture becomes ambiguous. This anomalous behavior is a new feature of the decay of an unstable state which is closely related to the possible existence of a scaling regime for the fluctuation correlation function.

Let us recall the mechanism underlying a scaling regime for a system with continuous symmetry in the case of nonlinear relaxation after a temperature quench in the absence of external magnetic field. In a discretized version of the model we have a local potential in each site with a continuous degeneracy of equilibrium states. Below the transition temperature the local thermal noise is not sufficient to prevent the growth of the spatial correlation of fluctuations which occurs through reorientation of the local magnetization. This is the mechanism underlying the growth of fluctuations that are macroscopic in size and their spatial correlations. In the case of orientational relaxation the system is close to the unstable state and the situation is not very different from the previous one if the magnetic field is sufficiently small. The main difference is indeed in the more limited time range in which scaling behavior can occur. Scaling is in fact verified in the intermediate time range in which the system has already lost memory of the initial spatial correlations, but is not yet affected by the presence of the new magnetic field. This time interval can be very small because of the following circumstances. The initial state can have quite large spatial correlations due to a small initial magnetic field. In this case field configurations with long-range spatial correlations have a statistical weight which is not negligible at intermediate times, and this prevents the emergence of scaling behavior in the correlation function. The spatial correlation of the final state is not related to the size of the system, as in the case of temperature quenches, but to the magnetic field and it is consequently expected to be a much smaller quantity.

Finally we want to mention the limitations of the theoretical approaches we used. The quasideterministic theory, in the simple form we adopted, does not describe the equilibrium fluctuations, although they can be taken into account introducing perturbative corrections.[5] Another limitation of the theory is the difficulty to extend it to systems where the spatial dependence of the fluctuations is important, although some attempt in this direction has been made.[5,19] As far as the spherical limit is concerned, the fact that the relevant stochastic processes for the components of the order parameter are Gaussian is the simplification that allows a consistent analytic solution of the model but at the same time constitutes its limitation.

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[1]See, for example, D. Gunton, M. San Miguel, and P. S. Sahni, in *Phase Transitions and Critical Phenomena,* edited by C. Domb and J. L. Lebowitz (Academic, London, 1983), Vol. 8.

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