Behavior of solutions in extended Klein-Gordon systems approaching asymptotically stationary outgoing solitary-wave solutions

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We present a new analytical method to see the exact dynamic behavior of solutions in extended Klein-Gordon systems describing a continuum containing a dissipation term and an external force term. The solutions are investigated geometrically in a state plane by transforming the equation into three basic equations each of which is associated with derivatives with respect to x, t, and ϕ, respectively, under the condition that the solutions approach asymptotically the stationary solitary wave solution as $|x|$ and $|t|$ approach infinity. The waves on ϕ_t are divided into the traveling-wave component, $V(x,t)$, and the others. The state plane is then constructed by $V(x,t)$ and ϕ. The method for analyzing the soliton-antisoliton interaction is described in detail. The analytical method is applied to the extended sine-Gordon system. Then, local distortions are found during the soliton-antisoliton interaction instead of the emission of radiation, and their mechanism and properties are clarified in detail from the singularities of the solution.

I. INTRODUCTION

Nonlinear Klein-Gordon equations such as the ϕ^4 equation, the sine-Gordon equation, the multiple sine-Gordon equation, and so on are applied to many areas of condensed-matter physics.[1] The more general forms of nonlinear Klein-Gordon equations are the equations in which an external force term and a dissipation term is taken into account. We refer to it as extended Klein-Gordon systems. As one of the typical examples of such systems, we exemplify a soliton (fluxon) dynamic system on the Josephson transmission line biased uniformly by dc current,[2] where the equation is constructed by adding a dissipation term and an external force term to the sine-Gordon equation. Another typical example of the systems is the dynamics of domain walls in ferrodistortive materials,[3,4] where the equation is constructed by adding a dissipation term and an external force term to the ϕ^4 equation. We may consider that the former follows essentially the differential equation which expresses a continuum system, and that the latter originates in the difference equation which expresses a discrete system. The investigation on dynamic solutions has been made mainly for the former systems by a computer simulation,[5-11] where we take advantage of the finite-difference equation, and by an analog simulation,[12-15] because of nonintegrability of the corresponding differential equation. In the above simulations, it is certain that we obtain an exact solution for a discrete system, but is not certain whether the result always agrees with the real solution, if we apply them to such continuum systems, for instance, as describing fluxon dynamics in the Josephson transmission line, and optical transparency in a degenerate atomic medium[16,17] and so on.[1,6] A perturbation method[18,19] is sometimes used for such continuum systems, if perturbation is small, but is powerless for such phenomena as large perturbation. Moreover, there may be phenomena not treated by this method.

The geometrical method in a state plane (phase plane), which consists of the relation between the derivative of the amplitude of waves ϕ with respect to the time coordinates t, i.e., ϕ_t, and ϕ itself, is a useful means for the investigation of exact stationary solutions to the extended sine-Gordon system.[2,20] This method may be applied to obtain a stationary solution in other extended Klein-Gordon systems. One of the typical features of this method is in that an autonomous first-order ordinary differential equation is treated on the state plane instead of treating an original second-order partial differential equation on real space. As a result, the region of the solution on the state plane is limited to a certain area, though it expands infinitely in the conventional treatment on real space. From the above fact, we can treat geometrically the exact solution easier than doing in real space. That is, once we know the field of directions in the state plane, which is the moved direction of solutions with increasing $\xi=x-ut$, where x is the one-dimensional space coordinate and u is the kink (phase) velocity of the wave in the stationary state, we can visually clarify the properties of solutions in the plane qualitatively almost with a freehand drawing, and then can easily calculate the necessary solution quantitatively.[20]

In this paper, we try to extend the above analytical method using a state plane to the method to obtain a dynamic solution in extended Klein-Gordon systems as a continuum described by

$$\phi_{xx}-\phi_{tt}-F(\phi)=G\phi_t-J_B \ , \tag{1.1}$$

under the condition that the solution approaches asymptotically stationary solitary wave solutions as both $|x|$ and $|t|$ approach infinity, where G is a dissipation coefficient (conductance) and J_B a uniformly applied external force (external current). Here, we do not try to analyze the solution to Eq. (1.1) with known functional forms, nor to associate the equation with a linear differential equation, nor to take advantage of a perturbation method, since the equation is nonintegrable. Instead, we make an effort to look for a type of equations equivalent to Eq. (1.1) which can easily be analyzed geometrically by using the state-plane technique.

There exist two kinds of phase planes for Eq. (1.1), describing the relation between ϕ_t and ϕ, and the relation between ϕ_x and ϕ, respectively. However, since ϕ_t and ϕ_x are not independent of each other, it is convenient to define a state plane which is common to these planes. To do so, we express ϕ_t and ϕ_x as

$$\phi_t = V(x,t)g(x,t) , \tag{1.2}$$

$$\phi_x = -V(x,t)h(x,t)/u , \tag{1.3}$$

introducing a state variable $V(x,t)$, and other variables $g(x,t)$ and $h(x,t)$. Then, we can construct the state plane representing the relation between $V(x,t)$ and ϕ. In Eqs. (1.2) and (1.3), if $g(x,t)$ and $h(x,t)$ are unity irrespective of x and t, ϕ_t and ϕ_x construct a stationary solution for Eq. (1.1) since they satisfy the following equation;

$$\phi_t + u\phi_x = 0 . \tag{1.4}$$

In this special case, the state plane is identical to the phase plane consisting of (ϕ,ϕ_t).

The most typical feature of this treatment is that we can derive three basic equations each of which is associated with the derivatives with respect to x, t, and ϕ, respectively, and is quite equivalent to Eq. (1.1). This is due to the fact that $V(x,t)$ is included in common in Eqs. (1.2) and (1.3). The equations associated with the derivatives with respect to x and t are regarded as ordinary differential equations, respectively, if t is fixed in the former and x is fixed in the latter, and if the functional forms of $g(x,t)$ and $h(x,t)$ are known. The functional forms of $g(x,t)$ and $h(x,t)$ are determined by imposing the initial and the boundary conditions on the state plane such that the solution approaches the previously well-defined stationary solitary-wave solution as both $|x|$ and $|t|$ approach infinity. Thus, we can construct the solution from any of the derived equations starting at the stationary state and developing to the nonstationary state by using a conventional analytical technique for the ordinary differential equation without losing any distinctive features of the analytical method in terms of the state plane technique for the stationary state. Moreover, we can obtain the exact solution in some cases, since we need in principle to use no perturbation technique in our method.

The plan of this paper is as follows. In Sec. II, we derive three basic equations equivalent to Eq. (1.1). In Sec. III, ϕ_t and ϕ_x are divided into the traveling-wave and the standing-wave component and/or the decaying radiation component by introducing nonlinear coordinates. In Sec. IV, the stationary state is described in association with the state plane. In Sec. V, the properties of $g(x,t)$ and $h(x,t)$ are investigated, and in Sec. VI the natures of the nonlinear coordinates at singular points are clarified in association with $g(x,t)$ and $h(x,t)$. In Sec. VII, the soliton-antisoliton and the soliton-soliton interaction solutions are derived from our basic equations for the pure sine-Gordon system to confirm the appropriateness of our theory. In Sec. VIII, an analytical method is developed for the soliton-antisoliton interaction. In Sec. IX, the method is applied to the extended sine-Gordon system, and the detailed behavior of solutions is discussed. Then, we find local distortions of waves produced by moving singularities, and its mechanism is clarified. In Sec. X, we summarize our results.

II. DERIVATION OF BASIC EQUATIONS

The solution ϕ to Eq. (1.1) is determined by designating x and t. We can then consider that ϕ_x and ϕ_t are determined through ϕ by designating x and t, if we keep phase planes consisting of (ϕ,ϕ_x) and (ϕ,ϕ_t) in mind. Thus, we can express ϕ_{xx} and ϕ_{tt} as

$$\phi_{xx} = \phi_x \frac{\partial \phi_x}{\partial \phi}, \quad \phi_{tt} = \phi_t \frac{\partial \phi_t}{\partial \phi} .$$

If we integrate Eq. (1.1) with respect to ϕ using above relations, we obtain the following equation:

$$\int \phi_x d(\phi_x) - \int \phi_t d(\phi_t) = \int [F(\phi) + G\phi_t - J_B] d\phi . \tag{2.1}$$

For convenience, we assume throughout this paper that $J_B \geq 0$ and $u^2 < 1$, and that $F(\phi)$ repeats exactly the same shape with a certain period of ϕ. The extended sine-Gordon system, the extended multiple sine-Gordon system, and so on satisfy the latter condition.

We insert, in part, Eqs. (1.2) and (1.3) into Eq. (2.1) and differentiate the result with respect to ϕ. Equation (2.1) is then rewritten as

$$\frac{1}{u^2} h^2(x,t) \left[\frac{\partial V}{\partial \phi}\right]_t - g^2(x,t) \left[\frac{\partial V}{\partial \phi}\right]_x - \zeta(x,t) = \frac{\eta(\phi)}{V(x,t)} , \tag{2.2}$$

where

$$\zeta(x,t) = (1/u)h_x + g_t + Gg(x,t) ,$$

$$\eta(\phi) = F(\phi) - J_B ,$$

$$\left[\frac{\partial V}{\partial \phi}\right]_t = \frac{V_x}{\phi_x} , \tag{2.3}$$

$$\left[\frac{\partial V}{\partial \phi}\right]_x = \frac{V_t}{\phi_t} , \tag{2.4}$$

and the subscripts t and x in $(\ )_t$ and $(\ )_x$ mean that they are fixed, respectively. We integrate Eqs. (2.3) and (2.4) with respect to ϕ, respectively. Then, we obtain $V(x,t)$ for either case. Accordingly, we can set

$$\left[\frac{\partial V}{\partial \phi}\right]_t = \left[\frac{\partial V}{\partial \phi}\right]_x = \frac{\partial V}{\partial \phi} . \tag{2.5}$$

Differentiating Eq. (1.2) with respect to x and Eq. (1.3) with respect to t, and equating them, i.e., $\phi_{xt} = \phi_{tx}$, we obtain the following conservative relation:

$$g_x + h_t/u = 0 \ . \tag{2.6}$$

Taking account of Eq. (2.5), we can rewrite Eq. (2.2) as

$$\beta(x,t)V(x,t)\frac{\partial V}{\partial \phi} - \zeta(x,t)V(x,t) = \eta(\phi) \ , \tag{2.7}$$

where

$$\beta(x,t) = h^2(x,t)/u^2 - g^2(x,t) \ .$$

If $\beta(x,t)$ becomes zero at a certain range of x and t, it is possible for a singularity to exist there. Using Eqs. (1.2), (1.3), and (2.3)–(2.5), we can transform Eq. (2.7) into the following expressions:

$$\phi_{xx} + \mathcal{X}_1(x,t)\phi_x = \frac{h^2(x,t)}{u^2}\frac{\eta(\phi)}{\beta(x,t)} \ , \tag{2.8}$$

$$\phi_{tt} - \mathcal{X}_2(x,t)\phi_t = g^2(x,t)\frac{\eta(\phi)}{\beta(x,t)} \ , \tag{2.9}$$

where

$$\mathcal{X}_1(x,t) = \frac{h(x,t)}{u}\frac{\zeta(x,t)}{\beta(x,t)} - \frac{h_x}{h(x,t)} \ ,$$

$$\mathcal{X}_2(x,t) = g(x,t)\frac{\zeta(x,t)}{\beta(x,t)} + \frac{g_t}{g(x,t)} \ .$$

If the functional forms of $g(x,t)$ and $h(x,t)$ are known, we can regard Eqs. (2.8) and (2.9) as the ordinary differential equations consisting of derivatives with respect to x and t, respectively, each of which is exactly equivalent to the original equation Eq. (1.1).

III. WAVE COMPONENT

Throughout this paper, we assume that there exists a stationary solitary wave at a position where both $|x|$ and $|ut|$ are effectively regarded as infinity. If there is no cause to disturb the solitary wave anywhere, the wave keeps the stationary state, where $g(x,t) = h(x,t) = 1$. On the other hand, if there is a cause to do it somewhere, the wave grows to deviate from the stationary state as approaching the source of disturbance, so that $g(x,t)$ and $h(x,t)$ also deviate from unity.

By expanding each of $g(x,t)$ and $h(x,t)$ into a power series consisting of only a function of x and only a function of t, and by using Eq. (2.6) under the condition that both $g(x,t)$ and $h(x,t)$ should at least approach only a function of t as $|x|$ approaches infinity, and should at least approach only a function of x as $|t|$ approaches infinity, since $g(x,t)$ and $h(x,t)$ approach unity with approaching both $|x|$ and $|t|$ to infinity, it is proved that $g(x,t)$ and $h(x,t)$ are finally expressed by[21]

$$g(x,t) = g(t) \ , \tag{3.1}$$

$$h(x,t) = h(x) \ . \tag{3.2}$$

In order to see the meaning of $V(x,t)$, we divide ϕ_t and ϕ_x, not ϕ, into two components expressed by

$$\phi_t = (\phi_t)^{(t)} + (\phi_t)^{(r)} \ ,$$

$$\phi_x = (\phi_x)^{(t)} + (\phi_x)^{(r)} \ ,$$

where

$$(\phi_t)^{(t)} = V(x,t) \ ,$$

$$(\phi_t)^{(r)} = [g(t) - 1](\phi_t)^{(t)} \ ,$$

$$(\phi_x)^{(t)} = -V(x,t)/u \ ,$$

$$(\phi_x)^{(r)} = [h(x) - 1](\phi_x)^{(t)} \ .$$

It is then noted that $(\phi_t)^{(t)}$ and $(\phi_x)^{(t)}$ satisfy

$$(\phi_t)^{(t)} + u(\phi_x)^{(t)} = 0 \ . \tag{3.3}$$

From the analogy between Eqs. (3.3) and (1.4), we can expect that $(\phi_t)^{(t)}$ and $(\phi_x)^{(t)}$ will construct a traveling-wave component. However, as they interact with $(\phi_t)^{(r)}$ and $(\phi_x)^{(r)}$, they constantly change their shape. Accordingly, the conservative relation such as Eq. (1.4) is not preserved in Eq. (3.3).

We express $(\phi_t)^{(t)}$ and $(\phi_x)^{(t)}$ by

$$(\phi_t)^{(t)} = \phi_T, \quad (\phi_x)^{(t)} = \phi_X \ , \tag{3.4}$$

by introducing nonlinear coordinates, $T(x,t)$ and $X(x,t)$. Then, $T(x,t)$ and $X(x,t)$ are given from Eqs. (1.2) and (1.3) by

$$T(x,t) = \int^t \frac{1}{V(x,t')}\frac{\partial \phi}{\partial t'}dt' = \int^t g(t')dt' + T_0(x) \ , \tag{3.5}$$

$$X(x,t) = -\int^x \frac{u}{V(x',t)}\frac{\partial \phi}{\partial x'}dx' = \int^x h(x')dx' + X_0(t) \ , \tag{3.6}$$

where $T_0(x)$ and $X_0(t)$ are arbitrary functions of x and t, respectively. We also introduce $\Xi(x,t)$ coordinates, which move with a constant velocity u, expressed by

$$\Xi(x,t) = -u\int^\phi \frac{d\phi}{V(x,t)} = X(x,t) - uT(x,t) \ . \tag{3.7}$$

By using Eq. (3.7), Eq. (3.4) is also written as

$$(\phi_t)^{(t)} = -u\frac{\partial \phi}{\partial \Xi}, \quad (\phi_x)^{(t)} = \frac{\partial \phi}{\partial \Xi} \ . \tag{3.8}$$

Accordingly, we can replace $V(x,t)$ by $V(\Xi)$. As the solution approaches the stationary state, $T(x,t)$ and $X(x,t)$ approach t and x, respectively (see also Sec. V). As a result, $\Xi(x,t)$ approaches ξ. Thus, we understand that $T(x,t)$, $X(x,t)$, and $\Xi(x,t)$ correspond to t, x, and ξ for the stationary state, respectively.

We refer to the components having (t) and (r) in the superscript as the (t) component and the (r) component of the wave, respectively. Then, the (r) component from its definition must disappear as $|x|$ and $|t|$ approach infinity at the same time, since $g(t)$ and $h(x)$ approach unity. Thus, we can expect that the (r) component constructs a standing wave and/or decaying radiation.

We can rewrite Eq. (1.1) using the $\Xi(x,t)$ coordinates as

$$V_\Xi = -u[F'(\Xi) + GV(\Xi) - J_B]/(1 - u^2) \ . \tag{3.9}$$

where

$$F'(\Xi)=F(\phi(\Xi))+F^{(r)}(\Xi)\ ,$$
$$F^{(r)}(\Xi)=-(\phi_x)_x^{(r)}+(\phi_t)_t^{(r)}+(\Xi_x-1)V_\Xi/u$$
$$+(\Xi_t+u)V_\Xi+G(\phi_t)^{(r)}\ .$$

Equation (3.9) is in agreement with the expression on the traveling wave in the stationary state, if $\Xi(x,t)$ and $F'(\Xi)$ are replaced by ξ and $F(\phi)$, respectively [see Eq. (4.1)]. Accordingly, from Eqs. (3.7) and (3.9), it may be considered that $(\phi_t)^{(t)}$ and $(\phi_x)^{(t)}$ construct the traveling wave having a constant velocity u in an extended Klein-Gordon system with $F'(\Xi)$ in the $\Xi(x,t)$ coordinates instead of $F(\phi)$ in the ξ coordinate. The equation on the state plane can be written from Eqs. (3.8) and (3.9) as

$$\frac{\partial V}{\partial \phi}=\frac{u^2}{1-u^2}\,\frac{F'(\Xi)+GV(\Xi)-J_B}{V(\Xi)}\ . \tag{3.10}$$

Equation (3.10) is compared to the expression for stationary solitary waves in the extended sine-Gordon system.[20] The solution to Eq. (3.10) has singular points if the denominator and the numerator vanish simultaneously. Such singular points correspond to $|\Xi|\to\infty$, where $V(\Xi)\to 0$.

IV. STATIONARY STATE

In Eq. (3.9) if both $|x|$ and $|ut|$ are much larger than unity, the (r) component disappears, and Eq. (3.7) is rewritten as

$$(1-u^2)\phi_{\xi\xi}+uG\phi_\xi=-u[F(\phi)-J_B]\ , \tag{4.1}$$

since $h(x)$ and $g(t)$ are then regarded as unity, and since $X_0(t)$ and $T_0(x)$ are regarded as constants (see Sec. V). Equation (4.1) describes the stationary state. The equation on the state plane is then expressed from Eq. (3.10) by

$$\frac{\partial V^{(s)}}{\partial \phi}=\frac{u^2}{1-u^2}\,\frac{F(\phi)+GV^{(s)}(\xi)-J_B}{V^{(s)}(\xi)}\ , \tag{4.2}$$

where (s) means the stationary state. The singular point corresponds to $|\xi|\to\infty$ in Eq. (4.2), where $V^{(s)}(\xi)\to 0$. The singularity is determined by setting $\phi=\phi_0+\psi$ and linearizing Eq. (4.1), where ϕ_0 is ϕ at the singular point. The solution to the linearized version of Eq. (4.1) around the singular point can be written as

$$\psi=\psi_{0,\pm}\exp(k_{0,\pm}\xi)\ , \tag{4.3}$$

where $\psi_{0,\pm}$ are constants,

$$k_{0,\pm}=\pm a_{0,\pm}/u\ ,$$
$$a_{0,\pm}=d_1\mp d_2$$
$$d_1=\frac{u}{2}\left[\left[\frac{uG}{1-u^2}\right]^2+\frac{4}{1-u^2}\,\frac{dF}{d\phi}\bigg|_{\phi=\phi_0}\right]^{1/2}, \tag{4.4}$$

$$d_2=\frac{u^2}{2}\,\frac{G}{1-u^2}\ , \tag{4.5}$$

and d_1^2 represents the discriminant. Since $u^2<1$, if $dF/d\phi\,|_{\phi=\phi_0}>0$ in Eq. (4.4), the singularity is a saddle point, and if $dF/d\phi\,|_{\phi=\phi_0}<0$, the singularity is a node or a spiral point depending upon whether the sum of two terms in the square root in Eq. (4.4) is positive or negative.

The stationary solitary-wave solution is constructed by finding a solution starting from a saddle point, and ending to another saddle point, for instance, as depicted in Fig. 1(a). The slopes at the saddle point, $\partial V^{(s)}/\partial\phi\,|_{\phi 0}$, are written as

$$\frac{\partial V^{(s)}}{\partial \phi}\bigg|_{\phi=\phi_{0,2n-2}}=a_{0,-}\ ,\quad \frac{\partial V^{(s)}}{\partial \phi}\bigg|_{\phi=\phi_{0,2n}}=-a_{0,+}\ , \tag{4.6}$$

if the solution is in the region between $(\phi,V^{(s)}(\phi))=(\phi_{0,2n-2},0)$ and $(\phi_{0,2n},0)$, where n is an integer denoting the position of the singular point counted from the origin.[20] The arrowhead on the curve in Fig. 1(a) denotes the field of directions, where we assume $u>0$, which means the forward wave. If $u<0$, which means the backward wave, the field of directions points to the opposite direction. The solution in the state plane leaves the saddle point $(\phi,V^{(s)}(\phi))=(\phi_{0,2n},0)$ toward the upper left with increasing ξ from $-\infty$, finally approaching the saddle point $(\phi_{0,2n-2},0)$ from the upper right as $\xi\to+\infty$. The relation between $V^{(s)}(\xi)$ and ξ and the relation between $\phi(\xi)$ and ξ are depicted in Fig. 1(b) and 1(c), respectively. The relations between Fig. 1(a)–1(c) are described in detail for the extended sine-Gordon system in Ref. 20.

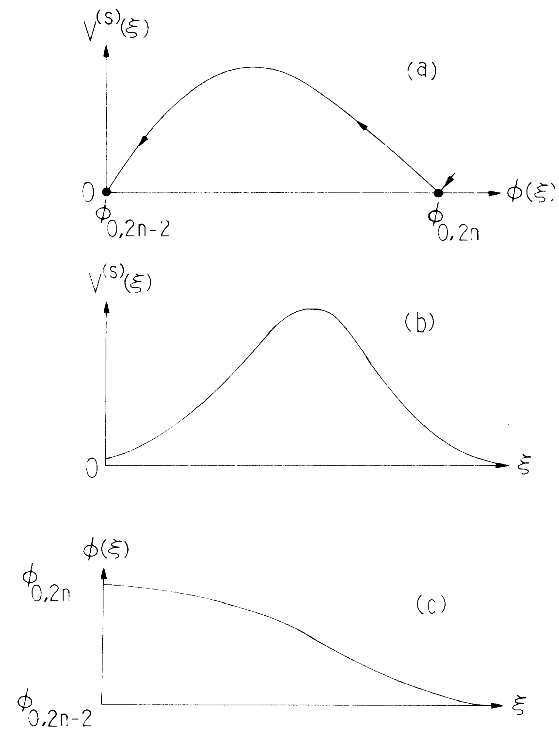

FIG. 1. Stationary-solitary-wave solution. (a) Solution in the state plane. The arrowheads on the curve denote the field of directions, and the dots the positions of the singular points at $|\xi|\to\infty$; (b) $V^{(s)}(\xi)$ vs ξ; (c) $\phi(\xi)$ vs ξ.

V. FUNCTIONAL FORMS OF $g(t)$ AND $h(x)$

We assume that the effect of the disturbance on the wave most dominates at the origin in real space, i.e., $(x,t)=(0,0)$. Then, the disturbance keeps weakened as a whole with increasing $|x|$ and $|ut|$ at the same time so that the wave tends toward the pure traveling wave having the phase velocity of u. This indicates that the (r) component is confined effectively in a region around $(x,t)=(0,0)$. This result also indicates that $\partial V^{(r)}/\partial\phi$ should approach zero even with increasing only $|x|$ or $|ut|$ beyond a certain value. Under such a condition, the (r) component can be disregarded in Eq. (3.9). As a result, $F'(\Xi)$ is replaced by $F(\phi)$ in Eq. (3.9). Thus, Eq. (3.9) is rewritten as

$$V_X=-u[F(\phi)+GV(\Xi)-J_B]/(1-u^2), \quad |x|\gg 1 \tag{5.1}$$

$$V_T=u^2[F(\phi)+GV(\Xi)-J_B]/(1-u^2), \quad |ut|\gg 1 \tag{5.2}$$

where we have used the relation between Eq. (3.4) and (3.8). It is noted however that the state where only $|x|$ or $|ut|\gg 1$ is generally different from the stationary state in Eq. (4.1), because even if $|x|$ approaches infinity, the state should be influenced by $g(t)$ and $X_0(t)$, and even if $|t|$ approaches infinity, the state should be influenced by $h(x)$ and $T_0(x)$. That is, we cannot then simply replace $\Xi(x,t)$ by ξ in Eqs. (5.1) and (5.2). Thus, our approach presents us with more extensive informations than the conventional theory even in the limit to small amplitude of waves. Under such a condition, we can eliminate $F(\phi)$ from Eqs. (2.7) and (3.10). As a result, we obtain the following expression:

$$[g^2(t)-1]\left.\frac{\partial V}{\partial\phi}\right|_{\phi=\phi_0}-G[g(t)-1]+\frac{dg}{dt}=0\,, \quad |x|\to\infty \tag{5.3}$$

$$\frac{1}{u}[h^2(x)-1]\left.\frac{\partial V}{\partial\phi}\right|_{\phi=\phi_0}-\frac{dh}{dx}=0, \quad |t|\to\infty\,. \tag{5.4}$$

Since $\partial V^{(s)}/\partial\phi\,|_{\phi=\phi_0}$ is independent of both x and t as expressed by Eq. (4.6), $\partial V/\partial\phi\,|_{\phi=\phi_0}$ in Eq. (5.3) should be at most only a function of t on account of $|x|\to\infty$. Thus, around the singular point corresponding to $|x|\to\infty$, we can write it by introducing a function of t, $\omega_\pm(t)$, being positive, as

$$\left.\frac{\partial V}{\partial\phi}\right|_{\phi=\phi_0}=\pm\omega_\pm(t)a_{0,\mp}, \quad |x|\to\infty \tag{5.5}$$

where the upper and the lower signs mean that the values of t are positive and negative, respectively.

Since $\partial V/\partial\phi\,|_{\phi=\phi_0}$ should be at most only a function of x in Eq. (5.4) on account of $|t|\to\infty$, we can also write it by introducing a function of x, $\gamma_\pm(x)$, being positive, as

$$\left.\frac{\partial V}{\partial\phi}\right|_{\phi=\phi_0}=\mp\gamma_\pm(x)a_{0,\pm}, \quad |t|\to\infty\,. \tag{5.6}$$

As $|t|$ approaches infinity in Eq. (5.5), $\omega_\pm(t)$ approaches unity, since we are always observing the far other side of the wave measured from the origin in this process. Then, the situation is in a stationary state. If $\omega_\pm(t)$ are equal to unity irrespective of the value of t as an ideal case, $g(t)$ for $t>0$ and $t<0$, $g_\pm(t)$, can be solved from Eq. (5.3), and are written as

$$g_\pm(t)=\begin{cases}\pm[B_\pm\tanh(B_\pm a_{0,\mp}t)-A_\pm], & (5.7)\\ \pm[B_\pm\coth(B_\pm a_{0,\mp}t)-A_\pm], & (5.8)\end{cases}$$

where $A_\pm=G/(2a_{0,\pm})$ and $B_\pm=1\pm A_\pm$.

Notice that if $B_-\le 0$ in Eq. (5.7) and (5.8), $g_-(t)$ does not satisfy the initial condition so that $g_-(t)$ does not converge to unity as $t\to-\infty$. Then, it is seen that the existence region of such a solution is limited so as to satisfy the condition $0\le A_-<1$.

Even if $|x|$ approaches infinity in Eq. (5.6), $\gamma_\pm(x)$ does not necessarily approach unity, since we are always observing the origin side of the wave in this process, and the wave is always influenced by disturbance around $x=0$. $h(x)$ for $t>0$ and $t<0$, $h_\pm(x)$, can be solved from Eq. (5.4), and are written as

$$h_\pm(x)=\begin{cases}(\pm)\tanh\left[\dfrac{a_{0,\pm}}{u}\displaystyle\int_0^x\gamma_\pm(x)dx\right], & (5.9)\\ (\pm)\coth\left[\dfrac{a_{0,\pm}}{u}\displaystyle\int_0^x\gamma_\pm(x)dx\right], & (5.10)\end{cases}$$

where $(\pm)$ signs are applied to $x>0$ and $x<0$, respectively, and the other $\pm$ signs are applied to $t>0$ and $t<0$, respectively, as well as the cases for Eqs. (5.5)–(5.8).

VI. SINGULAR POINTS IN NONLINEAR COORDINATES

We define the following quantity at infinity of t:

$$T_{0,\gamma}(x)=\mp\lim_{t\to\pm\infty}\int_{\pm\infty}^{t}\frac{1}{V(x,t')}\frac{\partial\phi}{\partial t'}dt' =\mp\frac{1}{\gamma_\pm(x)a_{0,\pm}}\,, \tag{6.1}$$

which is the reciprocal of the slope at the singular point at $t=\pm\infty$. Next, we define the following quantity at infinity of x;

$$X_{0,\omega}(t)=\pm u\lim_{x\to\pm\infty}\int_{\pm\infty}^{x}\frac{1}{V(x',t)}\frac{\partial\phi}{\partial x'}dx' =\mp\frac{u}{\omega_\pm(t)a_{0,\mp}}\,, \tag{6.2}$$

which is proportional to the reciprocal of the slope at the singular point at $x=\pm\infty$. We finally define $\Xi_{0,\infty}(x,t)$ from Eqs. (3.7), (6.1), and (6.2) as

$$\Xi_{0,\infty}(x,t)=\pm u\left[\frac{1}{\gamma_\pm(x)a_{0,\pm}}-\frac{1}{\gamma_\pm(t)a_{0,\mp}}\right]. \tag{6.3}$$

Since $T_{0,\gamma}(x)$, $X_{0,\omega}(t)$, and $\Xi_{0,\infty}(x,t)$ are derived from the definition of $T(x,t)$, $X(x,t)$, and $\Xi(x,t)$, respectively, they can be included in $T_0(x)$, $X_0(t)$, and $\Xi(x,t)$, respectively. If we need to choose the kind of the form of Eq. (5.8) for $g_\pm(t)$, then we see that $g_\pm(t)\to\infty$ as $t\to\pm 0$, which means $T(x,t)\to\mp\infty$, that is, the state at $t=0$ describes a singular point in Eq. (3.10). On the other hand, if we need to choose the form of Eq. (5.10) for $h_\pm(x)$, then $h_\pm(x)\to\infty$ as $x\to\pm 0$, which means $X(x,t)\to\mp\infty$. Then, the solution at $x=0$ is in a singular point. Thus, we can also define such expressions as Eqs. (6.1), (6.2), and (6.3) for the state at $t=0$ or $x=0$. There is another kind of singular points at the positions satisfying $\beta(x,t)=0$ as described in Sec. IX, which is also included in $T_0(x)$ and $X_0(t)$.

$V(\Xi)$ is generally not a single-valued function of ϕ on account of the variation of $T_0(x)$ and $X_0(t)$. However, in the stationary state, $\gamma_\pm(x)$ and $\omega_\pm(t)$ become constants, since the slope at the singular point in the state satisfying either $|x|$ or $|t|\to\infty$ is independent of both x and t. Thus, $\Xi(x,t)$ is replaced by ξ, and $V(\xi)$ becomes the single-valued function of ξ, i.e., of ϕ, being replaced by $V^{(s)}(\phi)$.

VII. PURE SINE-GORDON SYSTEM

The properties of the pure sine-Gordon system are well known. So we first apply our basic equations to the pure sine-Gordon system to verify the certainty of our theory. In this case, we can set $G=J_B=0$ and $F(\phi)=\sin\phi$. Then, $a_{0,\pm}$ are written as

$$a_{0,\pm}=a_0=u/(1-u^2)^{1/2}\ .$$

We assume in Eq. (3.10) that

$$F'(\Xi)=\sin\phi\ . \tag{7.1}$$

This assumption means that Eq. (3.10) is in agreement with Eq. (4.2), that is, the expression of the traveling-wave component in the state plane is not influenced by existence of the (r) component. Under such an assumption, Eq. (3.10) is expressed by

$$\frac{\partial V}{\partial\phi}=\frac{a_0^2\sin\phi}{V(\Xi)}\ . \tag{7.2}$$

However, it is again noted that the form of $V(\Xi)$ is quite different from that of $V^{(s)}(\xi)$ in real space. The solution to Eq. (7.2) is given by

$$V(\Xi)=\pm 2a_0\sin(\phi/2)\ . \tag{7.3}$$

Thus, $V(\Xi)$ becomes the single valued function of ϕ. Accordingly, we can replace $V(\Xi)$ by $V(\phi)$.

On the other hand, Eq. (2.7) reduces to

$$\frac{\partial V}{\partial\phi}=\frac{1}{\beta(x,t)}\left[\frac{\sin\phi}{V(\phi)}+\frac{1}{u}\frac{dh}{dx}+\frac{dg}{dt}\right]\ . \tag{7.4}$$

We see from Eqs. (7.2) and (7.3) that $\partial V/\partial\phi\,|_{\phi=\phi_0}$ is independent of both x and t at either $|T|$ or $|X|=\infty$. This means that we do not need to take account of the effect of neither $\gamma_\pm(x)$ nor $\omega_\pm(t)$, that is, $\gamma_\pm(x)=\omega_\pm(t)=1$ in Eq. (5.5) and (5.6). As a result, $\partial V/\partial\phi\,|_{\phi=\phi_0}$ becomes equal to $\pm a_0$, which is in agreement with the value in the stationary state.

Eliminating $\sin\phi$ from Eqs. (7.2) and (7.4), we obtain

$$\frac{\partial V}{\partial\phi}=a_0\frac{\Lambda^2(x,t)-1}{\Lambda^2(x,t)+1}\ , \tag{7.5}$$

where if we adopt Eq. (5.8) for $g(t)$ and Eq. (5.9) for $h(x)$,

$$\Lambda(x,t)=\frac{\sinh(a_0t)}{u\cosh(a_0x/u)}\ , \tag{7.6}$$

and if we adopt Eq. (5.7) for $g(t)$ and Eq. (5.10) for $h(x)$,

$$\Lambda(x,t)=\frac{\cosh(a_0t)}{u\sinh(a_0x/u)}\ . \tag{7.7}$$

Eliminating $V(\phi)$ from Eq. (7.3) and (7.5), we obtain well-known expressions,

$$\phi=\pm 4\tan^{-1}[\Lambda(x,t)]\ , \tag{7.8}$$

for two soliton interactions. The combination of Eqs. (7.6) with (7.8) represents the soliton-antisoliton interaction, and that of Eqs. (7.7) with (7.8) does the soliton-soliton interaction.[22] Thus, we understand that the assumption in Eq. (7.1) is correct for the pure sine-Gordon system.

However, for other nonlinear Klein-Gordon systems such as the ϕ^4 equation, the multiple sine-Gordon equations, and so on, we cannot obtain the correct solution using such an assumption as $F'(\Xi)=F(\phi)$. If $V(\Xi)$ is weakened or strengthened in the state plane during the interaction different from the pure sine-Gordon system, the equation should be deformed from

$$\frac{\partial V}{\partial\phi}=\frac{a_0^2F(\phi)}{V(\Xi)}\ , \tag{7.9}$$

which is obtained by replacing $\sin\phi$ by $F(\phi)$ in Eq. (7.2). This is the situation for the above nonlinear Klein-Gordon systems which are not integrable. Accordingly, we may regard Eq. (7.9) as a condition of the firm solidity of the wave in a pure nonlinear Klein-Gordon system.

ANALYTICAL METHOD FOR SOLITON-ANTISOLITON INTERACTION

A. Energy flow and its related quantities

In the soliton-antisoliton interaction, we choose ϕ in the region between $\phi_{0,2n-2}$ and $\phi_{0,2n+2}$. The net energy flow of the soliton and the antisoliton, $\epsilon^{(t)}$, is obtained by integrating the instantaneous power flow, $p^{(t)}(\Xi)$, on the equivalent transmission line defined by[23]

$$p^{(t)}(\Xi)=-(\phi_x)^{(t)}(\phi_t)^{(t)}\ , \tag{8.1}$$

with respect to t from $-\infty$ to $+\infty$ through $T(x,t)$ under the condition that there is no energy stored in the nonlinear element $F'(\Xi)$ after the whole process is complete. Here, we do not take account of the directionality of the energy flow. Then, the result of the integration is given by[23]

$$\epsilon^{(t)}=2J_B(\phi_{0,2n}-\phi_{0,2n-2})/(uG)\ . \tag{8.2}$$

In the limit to the stationary state, $(\phi_x)^{(t)}$ is replaced by

$-V^{(s)}(\xi)/u$, $(\phi_t)^{(t)}$ by $V^{(s)}(\xi)$, $X(x,t)$ by x, and $T(x,t)$ by t, and then we again obtain Eq. (8.2). Thus, we understand that $\epsilon^{(t)}$ becomes twice the energy of the stationary soliton.

If we differentiate Eq. (8.1) with respect to ϕ, we obtain

$$\frac{\partial p^{(t)}}{\partial \phi} = -\frac{2}{u}\frac{\partial V}{\partial T} . \tag{8.3}$$

We integrate Eq. (8.3) with respect to t from $-\infty$ to $+\infty$ through $T(x,t)$. The result shows zero, and is also in agreement with the result for the limit to the stationary state.

From the above two facts, we may also add the condition that the integration of $\partial^2 p^{(t)}/\partial\phi^2$ with respect to t from $-\infty$ to $+\infty$ through $T(x,t)$ is in agreement with the integration of it in the limit to the stationary state. Thus, we obtain the following expression exhibiting the relation between $\gamma_+(x)$ and $\gamma_-(x)$:

$$\gamma_+(x)a_{0,+} + \gamma_-(x)a_{0,-} = a_{0,+} + a_{0,-} = 2d_1 . \tag{8.4}$$

B. Coordinate transformation and solution at $t=0$

We rewrite Eq. (2.8) or (2.9) as

$$\frac{\partial \phi_t}{\partial x} + \frac{h_\pm(x)}{u\beta_\pm(x,t)}\left[\frac{1}{u}\frac{dh_\pm}{dx} + \frac{dg_\pm}{dt} + Gg_\pm(t)\right]\phi_t = -\frac{g_\pm(t)h_\pm(x)}{u\beta_\pm(x,t)}\eta(\phi) , \tag{8.5}$$

where

$$\beta_\pm(x,t) = h_\pm^2(x)/u^2 - g_\pm^2(t) .$$

We remove the assumption $\omega_\pm(t)=1$ here. From the analogy of soliton-antisoliton interaction solution for the pure sine-Gordon system, we can choose such a kind of the solution as Eq. (5.8) for $g_\pm(t)$, and can correctly do Eq. (5.9) for $h_\pm(x)$. Thus, we can impose the condition just around $t=0$ that $g_\pm^2(t) \gg h_\pm^2(x)$. Then, Eq. (8.5) reduces to

$$\frac{\partial \phi_t}{\partial x} - \frac{h_\pm(x)}{u}\frac{1}{g_\pm^2(t)}\frac{dg_\pm}{dt}\phi_t = 0 . \tag{8.6}$$

In Eq. (8.6), we impose the continuity condition at $t=\pm 0$ as

$$\phi_t|_{t\to+0} = \phi_t|_{t\to-0} = \phi_t|_{t=0} . \tag{8.7}$$

Then, we obtain the following expressions:

$$\left.\frac{\partial \phi_t}{\partial x}\right|_{t=0} (\pm)\frac{d_1}{|u|}h(x)\phi_t|_{t=0} = 0 , \tag{8.8}$$

$$h_-(x) = h_+(x) = h(x) ,$$

$$\omega_\pm(0)a_{0,\mp} = \gamma_\pm(x)a_{0,\pm} = d_1 . \tag{8.9}$$

Equation (8.9) indicates that $\gamma_\pm(x)$ are constants. The solution to Eq. (8.8) is given by

$$\phi_t|_{t=0} = \phi_{t,0}\exp\left[(\mp)\int_0^x \frac{d_1}{|u|}h(x')dx'\right] , \tag{8.10}$$

where $\phi_{t,0}$ is a constant.

In this stage, we do not know the exact form of $g_\pm(t)$ yet. Accordingly, we are obliged to use Eq. (5.8) instead of the exact expression. We rewrite the left side of Eq. (5.8) as $g'_\pm(t)$ for distinction from the exact form.

We try to derive the exact solution from Eq. (8.5) using $g'_\pm(t)$ for the exact value $g_\pm(t)$. We transform the t coordinate into newly introduced $T_\pm$ coordinates so as to satisfy $g'_\pm(t) = g_\pm(T_\pm)$. In this case, we also need to transform x coordinate into $X_\pm$ coordinates so that the states in the $(X_\pm, T_\pm)$ spaces are identical to the state for the exact solution in the (x,t) space. Then, we can also impose the following condition at the limit to $t\to 0$ for the phase velocity $w(x,t)$ defined by

$$\begin{aligned} w(x,t) &= \frac{dx}{dt} = -\frac{\phi_t}{\phi_x} , \\ \frac{dx}{dt} &= \frac{dX_+}{dT_+} = \frac{dX_-}{dT_-} , \end{aligned} \tag{8.11}$$

because of identity of these states.

The transformation from the x coordinate into the $X_\pm$ coordinates is made as follows:

$$a_{0,\pm}\gamma_\pm(x) = d_1 \mp d_2\alpha_\pm(X_\pm) \tag{8.12}$$

by introducing $\alpha_\pm(X_\pm)$. Since Eq. (8.12) should satisfy Eq. (8.4), we can set

$$\alpha_+(X_+) = \alpha_-(X_-) = \alpha(x) .$$

Taking account of Eq. (8.12) for $h_\pm(x)$ in Eq. (5.9), we obtain

$$\int_0^x a_{0,\pm}\gamma_\pm(x')dx' = \int_0^{X_\pm} d_1\left[1 \mp \frac{d_2}{d_1}\alpha_\pm(X'_\pm)\right]dX'_\pm . \tag{8.13}$$

By using Eq. (8.13), $h(x)$ is directly transformed into the value in the $X_\pm$ coordinates. The relation between x and $X_\pm$ is derived from Eq. (8.9) and (8.13) as

$$x = X_\pm \mp \frac{d_2}{d_1}\int_0^{X_\pm}\alpha_\pm(X'_\pm)dX'_\pm . \tag{8.14}$$

Again note that the states at x and $X_\pm$ are identical to each other. We can then transform Eq. (8.6) using $g'_\pm(t)$ at $t=0$ into[21]

$$\left.\frac{\partial \phi_t}{\partial X_\pm}\right|_{T_\pm\to 0} - \frac{h(x)}{u}\frac{1}{g_\pm^2(T_\pm)}\frac{dg_\pm}{dt}\frac{dx}{dX_\pm}\phi_t\Big|_{T_\pm\to 0} = 0 . \tag{8.15}$$

The solution to Eq. (8.15) is expressed by

$$\phi_t|_{T_\pm\to 0} = \phi_{t,0}\exp\left[(\mp)\int_0^{X_\pm}\frac{d_1}{|u|}h(x)\left[1 \mp \frac{d_2}{d_1}\alpha_\pm(X'_\pm)\right]dX'_\pm\right] , \tag{8.16}$$

where we use the relation in Eq. (8.13) for $h(x)$. Notice that

$$\int_0^{X_+} \alpha_+(X'_+)dX'_+ = \int_0^{X_-} \alpha_-(X'_-)dX'_- = \int_0^{x} \alpha(x')dx' , \quad (8.17)$$

because of identity of these states. It is then clear that Eq. (8.16) is in agreement with Eq. (8.10).

C. Analytical method

In Eq. (2.9) in which $g_\pm(t)$ is replaced by $g'_\pm(t)$, we transform x coordinate into $X_\pm$ coordinates using Eq. (8.14) and t coordinates into $T_\pm$ coordinates using the relation $g'_\pm(t)=g_\pm(T_\pm)$, and also use the identity condition, $dX_\pm/dT_\pm=dx/dt$. As a result, we obtain the following equation:[21]

$$\frac{\partial\phi_t}{\partial T_\pm} - \left[\frac{g_\pm(T_\pm)}{\beta'_\pm(x,t)} \left(\frac{1}{u}\frac{dh_\pm}{dX_\pm} + \frac{dg_\pm}{dT_\pm} + Gg(T_\pm) \right) + \frac{1}{g_\pm(T_\pm)}\frac{dg_\pm}{dT_\pm} \right] \phi_t = \frac{g_\pm^2(T_\pm)\eta(\phi)}{\beta'_\pm(x,t)} , \quad (8.18)$$

where

$$\beta'_\pm(x,t)=h_\pm^2(X_\pm)/u^2-g_\pm^2(T_\pm) . \quad (8.19)$$

We see that Eq. (8.18) is in agreement with Eq. (2.9) in the form.

Taking account of Eq. (8.17) in Eq. (8.14), we obtain

$$x=(X_+ +X_-)/2 , \quad (8.20)$$

$$t=T_\pm \mp \tau(x,t) , \quad (8.21)$$

where, $\tau(x,t)$ is the time difference in traveling between x and $X_\pm$, and is expressed by

$$\tau(x,t)=l(x)/w(x,t) ,$$

and

$$l(x)=\frac{d_2}{d_1}\int_0^x \alpha(x')dx' .$$

It is noted that $\tau(x,0)=0$ since $w(x,0)=\infty$.

The waveforms of ϕ_t in the $(X_\pm,T_\pm)$ spaces are calculated from Eq. (8.18) first by starting from the point $(\phi,\phi_t)=(\phi_{0,2n-2},0)$ with increasing T_- and ending to the point $(\phi_{0,2n},\phi_t\,|_{T_-\to 0})$, and next by starting the calculation from the point $(\phi_{0,2n+2},0)$ with decreasing T_+ and ending to the point $(\phi_{0,2n},\phi_t\,|_{T_+\to 0})$ so that $\phi_t\,|_{T_+\to 0}=\phi_t\,|_{T_-\to 0}$. The exact waveforms are obtained by transforming the result into the (x,t) space using Eqs. (8.20) and (8.21).

IX. APPLICATION TO EXTENDED SINE-GORDON SYSTEM

To obtain the soliton-antisoliton interaction solution for the extended sine-Gordon system, the numerical integration of Eq. (8.18) was made by using mainly the Runge-Kutta method, and partly the Euler's method for $G=0.018$ and $J_B=0.4$. As depicted in Fig. 2(a), the solutions for $|x|\gg 1$ in the (ϕ,ϕ_t) plane are completely separated into two isolated states, the stationary soliton and the stationary antisoliton. With decreasing $|x|$, the coupling of the soliton and the antisoliton increases and brings completely the one state at $t=0$. The solution curves in Fig. 2(a) are divided into the traveling-wave component and the (r) component as depicted in Figs. 2(b) and 2(c), respectively. The net waveforms in real space are depicted in Fig. 3(a), and the traveling-wave component and the (r) component are depicted in Figs. 3(b) and 3(c), respectively. It is seen that the (r) component constructs a standing wave. As the soliton and the antisoliton come near to each other, the traveling-wave component decays to be transformed into the standing-wave component. At $x=0$, it is completely replaced by the (r) component.

Next, we direct our attention to local distortions which occur at the positions where the condition $\beta(x,t)=0$ is satisfied: Once the soliton and the antisoliton come near to a certain extent, a wedge-shaped distortion is observed in each wave at the same time. The distortion moves to the direction opposing the corresponding soliton movement. As the waves come closer to each other, the distortion disappears. After the center of the wave passes through each other, a thorn-shaped distortion is generated in the front part of each wave this time. It moves to the direction opposing the corresponding soliton movement, and disappears when arriving around the center of the wave. As these waves go to each other, they approach the stationary state. As a result, $(\phi_t)^{(t)}$ approaches a single-valued function of ϕ.

The conventional computer simulation based upon the corresponding finite difference equation usually brings some emission of radiation in a soliton-soliton interaction on the extended sine-Gordon system,[24] and the soliton-antisoliton interaction for the ϕ^4 system,[5,7] for the double sine-Gordon system,[5,6] and even for pure sine-Gordon system.[6] On the other hand, some analyses have been made on the transmission of a soliton at a microshort put at a position in the extended sine-Gordon system by using a perturbation technique[18,19] and computer simulation.[25] Though in the perturbation technique no emission of radiation is observed, in the computer simulation it is observed. In our case, no emission of radiation takes place but local distortions.

We linearize Eq. (2.9) around a singular point for the extended sine-Gordon system. The discriminant $D(x)$ of the resultant equation is written as

$$D(x)=\left[\frac{u}{h^2(x)-u^2}\right]^2\left[\frac{dh}{dx}+uG\right]^2+\frac{4u^2}{h^2(x)-u^2}(\cos\phi_0) . \quad (9.1)$$

If $|x|$ is smaller than a certain value, say $|x_{c0}|$, where x_{c0} satisfies $h^2(x_{c0})=u^2$, the relation $h^2(x)<u^2g^2(t)$ is always preserved, since $h^2(x)$ is an increasing function of $|x|$ toward unity, and since $g^2(t)$ is a decreasing function of $|t|$ toward unity. The singular point at $|t|=\infty$ is then a node or a spiral point from Eq. (9.1). The former means that such a state as satisfying $h^2(x)=u^2g^2(t)$, i.e., $\beta(x,t)=0$, is never realized only with changing of t, and the latter means that an arbitrary solution around the singular point is always possible to reach the singular

point by increasing $|t|$ to infinity as seen from the property of solution curves for the node or the spiral point.[20] Thus, we obtain a smooth solution curve under this condition. The solution curves around $x=0$ in Fig. 2(a) correspond to this kind of solutions.

If $|x|$ is larger than $|x_{c0}|$, $h(x)$ becomes larger than u. In this case, the singularity at $|t|=\infty$ is a saddle point from Eq. (9.1). Then, almost every solution curve is repelled before reaching it, except four routes.[20] This suggests that the route connecting $\phi_{0,2n-2}$ and $\phi_{0,2n+2}$ should be very limited. Moreover, with decreasing $|t|$, the situation is possible to be changed from $h^2(x)>u^2g^2(t)$ to

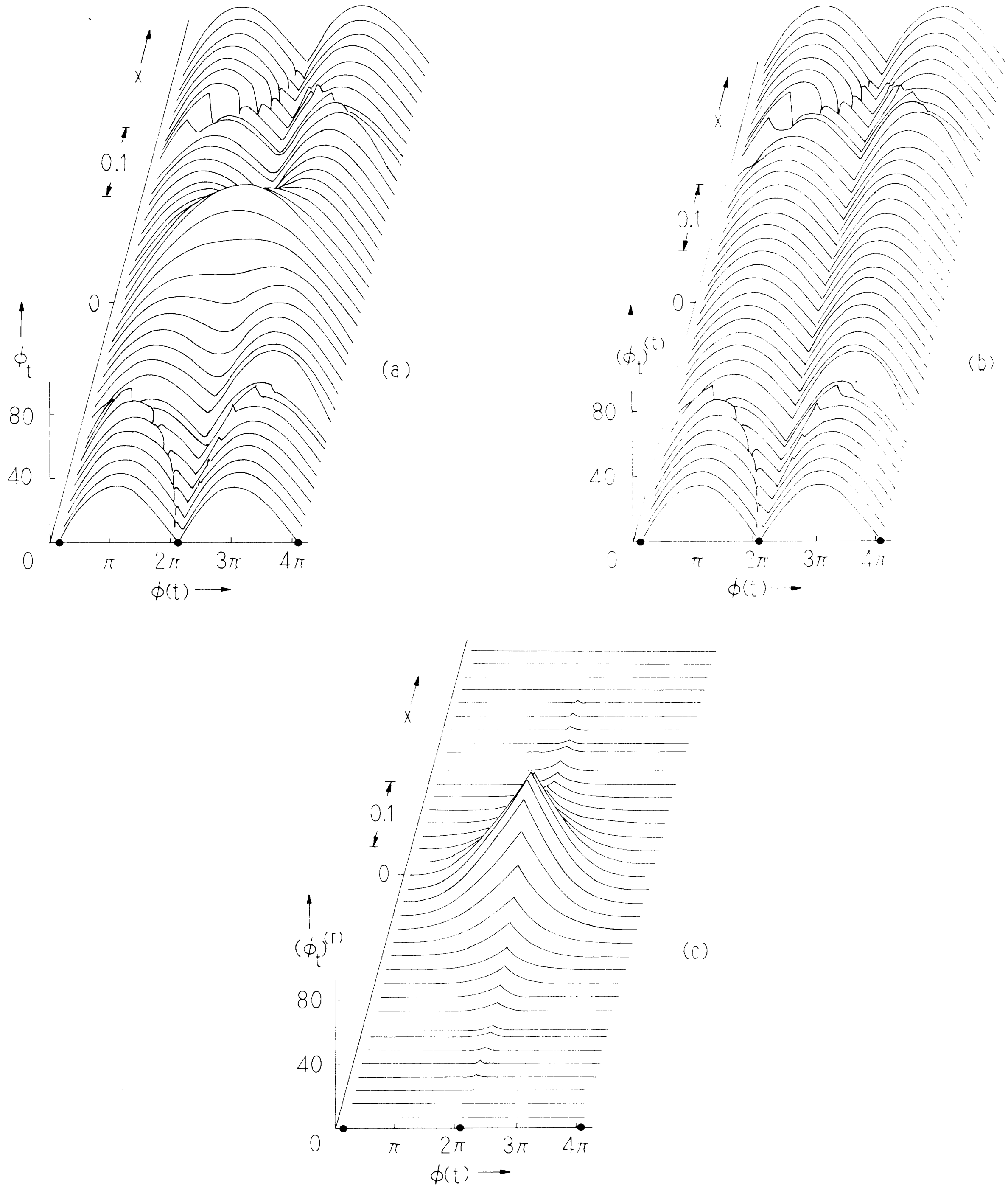

FIG. 2. Soliton-antisoliton interaction solution in the state plane to the extended sine-Gordon system, where $G=0.018$ and $J_B=0.4$. (a) Net solution. The dots denote the position of the singular points at $|T_{\pm}|\rightarrow\infty$. (b) Traveling-wave component. (c) Standing-wave component.

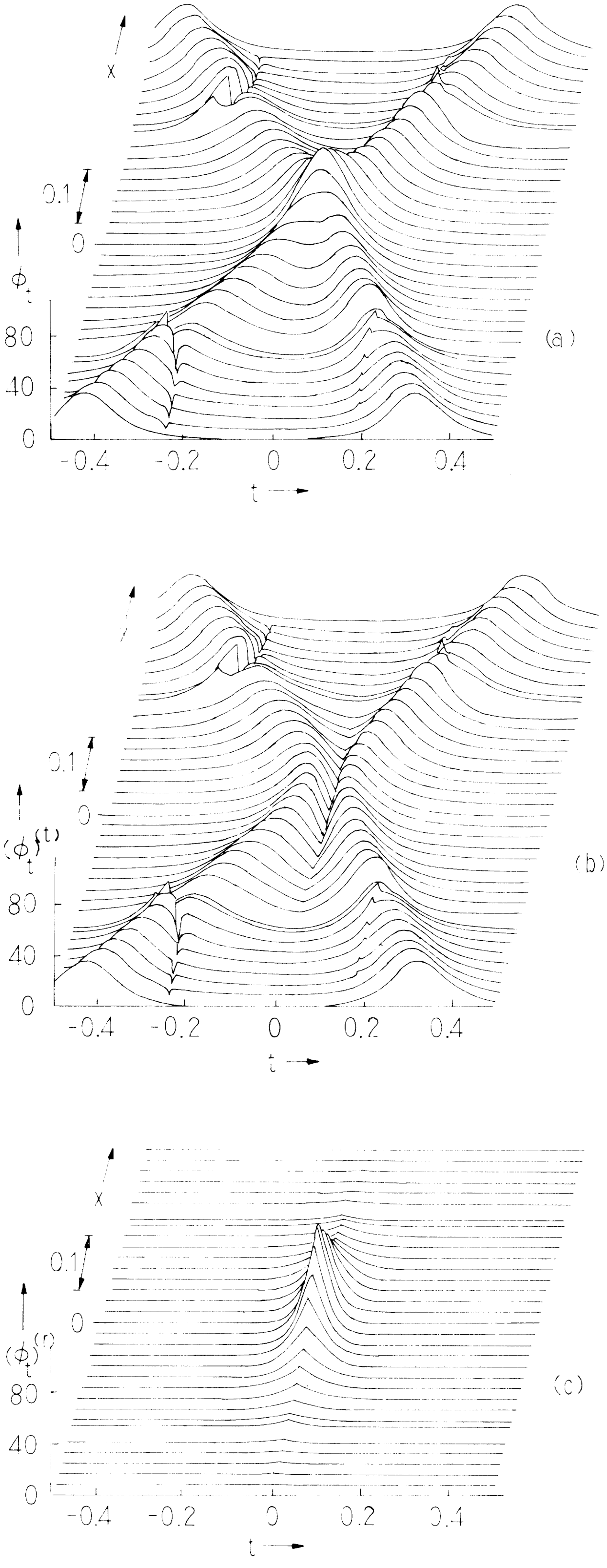

FIG. 3. Waveforms of the soliton-antisoliton interaction for the solution in Fig. 2. (a) Net waveforms. (b) Traveling-wave component. (c) Standing-wave component.

$h^2(x) < u^2 g^2(t)$ via the state

$$h^2(x_c) - u^2 g^2(t_c) = 0 \; , \tag{9.2}$$

at $t = t_c(x)$, where $x = x_c(t)$. It is also noted that the position satisfying Eq. (9.2) is moving. Around $t_c(x)$, the solution in the state plane is characterized from Eq. (2.7) by

$$[\eta(\phi) + \xi(x,t)V(x,t)]/\beta(x,t) \; . \tag{9.3}$$

When t approaches $t_c(x)$, $\beta(x,t)$ approaches zero. Then, the numerator in Eq. (9.3) should also approach zero in order to exhibit a definite value of the solution at $t_c(x)$. The way bringing the denominator to zero is generally different from the way doing the numerator to zero, and moreover, it depends upon the values of x, J_B, and G. Thus, the distortion is forced to be concentrated just around $t_c(x)$. It is rather exceptional that there appear no such distortions for the pure sine-Gordon system. This results from the fact that $V(\Xi)$ happens to become a single valued function of ϕ.

X. SUMMARY

Dynamic behavior of solutions in nonlinear Klein-Gordon systems with a dissipation and an external force, referred to as extended Klein-Gordon systems, are treated geometrically in a state plane by transforming the equation into three basic equations each of which is associated with the derivatives with respect to x, t, and ϕ, respectively.

The solutions are expressed by the forms of ϕ_t and ϕ_x. They are divided into the sum of the traveling-wave component and the other wave components, respectively, which one can analyze easier by using the state plane consisting of the relation between $V(x,t)$ and ϕ, where $V(x,t)$ denotes the traveling-wave component of ϕ_t and is included in common in the expressions of ϕ_t and ϕ_x in the form of multiplication. $V(x,t)$ is then expressed by the stationary form in a more generalized extended Klein-Gordon system on nonlinear coordinates.

The boundary and the initial conditions are imposed to the slopes at singular points on the traveling-wave component in the state plane under the conditions that there exists initially either a stationary soliton or an antisoliton at $|x| = \infty$, and that finally either the stationary soliton or the antisoliton is again established at $|x| = \infty$.

To verify the appropriateness of our model, the analytical method is first applied to the soliton-antisoliton interaction and the soliton-soliton interaction in the pure sine-Gordon system, and the well-known expressions are derived. Next, the generalized analytical method is developed for the soliton-antisoliton interaction in the extended Klein-Gordon system. As a numerical example, the extended sine-Gordon system is treated. It is found that when the soliton approaches an antisoliton up to a certain distance a local wedge-shaped distortion is generated in the soliton and in the antisoliton at the same time. The distortion moves to the direction opposing the soliton movement, and disappears before the collision at the center takes place. When they recede from each other up to a certain distance, a thorn-shaped distortion is generated in the front part of the wave. When the thorn reaches the vicinity of the center of the wave, it disappears. Such distortions appear around the position satisfying the condition $\beta(x,t)=0$, where the moving singularities is generated. Their properties and mechanisms are clarified. There appear no such distortions in the pure sine-Gordon system on account of disappearance of the moving singularity. This is because $V(x,t)$ becomes a single valued function of ϕ.

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