Largest current in a random resistor network

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The expected largest current in a random resistor network is shown to scale as $(\ln L)^\alpha$, where L is the size of the network and the exponent α depends on dimension and on the ratio of the smallest to the largest conductances in the network. This result follows from an analysis of configurations which carry large currents. The configurations which carry the largest currents in the network are shown to be funnel shaped. The proof that these configurations carry the largest currents is based upon a theorem concerning the minimum eigenvalue of a class of Sturm-Liouville problems.

I. INTRODUCTION

In this paper we study the expected value of the largest current flowing in a random resistor network. The specific problem that we address is stated as follows. Each bond in a square or cubic lattice of size L is assigned a conductance σ which is chosen independently from a probability law of the form

$$p(\sigma)=p\delta(\sigma-\sigma_>)+(1-p)\delta(\sigma-\sigma_<)$$

with $0<(\sigma_</\sigma_>)<1$. A current source is connected to the lattice via bus bars on opposite faces so that the average current density, measured in units of the lattice spacing, is j_{ave}. Let $i_{\max}$ be the largest of the currents flowing in the L^d resistors of the system and let $\langle i_{\max}\rangle$ be its average over configurations of the conductances. How does $\langle i_{\max}\rangle$ depend upon L? The main conclusion of this paper is that, for large L,

$$\langle i_{\max}\rangle \sim j_{\text{ave}}(\ln L)^\alpha \;, \tag{1.1}$$

where α depends on $\sigma_</\sigma_>$ and dimension. For $d=2$, α is given by

$$\alpha=\{1-(4/\pi)\tan^{-1}[(\sigma_</\sigma_>)^{1/2}]\}/2 \;. \tag{1.2}$$

For $d=3$, α is the solution of a more complicated transcendental equation. In general, $0<\alpha<1/d$, so that the maximum expected current always diverges with L, though extremely slowly if $\sigma_</\sigma_>$ is near unity.

The present investigation was stimulated by the work of Duxbury, Beale, and Leath,[1] hereafter referred to as DBL, who studied the limiting percolation case where $\sigma_<=0$. If $p>p_c$, the percolation threshold, they showed that Eq. (1.1) holds with $\alpha=1/(d-1)$. They obtained this result by considering a long, narrow region of insulating bonds oriented perpendicular to the average current. They found the current through the conducting bonds at the ends of such a defect and argued that the largest currents in the network occur at these locations. Our argument proceeds similarly in that we identify the most effective defect for producing large currents and calculate the current through it.

The primary physical motivation for studying the largest current in a random resistor network is to understand the electrical and mechanical breakdown properties of composite materials. The random fuse network, introduced by Arcangelis *et al.*[2] and studied by DBL, is a model system in which to study breakdown phenomena. The random fuse network is obtained from a random resistor network by introducing a critical current, i_c, at which the conductance of any resistor irreversibly drops to zero. As the voltage across the network is increased, breakdown first occurs at the resistor carrying the largest current; that is, when $i_{\max}=i_c$. Thus, if j_1 is the average current density at which breakdown begins, the results for the maximum current imply that

$$j_1\sim(\ln L)^{-\alpha} \;.$$

After the breakdown of the first resistor there is a new distribution of currents. If any of these exceed i_c the corresponding resistors are removed. Once a stable current distribution is achieved the external voltage is again increased and additional resistors removed until the system becomes an insulator.

The breakdown of the random fuse model illuminates several features of mechanical breakdown in real materials. In the random fuse model, the scalar voltage field replaces the tensor strain field in a mechanical system. Additionally, the nonlinearity associated with the breaking of a bond is a step function rather than the complicated stress-strain relation found beyond the Hooke's-law regime. Nonetheless, we expect the logarithmic size dependence of the breaking strength to be a robust feature of breakdown in a variety of models, though the exponent α may depend upon the field (voltage or strain) and on the breakdown law of a single bond. The present work sets the stage for the investigation of more complex models such as an elastic fuse network where each bond is a spring which may sustain both bending and stretching forces and which may be broken at a critical stress.

In Sec. II of this paper we present the arguments leading to Eqs. (1.1) and (1.2) for the square lattice. In Sec. III we consider three dimensions. In Sec. IV we outline a proof that the critical defects analyzed in Secs. II and III indeed lead to the maximum current in the network. This analysis leads us to an interesting theorem concerning the minimum nontrivial eigenvalue of a class of Sturm-Liouville operators. The paper closes with a discussion of the results and a comparison with the work of DBL.

II. CALCULATION OF THE LARGEST CURRENT IN A SQUARE LATTICE

The first step in estimating the largest current in a random resistor network is the identification of a class of configurations of resistors which give rise to large currents. These configurations, henceforth called defects, are necessarily large structures and we assume they can be analyzed using continuum methods. The second step in the analysis is an estimate of the size of the defects that are likely to be found in a system of size L. As the system size increases, larger defects appear and the maximum current in the network increases.

An example of a defect is shown in Fig. 1. It is a rectangular region divided into quadrants having conductivities $\sigma_<$ and $\sigma_>$. This defect, which we call a funnel, is embedded in an environment having a conductivity near the average conductivity of the system. When it is oriented so that the far field is parallel to the z axis, current is funneled though the high-conductivity region and the current density is greatest near the vertex at the center of the defect. In the continuum version of the funnel the current density diverges at the vertex. For a funnel of size l in a lattice of spacing a, the current density reaches a finite maximum in the bonds near the vertex. This maximum increases as a power law in l/a and it is this power law which determines α. In this section we carry out an analysis of $\langle i_{\max} \rangle$ considering funnel-shaped defects.

The continuum version of the funnel may be analyzed using elementary methods. Let $\Phi(r,\theta)$ be the potential at position r and θ, where r is the distance from the vertex and the angle θ is measured from the z axis as shown in Fig. 1. The current density $\mathbf{j}(r,\theta)$ satisfies the continuity equation $\nabla \cdot \mathbf{j}=0$ and Ohm's law, $\mathbf{j}(r,\theta) = -\sigma(r,\theta)\nabla\Phi(r,\theta)$. For small r, σ is a function of θ only and we may separate variables, $\Phi(r,\theta)=g(r)\Psi(\theta)$. The radial equation is satisfied by $g(r)=r^{\nu}$, where $\nu \geq 0$ is required since there are no current sources. The exponent ν is determined by the Sturm-Liouville eigenvalue problem,

$$[(\partial_\theta \sigma \partial_\theta)+\nu^2\sigma]\Psi=0 \ , \tag{2.1}$$

subject to periodic boundary conditions, $\Psi(\theta)=\Psi(\theta+2\pi)$. Across discontinuities in $\sigma(\theta)$, continuity of the normal component of the current and parallel component of the field leads to the requirement that

$$\Psi(\theta^+)=\Psi(\theta^-) \tag{2.2}$$

and

$$\sigma(\theta^+)\partial_\theta\Psi(\theta^+)=\sigma(\theta^-)\partial_\theta\Psi(\theta^-) \ , \tag{2.3}$$

where $+$ and $-$ refer to the two sides of a discontinuity in σ.

Consider the funnel configuration with $\beta=\pi/4$. In this symmetric case, $\sigma(\theta)$ is given by

$$\sigma(\theta)=\begin{cases}\sigma_>, & -\pi/4<\theta<\pi/4 \text{ or } 3\pi/4<\theta<5\pi/4 \\ \sigma_<, & \text{otherwise} \ .\end{cases} \tag{2.4}$$

Between discontinuities, the solutions are linear combinations of $\cos(\nu\theta)$ and $\sin(\nu\theta)$. An eigenfunction can be found which satisfies symmetry conditions that the potential along the y axis vanish, $\Psi[(2n+1)\pi/2]=0$, and current across the z axis vanish, $\partial_\theta\Psi(n\pi)=0$. This solution is $\cos(\nu\theta)$ in the high-conductivity region and $\sin[\nu(\pi/2-\theta)]$ in the low-conductivity region. Dividing Eq. (2.2) by Eq. (2.3) at the $\pi/4$ boundary yields a transcendental equation for ν,

$$\sigma_<\tan(\nu\pi/4)=\sigma_>\cot(\nu\pi/4) \ , \tag{2.5}$$

whose nonvanishing solutions are

$$\nu=(4/\pi)\tan^{-1}[(\sigma_</\sigma_>)^{1/2}] \ . \tag{2.6}$$

Although the full solution for Φ is a sum over the complete set of eigenfunctions, sufficiently near the vertex the solution is dominated by the eigenfunction with the strongest singularity in r and, thus, the smallest eigenvalue. The smallest nonvanishing eigenvalue, which we henceforth refer to as ν, is given by the first branch of the arctangent and lies in the range $0<\nu<1$.

The current density is the gradient of Φ and thus diverges near the vertex as

$$|\mathbf{j}| \sim j_{\text{ave}}(r/l)^{\nu-1} \ . \tag{2.7}$$

In the corresponding lattice the largest currents occur in bonds near the vertex; thus the largest current in the funnel behaves like $j_{\text{ave}}a^{d-1}(a/l)^{\nu-1}$.

The second step in the calculation is an estimate of the size of the largest symmetric funnel in a network of size L. In this we follow the arguments of DBL. Given a square network of size l, the probability, $K(l)$, that the network is configured as a symmetric funnel is

$$K(l)=[p(1-p)]^{l/2} \ . \tag{2.8}$$

For large l, K is small and the likelihood that two funnels are near one another is very small. Thus, for $1 \ll l \ll L$ we may treat funnels of size l as if they are uniformly distributed through the system with density K. The probability, $p(l)$, of finding at least one funnel of size l is

$$p(l)=1-\exp(-KL^2) \ . \tag{2.9}$$

Assuming that the largest current in the network occurs in a funnel, we have, setting $a=1$,

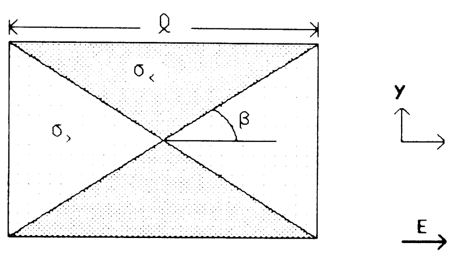

FIG. 1. The funnel configuration. Inside the rectangular region, the conductivity depends only on the angle $\sigma(\theta)=\sigma_>$ for $-\beta \leq \theta \leq \beta$ and $\pi-\beta \leq \theta \leq \pi+\beta$, while $\sigma(\theta)=\sigma_<$ for $\beta \leq \theta \leq \pi-\beta$ and $\pi+\beta \leq \theta \leq -\beta$. θ is measured from the z direction. The $\mathbf{E}$ field is taken to be in the z direction far from the funnel.

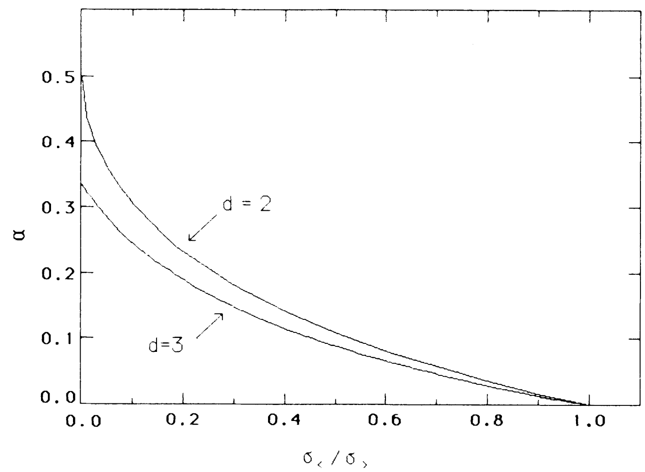

FIG. 2. α vs $\sigma_</\sigma_>$ for $d=2$ [see Eq. (2.6)] and $d=3$ [see Eq. (3.2)].

$$\langle i_{\max}\rangle \sim j_{\text{ave}} \int_0^\infty dl\, l^{1-\nu} p(l) \int_0^l dx\, p(x) \Big/ \int_0^\infty dx\, p(x) \ . \tag{2.10}$$

The integrand in the numerator is dominated by values of l for which $K(l)L^2$ is of order 1; that is, where l is of order $(\ln L)^{1/2}$. Thus,

$$\langle i_{\max}\rangle \sim j_{\text{ave}}(\ln L)^{(1-\nu)/2} \ , \tag{2.11}$$

which, in combination with Eq. (2.6), yields the results given in Eqs. (1.1) and (1.2). The exponent α is plotted as a function of $\sigma_</\sigma_>$ in Fig. 2.

III. THE LARGEST CURRENT IN A THREE-DIMENSIONAL RANDOM NETWORK

The calculation of the largest current for $d=3$ is very similar to the $d=2$ calculation. We propose that the optimal defect is a solid funnel obtained by rotating the two-dimensional funnel shown in Fig. 1 about the z axis. In Sec. IV we argue that this is indeed the best shape.

Choosing polar coordinates r,θ,ϕ and separating variables, we find that the potential takes the form $\Phi(r,\theta,\phi)=r^\nu\Psi(\theta)$. The exponent ν is determined by the Sturm-Liouville eigenvalue problem,

$$\{[\partial_\theta \sin(\theta)\sigma(\theta)\partial_\theta]+\nu(\nu+1)\sin(\theta)\sigma(\theta)\}\Psi(\theta)=0 \ , \tag{3.1}$$

subject to the condition that $\Psi(\theta)$ remain finite at $\theta=0$ and π. Across discontinuities in $\sigma(\theta)$, the potential must satisfy conditions (2.2) and (2.3). For the $d=3$ funnel, $\sigma(\theta)$ takes the values $\sigma_<$ and $\sigma_>$, with jumps at $\theta=\beta$ and $\pi-\beta$, where β is chosen to minimize the first nontrivial eigenvalue of Eq. (3.1). Between discontinuities in σ the solutions are linear combinations of Legendre functions of the first and second kind, $P_\nu(\cos\theta)$ and $Q_\nu(\cos\theta)$. The finiteness condition restricts the solution in the $\sigma_<$ region to be a Legendre function of the first kind, while the symmetry of the conductivity about $\theta=\pi/2$ means that the eigenfunction vanishes at $\theta=\pi/2$. Using known properties of the Legendre functions, this symmetry condition implies that the eigenfunction in the $\sigma_>$ region is proportional to

$$(2/\pi)\cot(\pi\nu/2)P_\nu(\cos\theta)+Q_\nu(\cos\theta) \ .$$

Matching these solutions across the discontinuity at $\theta=\beta$, we obtain the following transcendental equation for ν:

$$0=[1-(\sigma_</\sigma_>)]P_\nu P_\nu' + (2/\pi)\cot(\pi\nu/2)[Q_\nu P_\nu' - (\sigma_</\sigma_>)P_\nu Q_\nu'] \ , \tag{3.2}$$

where the Legendre functions and their derivatives are evaluated at $\cos(\beta)$.

We have studied Eq. (3.2) numerically. For a fixed ratio $\sigma_</\sigma_>$ the optimal defect is determined by choosing the β which minimizes ν. We find that this β varies by a small amount as $\sigma_</\sigma_>$ varies from 0 to 1 and is always close to 0.96. Equation (3.2) may be solved analytically in the two limits $\nu\to 0$ and $\nu\to 1$. For $\nu\to 0$ the result is that the best value for $x=\cos(\beta)$ satisfies

$$2=(1+x)\ln[(1+x)/(1-x)]$$

or $\beta=0.971$ and that, with this choice, $\nu=0.278\times(\sigma_</\sigma_>)$. For $\nu\to 1$ we find that $\cos^2(\beta)=\frac{1}{3}$ or $\beta=0.955$ and that $1-\nu=3.59[1-(\sigma_</\sigma_>)]$. As $\sigma_</\sigma_>$ increases from 0 to 1, the optimal β decreases from 0.971 to 0.955. It is a curious fact that the optimal β varies by less than 2% over the whole range of conductivity ratios.

The remainder of the calculation of the size dependence of the largest current in a three-dimensional network proceeds as in the two-dimensional case, except that the number of bonds which must be specified to construct a defect of size l is now l^3 rather than l^2. Following the reasoning leading to Eq. (2.11) yields

$$\langle i_{\max}\rangle \sim j_{\text{ave}}(\ln L)^{(1-\nu)/3} \ . \tag{3.3}$$

The exponent α defined in Eq. (1.1) is plotted in Fig. 2.

IV. WHY IS THE FUNNEL SHAPE BEST?

In Secs. II and III we considered only funnel-shaped defects. Are there other defects which typically occur in a system of size L and which have current densities exceeding those in the funnel? In this section we examine this question for the $d=2$ case; the analysis for $d>2$ should be similar.

Consider the Sturm-Liouville problem defined by Eqs. (2.1)–(2.3). Among all functions $\sigma(\theta)$ which are piecewise differentiable and bounded by

$$0<\sigma_< \le \sigma(\theta) \le \sigma_> \ , \tag{4.1}$$

we claim that the function associated with the funnel having $\beta=\pi/4$ has the smallest nontrivial eigenvalue.

To prove this claim, first note that for an arbitrary function, $\sigma_0(\theta)$, the operator

$$\mathcal{L}_0 \equiv (1/\sigma_0)\partial_\theta \sigma_0 \partial_\theta \tag{4.2}$$

is self-adjoint with respect to the inner product

$$\langle f \,|\, g\rangle_0 \equiv \int_0^{2\pi} d\theta\, \sigma_0(\theta) f^*(\theta) g(\theta) \ . \tag{4.3}$$

Thus the eigenfunctions of $\mathcal{L}_0$ with periodic boundary conditions can be chosen to be a complete orthonormal set. We call the first nontrivial eigenfunction and eigenvalue Ψ_0 and $-\lambda_0$, respectively,

$$\mathcal{L}_0\Psi_0=-\lambda_0\Psi_0 , \tag{4.4}$$

where $\lambda_0>0$. The method of our proof is to use perturbation theory to determine how the eigenvalue varies as σ is varied and then to eliminate classes of functions σ for which variations exist which reduce λ. Suppose we write σ as a variation of σ_0,

$$\sigma=\sigma_0+\sum_{j=1}^{\infty}\varepsilon^j\sigma_j , \tag{4.5}$$

where ε is a small positive number. This leads to variations in the operator $\mathcal{L}$, the first nontrivial eigenfunction Ψ, and its eigenvalue $-\lambda$,

$$\mathcal{L}=\mathcal{L}_0+\sum_{j=1}^{\infty}\varepsilon^j\mathcal{L}_j , \tag{4.6a}$$

$$\lambda=\lambda_0+\sum_{j=1}^{\infty}\varepsilon^j\lambda_j , \tag{4.6b}$$

and

$$\Psi=\Psi_0+\sum_{j=1}^{\infty}\varepsilon^j\Psi_j . \tag{4.6c}$$

By considering the equation $\langle\Psi_0\,|\,\mathcal{L}+\lambda\,|\,\Psi\rangle_0=0$ and collecting terms of first order in ε, we obtain the following result for the first-order shift in the eigenvalue,

$$\begin{aligned}\lambda_1&=-\langle\Psi_0\,|\,\mathcal{L}_1+\lambda(\sigma_1/\sigma_0)\,|\,\Psi_0\rangle_0\\&=\int d\theta\,\sigma_1(\theta)\{[\Psi_0'(\theta)]^2-\lambda_0[\Psi_0(\theta)]^2\} .\end{aligned} \tag{4.7}$$

We first eliminate all functions σ, except those for which either $\sigma(\theta)=\sigma_<$ or $\sigma(\theta)=\sigma_>$ for all θ. For functions which take other values the sign of σ_1 may be freely chosen so that we can make λ_1 negative.

Thus we can restrict ourselves to functions which are piecewise constant taking the two values $\sigma_<$ and $\sigma_>$. Within this class, we can eliminate all functions except those which have exactly four discontinuities. To see this consider a variation $\delta\sigma$ which leaves $\sigma=\sigma_0+\delta\sigma$ two valued,

$$\delta\sigma=\pm(\sigma_>-\sigma_<)[H(\theta-\theta_1-\varepsilon/2)-H(\theta-\theta_1+\varepsilon/2)] , \tag{4.8}$$

where $H(\theta)$ is the Heaviside step function, θ_1 is an arbitrary angle, and the $\pm$ is chosen to be the sign of $\sigma_0(\theta_1)-(\sigma_>+\sigma_<)/2$ so that the condition (4.1) is satisfied by σ. For example, if $\sigma_0=\sigma_<$ at θ_1 then the graph of $\delta\sigma$ is a rectangle of height $\sigma_>-\sigma_<$ and width ε centered at θ_1. Formally expanding $\delta\sigma$ in powers of ε we find that σ_1 is proportional to a δ function at θ_1 with the same overall sign as in Eq. (4.8). Thus the first-order change in the eigenvalue due to this variation is given by

$$\lambda_1=\pm(\sigma_>-\sigma_<)f(\theta_1) , \tag{4.9a}$$

where

$$f(\theta)=[\Psi_0'(\theta)^2-\lambda_0\Psi_0(\theta)^2] . \tag{4.9b}$$

In order to have λ_1 positive we require $\sigma_0=\sigma_>$ wherever f is positive and vice versa. Thus it suffices for us to learn how many sign changes f has. Using the Sturm-Liouville equation to rewrite f, we find that a sign change in f corresponds to a zero crossing of $(\sigma\Psi_0'\Psi_0)'$. Using standard arguments from Sturm-Liouville theory[3] it is possible to show that both $\sigma\Psi_0'$ and Ψ_0 are piecewise differentiable, have two zeros, and that the zeros of $\sigma\Psi_0'$ are interleaved with those of Ψ_0. Thus $\sigma\Psi_0'\Psi_0$ has four extrema and f has four sign changes. This proves that if σ has other than four discontinuities, it cannot yield a minimum λ.

It remains for us to show that the four discontinuities of σ must be equally spaced in order to achieve the minimum in λ. The functions under consideration are defined by four intervals in which σ is constant. These intervals begin at θ_j and have widths $\gamma_j=\theta_{j+1}-\theta_j$ for $j=1,2,3,4$ such that $\gamma_1+\cdots+\gamma_4=2\pi$, $\sigma=\sigma_>$ in the first interval, and $\theta_5\equiv\theta_1+2\pi$. The corresponding eigenfunction takes the form $A_j\cos[\lambda^{1/2}(\theta-\theta_j)+\alpha_j]$ for $j=1,2,3,4$. Using the conditions (2.2) and (2.3) across the discontinuities in σ, we obtain the following set of simultaneous transcendental equations for λ,

$$\alpha_{j+1}=\tan^{-1}[(\sigma_</\sigma_>)^{\pm1}\tan(\lambda^{1/2}\gamma_j+\alpha_j)] , \tag{4.10}$$

where the $+$ $(-)$ holds for j even (odd) and where $\alpha_5=\alpha_1$. The arctangent is defined so that α_{j+1} and $\lambda^{1/2}\gamma_j+\alpha_j$ are in the same quadrant. We have not had the courage to attack Eq. (4.10) analytically. Rather, we solved for λ numerically for 200 000 randomly chosen sets $\{\gamma_j\}$ to show that the symmetric funnel $\gamma_j\equiv\pi/2$ yields the smallest eigenvalue.

As an alternative to the above variational approach, we have examined a variety of smooth functions $\sigma(\theta)$ by converting the Sturm-Liouville equation (2.1) into a Schrödinger equation. For example, we have solved for the eigenvalue ν^2 associated with the continuous function

$$\sigma_K(\theta)=\exp\{\Lambda\,\tanh[K\cos(2\theta)]/\tanh(K)\} ,$$

where $\sigma_</\sigma_>=\exp(-2\Lambda)$. As $K\to\infty$, σ_K approaches the symmetric funnel function of Eq. (2.4). Table I gives ν^2 versus K and illustrates the fact that ν^2 is minimized for piecewise constant functions.

For $d=3$ we believe that similar arguments may be made based upon Courant's nodal domain theorem, which shows that the lowest nontrivial eigenfunction on the unit sphere has one node line.

TABLE I. The eigenvalue ν^2 vs K for the Sturm-Liouville problem of Eq. (2.1) for $\sigma=\sigma_K(\theta)=\exp\{\Lambda\tanh[K\cos(2\theta)]/\tanh(K)\}$ with $\sigma_</\sigma_>=\exp(-2\Lambda)=0.03$. As $K\to\infty$, σ_K approaches the symmetric funnel function of Eq. (2.4).

K	ν^2
0.0	0.12
2.0	0.079
4.0	0.061
6.0	0.056
8.0	0.054
∞	0.048

V. DISCUSSION

Let us first compare our results to those of DBL. In the limit $\sigma_{<}/\sigma_{>}\rightarrow 0$ we predict that the exponent α in Eq. (1.1) goes to $1/d$. In contrast, for $\sigma_{<}=0$, DBL find that $\alpha=1/(d-1)$. We believe that there is a discontinuity in α at $\sigma_{<}/\sigma_{>}=0$ and that this discontinuity is the signature of a crossover in the $\ln L$ dependence of $\langle i_{\max}\rangle$.

In DBL the critical defect is an insulating line (for $d=2$) or disk (for $d=3$) which is oriented perpendicular to the average current flow. For the purpose of analysis they treat these defects as elliptical or ellipsoidal regions of insulating material with $d-1$ long axes of length l and one axis of unit length. Their result follows from the fact that the maximum current associated with the defect, found at its edge, scales like l, while the typical size of the largest defect scales like $l^{d-1}\sim\ln L$.

For finite $\sigma_{<}/\sigma_{>}$ the current at the edge of an elliptical or ellipsoidal region of low-conductivity material can be shown to approach the finite value $j_{\text{ave}}(\sigma_{>}/\sigma_{<})$ as $l\rightarrow\infty$. However, if $\sigma_{>}/\sigma_{<}$ is large or l is small, these $(d-1)$-dimensional defects behave as if they were perfectly insulating and yield the largest currents in the network until L reaches L_c such that $\langle i_{\max}\rangle\approx j_{\text{ave}}(\sigma_{>}/\sigma_{<})$; that is, $\ln L_c\approx(\sigma_{>}/\sigma_{<})^{d-1}$. For $L>L_c$ there is no further growth in the maximum currents in the $(d-1)$-dimensional defects and $\langle i_{\max}\rangle$ is dominated by d-dimensional defects. As $\sigma_{>}/\sigma_{<}\rightarrow\infty$, the crossover length L_c goes to infinity, leading to the discontinuity in α at $\sigma_{<}/\sigma_{>}=0$. As a practical matter, if $\sigma_{>}/\sigma_{<}$ is large the exponent $\alpha=1/(d-1)$ will be observed for all accessible L. We note that for $\sigma_{>}/\sigma_{<}$ large there may be an additional crossover near the percolation threshold when $L<\xi$, the correlation length.

In summary, we have studied the question of the expected value of the largest current flowing in the bonds of a two- or three-dimensional random resistor network. The analysis is based on the notion that the leading size dependence of the largest current is determined by the most critical defect in the system. That is, the resistor carrying the largest current is surrounded by a large region, the defect, in which the conductivity is configured in a way which funnels current through the resistor. The critical defect is the one which yields the highest current using the smallest number of bonds. We assumed that this defect could be analyzed using continuum methods. The surprising result is that the expected value of the largest current grows as the α power of the logarithm of the system size and that α depends on $\sigma_{<}/\sigma_{>}$ in a complicated way.

We believe our results are exact, but they have not been arrived at using rigorous methods. Unfortunately, numerical experiments will be difficult because of the extremely slow growth in the largest current. Conceivably, physical experiments could span a sufficiently large range of sizes.

ACKNOWLEDGMENTS

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