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Renormalization-group analysis of Lifshitz tricritical behavior

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A renormalization-group analysis of critical behavior near a Lifshitz tricritical point (LTP) is presented, with emphasis on the role played by new, momentum-dependent quartic terms. These result in new stable fixed points which determine the critical behavior. For some values of n (the number of order-parameter components) and m (the dimensionality of the "soft" subspace, characterized by quartic momentum-dependent inverse correlation functions), the renormalization-group recursion relations have two stable and accessible fixed points. However, one of these can never be reached in practice, due to a thermodynamic instability which results in a first-order phase transition. For $m=d-1$, one of the fixed points describes the critical dynamics of the usual n-vector spin model in $(d-1)$ dimensions. This dynamic fixed point also characterizes LTP behavior for large n and $n=1$. In all other cases, the LTP has new exponents, which are not related to the dynamic model. Our results may be relevant to Lifshitz tricritical behavior in $RbCaF_3$ and in some liquid-crystal systems.

I. INTRODUCTION

Multicritical points have been a subject of theoretical and experimental interest for many years. Much of this interest has centered upon *tricritical points*,[1] at which a second-order phase transition becomes first order, and *Lifshitz points*,[2] at which a transition into a commensurate ordered phase (e.g., ferromagnetic) turns into a transition into an incommensurate one (with a wave vector which varies continuously along the transition line). Under special conditions, a line of tricritical points can cross a line of Lifshitz points, thereby yielding a higher-order multicritical point, which we shall call a *Lifshitz tricritical point* (LTP). A possible case in which this was both expected theoretically and observed experimentally is the structural antiferrodistortive transition in the perovskite $RbCaF_3$, where the usual transition point appears to be very close to the Lifshitz point, and which goes through a tricritical point under uniaxial pressure.[3] Other realizations possibly occur in liquid-crystal phase diagrams, e.g., near the *NAC* point or for the smectic-*A* to stacked-hexatic-*B* transition.[4]

Recently, it has also been shown that the critical dynamics of certain systems in $(d-1)$ dimensions is closely related to the equilibrium properties of related systems in d dimensions, at a special type of LTP.[5–9] As a consequence of this work, it was recognized that the earlier analysis of the LTP by Nicoll *et al.*[10] is inadequate, and that a more general Landau-Ginzburg-Wilson effective Hamiltonian must be considered. In particular, it was shown to be necessary to take additional invariants, involving momentum-dependent interactions, into account.[8,11]

Since the renormalization-group analysis of such Hamiltonians is of interest in itself and may also be relevant to the interpretation of experimental results,[3] we present here a complete study of the LTP, including *all* relevant interactions. We proceed as follows: In Sec. II, we present the effective Hamiltonian to be studied and point out the new features which were not considered in earlier work. Next, in Sec. III, we carry out a renormalization group, ϵ-expansion analysis of the Hamiltonian for a general n-vector order parameter to first order in ϵ. The question of thermodynamic stability is addressed in Sec. IV and the $n\to\infty$ limit is considered explicitly in Sec. V, where results are obtained for general dimensionality. Finally, in Sec. VI, we summarize our findings and discuss their implications.

II. EFFECTIVE HAMILTONIAN

In order to study simultaneously both the tricritical and Lifshitz multicritical behavior of a d-dimensional system characterized by an n-component vector order parameter $\boldsymbol{\phi}(\mathbf{x})$, we consider the effective Landau-Ginzburg-Wilson Hamiltonian[12]

$$\mathcal{H}=\tfrac{1}{2}\int d^dx\,[r\phi^2+\mu(\nabla_\alpha\boldsymbol{\phi})^2+(\nabla_\alpha^2\boldsymbol{\phi})^2+(\nabla_\beta\boldsymbol{\phi})^2] + \int d^dx\,[u\phi^4+\bar{y}_1\phi^2\boldsymbol{\phi}\cdot(\nabla_\alpha^2\boldsymbol{\phi})+\bar{y}_2\phi^2(\nabla_\alpha\boldsymbol{\phi})^2+w\phi^6]\ , \tag{1a}$$

and the associated partition function

$$Z=\int D[\phi]\exp[-\mathcal{H}]\ . \tag{1b}$$

Here α is an m-dimensional subspace with $m\le d$ and $\nabla_\alpha=\sum_{i=1}^m\hat{\mathbf{i}}\partial/\partial x_i$. Similarly, β is the complementary $(d-m)$-dimensional subspace and $\nabla_\beta=\nabla-\nabla_\alpha$. The two subspaces are each fully isotropic and $\boldsymbol{\phi}$ has $O(n)$ symmetry.

The usual tricritical point[1] is obtained when the scaling field $\tilde{u}$ associated with u vanishes at the critical point $t=0$ [t is the scaling field associated with r in Eq. (1a)], while that associated with μ (i.e., $\tilde{\mu}$) is nonzero. For $d \geq 3$, tricritical behavior is governed by the Gaussian fixed point, which becomes unstable with respect to w when $d<3$. At the Lifshitz multicritical point,[2] on the other hand, $\tilde{\mu}$ vanishes at $t=0$ while $\tilde{u}$ does not. In this case, u is the only relevant field when $d<4+m/2$. The LTP is reached when t, $\tilde{u}$, and $\tilde{\mu}$ all vanish simultaneously. In this case, in addition to w, we shall show that $\bar{y}_1$ and $\bar{y}_2$ are also relevant to describing the system's critical behavior in $d<3+m/2$ dimensions. These terms were not included in earlier studies of this problem.[10]

In reciprocal space (1) can be written in the form

$$\mathcal{H}=\frac{1}{2}\int_{\mathbf{q}} V_2(\mathbf{q})\boldsymbol{\phi}_{\mathbf{q}}\cdot\boldsymbol{\phi}_{-\mathbf{q}}+\int_{\mathbf{q}_1}\int_{\mathbf{q}_2}\int_{\mathbf{q}_3}\int_{\mathbf{q}_4} V_4(\mathbf{q}_1,\mathbf{q}_2,\mathbf{q}_3,\mathbf{q}_4)\boldsymbol{\phi}_{\mathbf{q}_1}\cdot\boldsymbol{\phi}_{\mathbf{q}_2}\boldsymbol{\phi}_{\mathbf{q}_3}\cdot\boldsymbol{\phi}_{\mathbf{q}_4}\delta\left[\sum_{i=1}^{4}\mathbf{q}_i\right]$$
$$+\int_{\mathbf{q}_1}\int_{\mathbf{q}_2}\int_{\mathbf{q}_3}\int_{\mathbf{q}_4}\int_{\mathbf{q}_5}\int_{\mathbf{q}_6} w\boldsymbol{\phi}_{\mathbf{q}_1}\cdot\boldsymbol{\phi}_{\mathbf{q}_2}\boldsymbol{\phi}_{\mathbf{q}_3}\cdot\boldsymbol{\phi}_{\mathbf{q}_4}\boldsymbol{\phi}_{\mathbf{q}_5}\cdot\boldsymbol{\phi}_{\mathbf{q}_6}\delta\left[\sum_{i=1}^{6}\mathbf{q}_i\right], \tag{2a}$$

with

$$V_2=r+\mu k^2+k^4+p^2 , \tag{2b}$$

$$V_4=u+y_1k_1^2+y_2(\mathbf{k}_1+\mathbf{k}_2)^2 , \tag{2c}$$

and $\int_{\mathbf{q}}\equiv(2\pi)^{-d}\int d^dq$ over a d-dimensional volume bounded by the surface $k^4+p^2=1$. Here $\mathbf{k}$ and $\mathbf{p}=\mathbf{q}-\mathbf{k}$ are, respectively, m- and $(d-m)$-dimensional wave vectors in the α and β subspaces and $y_1=\bar{y}_1-\bar{y}_2$, $y_2=\frac{1}{2}\bar{y}_2$. Note that an additional invariant, having the form

$$\int_{\mathbf{q}_1}\int_{\mathbf{q}_2}\int_{\mathbf{q}_3}\int_{\mathbf{q}_4} y_3(\mathbf{k}_1+\mathbf{k}_3)^2\boldsymbol{\phi}_{\mathbf{q}_1}\cdot\boldsymbol{\phi}_{\mathbf{q}_2}\boldsymbol{\phi}_{\mathbf{q}_3}\cdot\boldsymbol{\phi}_{\mathbf{q}_4}\delta\left[\sum_{i=1}^{4}\mathbf{q}_i\right] \tag{3}$$

is reducible to those appearing in (2) upon integration by parts. Adding (3) to (2a) is equivalent to replacing y_1 in (2c) by $y_1'=y_1+2y_3$ and y_2 by $y_2'=y_2-\frac{1}{2}y_3$. Also, when $n=1$, the two momentum-dependent quartic terms are simply different forms of the same invariant[8,11] and one can delete the y_2 term by replacing y_1 by $y_1''=y_1+\frac{4}{3}y_2$.

Near the LTP, we expect the spin correlation function to have the scaling form

$$G(k,p,t,\tilde{\mu},\tilde{u})=\langle\boldsymbol{\phi}_{\mathbf{q}}\cdot\boldsymbol{\phi}_{-\mathbf{q}}\rangle$$
$$=t^{-\gamma_{LT}}g(kt^{-\nu_{\parallel}},pt^{-\nu_{\perp}},\tilde{\mu}t^{-\Phi_\mu},\tilde{u}t^{-\Phi_u}) , \tag{4a}$$

where Φ_μ and Φ_u are crossover exponents. At the LTP, $t=\tilde{\mu}=\tilde{u}=0$ and (4a) reduces to

$$G(k,p,0,0,0)=k^{-(4-\eta_{\parallel})}\tilde{g}[p/k^{(4-\eta_{\parallel})/(2-\eta_{\perp})}] , \tag{4b}$$

with the scaling relations[2]

$$\gamma_{\rm LT}=\nu_{\parallel}(4-\eta_{\parallel})=\nu_{\perp}(2-\eta_{\perp}) . \tag{4c}$$

III. RENORMALIZATION-GROUP ϵ-EXPANSION ANALYSIS

To determine the upper critical dimensionality at the LTP and the relevance of the various operators appearing in $\mathcal{H}$ when the scaling fields $\tilde{\mu}=\tilde{u}=0$, we apply standard renormalization-group methods[12,13] based upon anisotropic rescaling[2] of the α and β subspaces. We thus rescale lengths in α by a and in β by b, and rescale the order parameter $\boldsymbol{\phi}$ by ζ. In order to keep the coefficients of *both* the $k^4\phi^2$ and $p^2\phi^2$ terms in (2) at unity, it is necessary that the bare rescaling factors satisfy

$$\zeta^2a^{-4}a^{-m}b^{-(d-m)}=\zeta^2b^{-2}a^{-m}b^{-(d-m)}=1 . \tag{5}$$

Thus, to this order,

$$b=a^2 , \tag{6a}$$

$$\zeta^2=b^{d+2-m/2} . \tag{6b}$$

The bare rescaling factors of the operators associated with y_1,y_2 and w are

$$y_1,y_2:\ \zeta^4a^{-2}a^{-3m}b^{-3(d-m)}=b^{3+m/2-d} , \tag{7a}$$

$$w:\ \zeta^6a^{-5m}b^{-5(d-m)}=b^{2(3+m/2-d)} . \tag{7b}$$

Thus the upper critical dimensionality d_u associated with the LTP is

$$d_u=3+m/2 , \tag{8}$$

and the operators associated with y_1,y_2, and w all, in principle, become relevant for $d\leq d_u$.

Defining

$$\epsilon=d_u-d=3+m/2-d , \tag{9}$$

it is straightforward to show that, to $O(\epsilon)$, the relations (6) are still valid. The $O(\epsilon)$ recursion relations for y_1,y_2, and w will have the form

$$y_1'=b^{\epsilon}[y_1-(X_1y_1^2+X_2y_2^2+X_3y_1y_2$$
$$+X_1'wy_1+X_2'wy_2+X_1''w^2)I\ln b] , \tag{10a}$$

$$y_2'=b^{\epsilon}[y_2-(X_4y_1^2+X_5y_2^2+X_6y_1y_2$$
$$+X_3'wy_1+X_4'wy_2+X_2''w^2)I\ln b] , \tag{10b}$$

$$w'=b^{2\epsilon}[w-(X_7wy_1+X_8wy_2+X_5'w^2)I\ln b] . \tag{10c}$$

Here the reduced prefactors X_i arise from one-loop diagrammatic contributions, X_i' from two-loop diagrams, and X_i'' from three-loop ones (see Fig. 1). The common

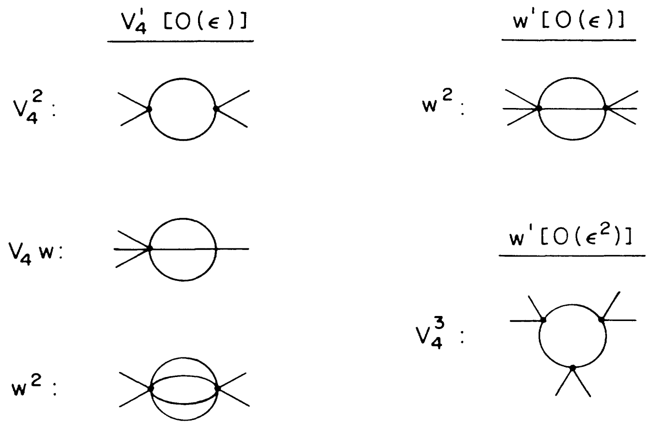

FIG. 1. Diagrammatic contributions to the renormalized couplings, V_4' and w' to $O(\epsilon)$. The final diagram (V_4^3) is needed when w is $O(\epsilon^2)$ (see text). The vertices V_4 and w are defined in Eq. (2).

factor I is defined in Table I.

Note that the X_i' and X_i'' terms in (10) are relevant, to $O(\epsilon)$, *only* when w^*, the fixed point value of w, is of $O(\epsilon)$. We shall argue (see Sec. VI) that y_1, y_2 are $O(\epsilon)$ and w^* is $O(\epsilon^2)$ at all physically relevant fixed points. These can all be found by dropping the X_i', X_i'' terms in (10a) and (10b) and replacing (10c) by

$$w' = b^{2\epsilon}[w - (X_7 w y_1 + X_8 w y_2 - X_9 y_1^3 - X_{10} y_1^2 y_2 - X_{11} y_1 y_2^2 - X_{12} y_2^3) I \ln b] \ . \qquad (10d)$$

To obtain $w^* = O(\epsilon^2)$, it is not necessary to calculate $O(\epsilon^2)$ contributions to the order-parameter rescaling parameter ζ.

The one-loop diagrammatic contributions to (10) are evaluated in the standard way,[13] using the propagator

$$G_{ij} = \delta_{ij} / (r + \mu k^2 + k^4 + p^2) \ . \qquad (11)$$

To obtain the X_i it is sufficient to set $r = \mu = 0$ in (11) and to evaluate all integrals in $d_u = 3 + m/2$ dimensions. Details are given in the Appendix and the resulting expressions are listed in Table I. Note particularly that $X_4(m = d_u - 1 = 4) = 0$. This is *not* accidental but rather a consequence of the connection between $(d-1)$-dimensional critical dynamics and the equilibrium properties of the d-dimensional LTP.[7,8]

From Eqs. (10a) and (10b) it is clear that the $O(\epsilon)$ fixed point values y_1^*, y_2^* are independent of w^* when the latter is $O(\epsilon^2)$. Defining

$$y_i^* = \epsilon z_i / I \quad (i = 1, 2) \ , \qquad (12a)$$

we obtain the following relations for the $O(\epsilon)$ reduced fixed points z_1, z_2

$$z_1 = X_1 z_1^2 + X_2 z_2^2 + X_3 z_1 z_2 \ ,$$
$$z_2 = X_4 z_1^2 + X_5 z_2^2 + X_6 z_1 z_2 \ . \qquad (13a)$$

For each solution of (13a) the corresponding $O(\epsilon^2)$ fixed point value of w^* can be found by defining

$$w^* = \epsilon^2 z_3 / I^2 \ , \qquad (12b)$$

and solving (10d) to obtain

$$z_3 = \frac{X_9 z_1^3 + X_{10} z_1^2 z_2 + X_{11} z_1 z_2^2 + X_{12} z_2^3}{X_7 z_1 + X_8 z_2 - 2} \ . \qquad (13b)$$

We shall need w^* to $O(\epsilon^2)$ in Sec. IV.

Equation (13) determines all $y_i = O(\epsilon)$, $w^* = O(\epsilon^2)$ fixed points. In general, there are four solutions of these equations, one of which is the Gaussian fixed point $z_i = 0$. The others may be found by setting $z_2 = \sigma z_1$ in (13a), solving the cubic equation

$$X_2 \sigma^3 + (X_3 - X_5)\sigma^2 + (X_1 - X_6)\sigma - X_4 = 0 \ , \qquad (14)$$

and using

$$z_2 = \sigma z_1 = \sigma / (X_2 \sigma^2 + X_3 \sigma + X_1) \ . \qquad (15)$$

In this section we shall be interested only in results to $O(\epsilon)$ and thus set $w^* = 0$.

Having found a fixed point $(y_1^*, y_2^*, w^* = 0)$, the next step is to examine its stability with respect to small deviations. This is done[13] by linearizing the recursion relations (10) about a given fixed point and calculating the relevant eigenvalues $b^{\lambda\epsilon}$. For all $w^* = 0$ fixed points, two of the eigenvalue exponents λ are given by the solutions of the determinantal equation

$$\begin{vmatrix} 1 - 2X_1 z_1 - X_3 z_2 - \lambda & -(2X_2 z_2 + X_3 z_1) \\ -(2X_4 z_1 + X_6 z_2) & 1 - 2X_5 z_2 - X_6 z_1 - \lambda \end{vmatrix} = 0 \ , \qquad (16a)$$

and the third is

$$\lambda_w = 2 - X_7 z_1 - X_8 z_2 \ . \qquad (16b)$$

A given fixed point is locally stable when the three eigenvalue exponents in (16) are negative, unstable when they

TABLE I. One-loop diagrammatic contributions to the renormalization-group recursion relations.

$X_1 = (n+8)$	$X_2 = 4(m+2)(6-m)/3m$	$X_3 = 4(16-m)/3$
$X_4 = (n+8)(4-m)/24$	$X_5 = (m^2 + 32m - 36)/3m$	$X_6 = 2(n+2)$
$X_7 = 3(n+14)$	$X_8 = 60$	$X_9 = (n+26)(m+2)/3$
$X_{10} = 18(m+2)$	$X_{11} = 12(m+2)$	$X_{12} = 8(m+2)/3$

$$I = \tfrac{1}{2} K_m K_{d-m} \Gamma[(m+2)/4] \Gamma[(6-m)/4]$$
$$K_d^{-1} = 2^{d-1} \pi^{d/2} \Gamma(d/2)$$

are positive, and a saddle point when they have mixed algebraic signs.

As at the usual Lifshitz point, the critical exponents η and ν are each replaced by the pairs $\eta_\parallel,\eta_\perp$ and $\nu_\parallel,\nu_\perp$, respectively. To $O(\epsilon)$, we have[2]

$$\eta_\parallel,\eta_\perp=0+O(\epsilon^2) . \tag{17}$$

The other exponents $\nu_\parallel,\nu_\perp$ associated with a given fixed point are obtained by considering the linearized recursion relation for r in the neighborhood of the fixed point.[13] To $O(\epsilon)$, for $w^*=0$ fixed points, we have

$$\Delta r'=b^{2-[(n+2)z_1+4z_2]\epsilon}\Delta r , \tag{18a}$$

and thus

$$\nu_\perp=2\nu_\parallel=\{2-[(n+2)z_1+4z_2]\epsilon\}^{-1} . \tag{18b}$$

Finally, consider the crossover exponents Φ_μ and Φ_u defined in (4a). The relevant diagrams are shown in Fig. 2 and the linearized recursion relations for μ and u are

$$\mu'=\mu b\{1-[(n+2)y_1^*+4y_2^*]I\ln b\} , \tag{19a}$$

$$u'=ub^{1+\epsilon}\{1-[2(n+8)y_1^*+24y_2^*]I\ln b\} . \tag{19b}$$

We thus obtain

$$\lambda_\mu=\Phi_\mu/\nu_\perp=1-[(n+2)z_1+4z_2]\epsilon , \tag{20a}$$

$$\lambda_u=\Phi_u/\nu_\perp=1+[1-2(n+8)z_1-24z_2]\epsilon . \tag{20b}$$

The crossover exponent associated with w is, of course, simply

$$\Phi_w=\lambda_w\nu_\perp . \tag{20c}$$

Before considering general solutions of the fixed point Eqs. (14) and (15) for arbitrary (n,m), it is worthwhile to examine the particular cases $(n=1,m)$ and $(n,m=4)$. For the Ising $(n=1)$ case, we pointed out earlier that the operators associated with the coefficients y_1,y_2 are identical and it thus suffices to consider either of them. Taking y_1 as the relevant parameter, we straightforwardly obtain the fixed point equation

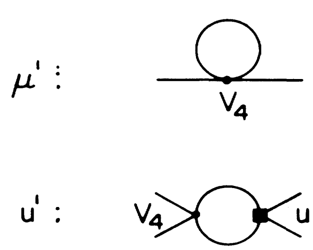

FIG. 2. Diagrammatic contributions needed to calculate the crossover exponents Φ_μ and Φ_u. The square is used to distinguish the u from the V_4 vertex.

$$z_1=(X_1+\tfrac{4}{3}X_4)z_1^2 , \tag{21a}$$

which has the $z_1\neq0$ solution

$$z_1=(X_1+\tfrac{4}{3}X_4)^{-1}=2/(22-m) . \tag{21b}$$

The associated eigenvalue exponents are

$$\lambda_{y_1}=1-2(X_1+\tfrac{4}{3}X_4)z_1=-1 ,$$
$$\lambda_w=2-X_7z_1=-2(23+m)/(22-m) . \tag{22}$$

Thus this fixed point is stable. The critical exponents $\nu_\parallel,\nu_\perp$ for $n=1$ are, from (18b),

$$\nu_\perp=2\nu_\parallel=[2-6\epsilon/(22-m)]^{-1}$$
$$=\tfrac{1}{2}+\frac{3}{2(22-m)}\epsilon+O(\epsilon^2) . \tag{23}$$

These results, given by us earlier,[8] were obtained independently by Dengler.[11]

The case $(n,m=4)$ is of particular interest due to its connection with the problem of critical dynamics in $(d-1)$ dimensions.[8] For $m=4$, the solutions of (14) and (15) are

$$\sigma=0;\quad z_1=1/(n+8),\quad z_2=0 , \tag{24a}$$

and

$$4\sigma^2+7\sigma+(4-n)=0 ,$$
$$z_2=\sigma z_1=\sigma/(9\sigma+4+2n) . \tag{24b}$$

For $\sigma=0$ the corresponding eigenvalue exponents are

$$\lambda_{y_1}=-1,\quad \lambda_{y_2}=(4-n)/(n+8) ,$$
$$\lambda_w=-(n+26)/(n+8) . \tag{25}$$

This fixed point is stable only for $n>4+O(\epsilon)$. Note, however, that when the initial value of y_2 in the Hamiltonian (3) is zero, it remains zero under renormalization-group transformations. In this case the fixed point at $z_1=1/(n+8)$, $y_2^*=w^*=0$ is "stable" for all n as only stability in the y_1-w parameter space is relevant. This is precisely the parameter space for the critical dynamics problem[8] and this *special* LTP determines the static critical exponents related to those of the dynamic model for *all n*. From (18b) and (24a), we obtain

$$\nu_\perp=2\nu_\parallel=[2-(n+2)\epsilon/(n+8)]^{-1}$$
$$=\tfrac{1}{2}+\frac{(n+2)}{4(n+8)}\epsilon+O(\epsilon^2) . \tag{26}$$

Note that $2\nu_\parallel$ coincides with the usual static correlation length exponent of the n-vector model.[7,8]

We now return to the general case. Here the $w^*=0$ fixed-point equations (13a) have four solutions, one of which, $z_1=z_2=0$, is *unstable* for all $\epsilon>0$. The other three are given in Table II for $2\le n\le4$, $1\le m\le4$. For each of these fixed points, the associated eigenvalue exponents are given in Table III. As usual,[13] one of the (y_1-y_2) plane eigenvalues is always $-\epsilon$ (to leading order in ϵ); its associated eigenvector is along the line joining the fixed point to the origin.

TABLE II. Fixed points for the LTP Hamiltonian in the $w^*=0$ plane. There is an additional point at $z_1=z_2=0$.

		n = 2		3		4	
m		z_1	z_2	z_1	z_2	z_1	z_2
1	(a)	0.191 44	−0.075 47	0.155 27	−0.054 59	0.130 84	−0.041 96
	(b)	0.135 83	−0.114 60	0.106 31	−0.097 03	0.087 53	−0.084 94
	(c)	0.069 14	0.012 99	0.061 72	0.0132 2	0.055 61	0.013 40
2	(a)	0.266 93	−0.120 52	0.180 19	−0.066 74	0.139 77	−0.044 30
	(b)	0.515 01	−0.502 55	0.299 75	−0.327 36	0.206 39	−0.244 16
	(c)	0.073 29	0.012 99	0.064 74	−0.013 75	0.057 69	0.014 22
3	(a)	0.270 70	−0.118 40	0.163 74	−0.052 76	0.123 71	−0.030 93
	(b)	−0.915 68	1.042 21	−3.754 81	4.854 38	−1.223 15	1.727 58
	(c)	0.081 42	0.010 22	0.071 21	0.011 75	0.062 57	0.013 29
4	(a)	0.209 93	−0.075 49	0.116 45	−0.018 27	0.083 33	0
	(b)	−0.221 56	0.308 05	−0.230 54	0.367 27	−0.266 67	0.466 67
	(c)	0.100 00	0	0.090 91	0	0.083 93	0

The results of Table II can be summarized as follows for $\epsilon > 0$.

(1) The Gaussian fixed point at the origin is always unstable.

(2) For $m=1,2$, there is one stable fixed point, given by (a). The points (b) and (c) are saddle points, with two and one positive eigenvalues, respectively. This is also the case for $m=3$, $n=4$. Typical renormalization-group flows are shown in Fig. 3(a).

(3) For $m=3$, $n=2,3$ and for $m=4$, $n=2,3,4$, there are *two stable fixed points,* given by (a) and (b). The points (c) are saddle points with one positive eigenvalue. [For $m=n=4$, the degeneracy would be lifted at $O(\epsilon^2)$, making the above statement valid.] Renormalization-group flows for this case are illustrated in Figs. 3(b) and 3(c).

In addition to the values $1 \leq n \leq 4$, it will be of interest to consider explicitly the limit $n \rightarrow \infty$. The relevant solution of (13a) is

$$z_1 = 1/n, \quad z_2 = -(4-m)/24n \ , \tag{27a}$$

with associated eigenvalue exponents

$$\lambda_{y_1} = \lambda_{y_2} = \lambda_w = -1 \ , \tag{27b}$$

and critical exponents

$$\nu_\perp = 2\nu_\parallel = (2-\epsilon)^{-1} \ . \tag{27c}$$

$$\Phi_\mu = \Phi_u = (1-\epsilon)\nu_\perp \ . \tag{27d}$$

Note that the y_2 invariant is irrelevant in this limit (see Sec. V).

TABLE III. Eigenvalue exponents associated with each of the fixed points of Table II. The eigenvector of λ_w is along w while λ_1 is in the y_1-y_2 plane. The second eigenvalue in this plane, $\lambda_2=-1$ for all (n,m) (see Ref. 13).

		n = 2		3		4	
m		λ_w	λ_1	λ_w	λ_1	λ_w	λ_1
1	(a)	−2.661	−1.002	−2.643	−0.986	−2.548	−0.955
	(b)	2.356	1.259	2.400	1.345	2.369	1.378
	(c)	−2.098	0.830	−1.941	0.787	−1.807	0.757
2	(a)	−3.581	−0.939	−3.186	−0.809	−2.889	−0.732
	(b)	7.432	3.321	6.354	3.011	5.505	2.732
	(c)	−2.297	0.567	−2.127	0.525	−1.981	0.500
3	(a)	−3.890	−0.712	−3.185	−0.516	−2.825	−0.443
	(b)	−16.580	−5.406	−97.768	−35.423	39.605	15.401
	(c)	−2.522	0.386	−2.337	0.337	−2.176	0.313
4	(a)	−3.547	−0.311	−2.842	−0.105	−2.500	0
	(b)	−5.848	−1.270	−8.279	−2.110	−11.600	−3.267
	(c)	−2.800	0.200	−2.636	0.091	−2.500	0

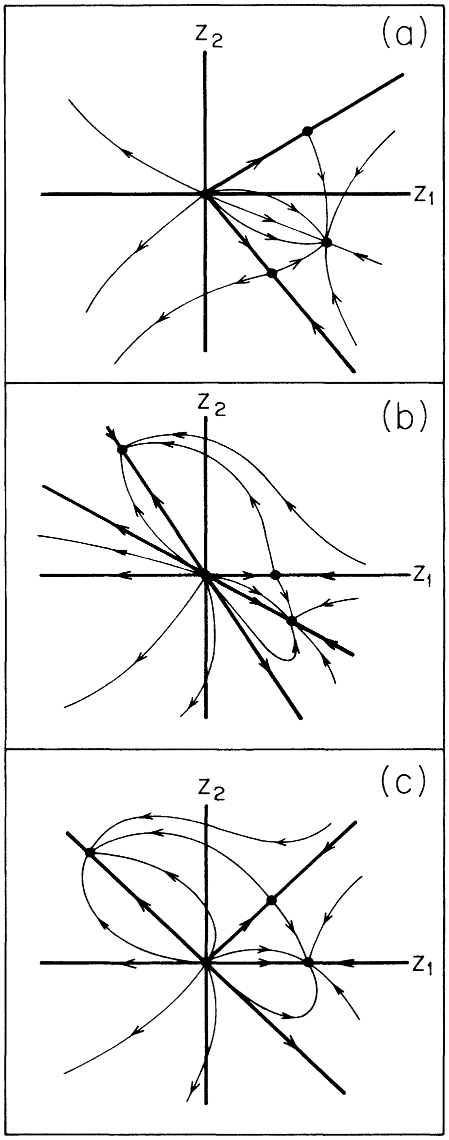

FIG. 3. Schematic flow diagrams on the critical surface for Lifshitz tricritical points. An example with a single stable fixed point is given in (a). Parts (b) and (c) illustrate the $m=4$ case relevant to critical dynamics, where the axis $z_2=0$ is invariant. In (b) the $z_2=0$ fixed point characterizes critical dynamics while behavior at a general LTP is described by the stable fixed point in the fourth quadrant. In (c), the $z_2=0$ fixed point is relevant to both dynamics and the general LTP. The second fixed point in (b) and (c) (in the second quadrant) is always thermodynamically unstable. For $m\neq4$, the z_1 axis is not invariant and the $z_2=0$ fixed point in (b) and (c) moves off this axis. Here also the stable fixed point in the second quadrant (when it exists) is always thermodynamically unstable.

IV. THERMODYNAMIC STABILITY

In Sec. III we found that for certain values of the order parameter and spatial dimensionalities (n,d), there exist *two* stable fixed points. This is unusual and might be interpreted as indicating that the initial values of the coefficients y_1 and y_2 are relevant to determining critical behavior at a LTP. We believe, however, that only one of the two stable fixed points, in fact, describes a general LTP. When the initial values of the parameters in the Hamiltonian are such that this fixed point is not reached, we predict a first-order phase transition, regardless of whether or not the alternate stable fixed point (when it exists) is within the range of attraction of our renormalization-group recursion relations.

Our argument is as follows. From (2), the classical Landau free energy density of our model system can be written as

$$f(\boldsymbol{\phi},\mathbf{k})=(r+\mu k^2+k^4)\phi^2 + 2[3(u+y_1k^2)+4y_2k^2]\phi^4+20w\phi^6 \ , \qquad (28)$$

where $\boldsymbol{\phi}$ are the Fourier components of the order parameter with wave vector $\mathbf{q}=\mathbf{k}$ (i.e., $\mathbf{p}=0$) and $\phi=|\boldsymbol{\phi}|$. At the classical LTP, $r=\mu=u=0$, and thermodynamic stability requires that f be positive for all infinitesimal values of $\boldsymbol{\phi}$ and/or $\mathbf{k}$. Thus, for $\mathbf{k}=0$, it follows that stability is guaranteed provided that

$$w>0 \ . \qquad (29)$$

The condition (29), while necessary, is not sufficient. When $(3y_1+4y_2)<0$, thermodynamic stability requires that (29) be replaced by

$$20w-(3y_1+4y_2)^2>0 \ . \qquad (30)$$

Of course, the thermodynamic stability conditions (29) and (30) are not directly relevant at fixed points of the renormalization-group recursion relations since mean-field theory is not valid there. However, if the fixed point Hamiltonian does not satisfy these conditions, there will exist renormalization group trajectories on which (a) the LTP conditions $\mu=u=0$ are satisfied and (b) w, y_1, and y_2 remain arbitrarily close to their fixed point values while r flows away from r^*. On such a trajectory, r eventually becomes $O(1)$ and the mean-field thermodynamic considerations (29) and (30) become valid. We therefore conclude that, in addition to being stable under renormalization-group operations on the critical surface (r^*,μ^*,u^*), a fixed point of (2) can describe an accessible LTP if and only if (w^*,y_1^*,y_2^*) satisfy the thermodynamic stability conditions given in (29) and (30).

Since the values of y_1^* and y_2^* found in Sec. III are $O(\epsilon)$, it is necessary to calculate w^* to $O(\epsilon^2)$ in order to examine the thermodynamic stability of a fixed point. This is easily done for all the fixed points in Table II using (12b) and (13b). We are, of course, interested particularly in those fixed points which are stable (on the critical surface) with respect to renormalization-group operations, as summarized in Sec. III. The results are as follows. Consider first the fixed points given by lines (a) in Table II. For all of these points $(3z_1+4z_2)>0$ and $z_3>0$. We conclude that *all the fixed points on lines (a) in Table II are thermodynamically stable.*

Consider now the other stable (in the renormalization-group sense) fixed points given in Table II on lines (b) (for $m=3$, $n=2,3$ and $m=4$, $n=2,3,4$). All these points are characterized by $z_3<0$. We conclude that while several of the fixed points on lines (b) in Table II are stable under renormalization, *none of these are thermodynamically stable. They cannot therefore describe critical behavior at a LTP.*

Finally, consider the special fixed point relevant to the problem of critical dynamics in $(d-1)$ dimensions. Since here $z_1=1/(n+8)>0$ and $z_2=0$, $z_3>0$ and this point is always thermodynamically stable, as would be expected.

V. $n\to\infty$ LIMIT: RENORMALIZATION-GROUP ANALYSIS

We here consider the LTP in the limit in which the number of components of the order parameter approaches infinity ($n\to\infty$) for general dimensionality d. As usual,[12] we take the coefficients u, y_1, y_2, w to be $O(n^{-1})$ and, to avoid additional notation, assume that this has been explicitly done in (2) (i.e., $u\to u/n$, etc.). Following the standard procedure,[12] we obtain

$$\tilde{r}=r+uJ_0(\tilde{r},\tilde{\mu})+y_1J_2(\tilde{r},\tilde{\mu}) \ , \tag{31a}$$

$$\tilde{\mu}=\mu+y_1J_0(\tilde{r},\tilde{\mu}) \ , \tag{31b}$$

$$\tilde{u}=u+wJ_0(\tilde{r},\tilde{\mu}) \ , \tag{31c}$$

where

$$J_\ell(\tilde{r},\tilde{\mu})=\int d^dq k^\ell(\tilde{r}+\tilde{\mu}k^2+k^4+q^2)^{-1} \ . \tag{31d}$$

Note that y_2 does not appear in the $n\to\infty$ limit, in agreement with the ϵ-expansion results. At the LTP, $\tilde{r}$, $\tilde{\mu}$, and $\tilde{u}$ all vanish simultaneously. Defining

$$r_{\rm LT}=-uJ_0(0,0)-y_1J_2(0,0) \ , \tag{32a}$$

$$\mu_{\rm LT}=-y_1J_0(0,0) \ , \tag{32b}$$

$$u_{\rm LT}=-wJ_0(0,0) \ , \tag{32c}$$

it is necessary to evaluate the ratio $\tilde{r}/\tilde{\mu}^2$ as $\tilde{r},\tilde{\mu}\to 0$. Setting $\mu=\mu_{\rm LT}$ (its critical value) we consider, from (31b) and (32b),

$$\tilde{\mu}=y_1[J(\tilde{r},\tilde{\mu})-J(0,0)] \tag{33}$$

as $\tilde{\mu}\to 0^-$, $\tilde{r}\to 0^+$. Careful analysis shows that, in this limit, $\tilde{r}/\tilde{\mu}^2$ reaches a unique, nonzero, value for $d<3+m/2$. Substituting into (31b), we obtain to leading order in this limit

$$\tilde{r}=t-c_1\tilde{u}\,\tilde{r}^{\,1/2}+c_2w\tilde{r}-c_3y_1\tilde{r}^{\,(2d-m-2)/4} \ , \quad d<3+m/2 \ , \tag{34}$$

with $c_i>0$ and $t=r-r_{\rm LT}$. We thus have

$$\gamma_{\rm LT}=4/(2d-m-2) \ ,$$
$$\Phi_\mu=\Phi_u=(2d-m-4)/(2d-m-2) \ , \tag{35}$$
$$\Phi_w=-(6+m-2d)/(2d-m-2) \ ,$$

in agreement with the ϵ-expansion results given in Sec. III.

Consider now, in the $n\to\infty$ limit, fixed points at which $y_1^*=0$ and $w^*\neq 0$. Then, instead of (34), we would obtain

$$\tilde{r}=t-\tilde{c}_1\tilde{u}\,\tilde{r}^{\,(2d-m-4)/4}+\tilde{c}_2w\tilde{r}^{\,(2d-m-4)/2} \ , \tag{36}$$

with $\tilde{c}_i>0$. Since $\tilde{c}_2>0$, this equation has *no solution* for $\tilde{r}$ which vanishes as $t\to 0$. As in the case of the ordinary tricritical point,[14] we interpret this as indicating that the phase transition associated with such $n\to\infty$ fixed points is of first order. Clearly, it is the y_1 term, via the renormalization of μ given by (33), which changes the nature of the phase transition and results in an accessible, stable fixed point characterizing Lifshitz tricritical behavior in the $n\to\infty$ limit.

VI. DISCUSSION

Our objective in this paper was to present a comprehensive renormalization-group treatment of Lifshitz tricritical behavior. The interest in such an analysis is twofold. First, the possibility of experimental measurements at or near such a multicritical point, and second, the close connection between the Lifshitz tricritical point in d dimensions and dynamical critical behavior at an ordinary critical point in $(d-1)$ dimensions.

The essential difference between our study and earlier work is that we have included *all* relevant invariants in the effective Hamiltonian. In particular, quartic momentum-dependent interactions, which are irrelevant near the ordinary critical point, *must* be considered *ab initio*. The result, within the ϵ-expansion framework, is that there are, in principle, three relevant fields on the critical surface of a LTP. This results in a maximum of eight fixed points. At four of these, the fixed point value w^* of the sixth-order coupling constant is either zero (Gaussian fixed point) or $O(\epsilon^2)$. These were found by us explicitly. At the others (if they exist), w^* is presumably $O(\epsilon)$. These latter points are extremely difficult to determine, even to $O(\epsilon)$, and we therefore used an alternate method, the $n\to\infty$ or spherical model limit, to examine their relevance to Lifshitz tricritical behavior. Here we found that for any fixed point at which the momentum-dependent interaction term coupling constant y_1^* is nonvanishing, w is *irrelevant*. Conversely, any $y_1^*=0$, $w^*\neq 0$ *cannot* describe a LTP as such a fixed point is not accessible in this limit. We regard this as a strong indication that, within the ϵ-expansion framework, it is *only* the $w^*=0+O(\epsilon^2)$ fixed points which are relevant to describing Lifshitz tricritical behavior. Certainly, the $O(\epsilon)$ stable (with respect to both renormalization-group and thermodynamic criteria) fixed point we have found for general n does not lose its stable character as $n\to\infty$. In any case, the fixed point given earlier,[10] with $w^*=O(\epsilon)$ and $y_i^*=0$, as characterizing LTP behavior *cannot* be relevant as $n\to\infty$.

An interesting aspect of our analysis was that for certain parameter values, there exist *two* $O(\epsilon)$ fixed points which are stable and accessible on the critical surface. This has not, to our knowledge, been found previously in an ϵ-expansion calculation. Two stable fixed points had been found for the random exchange Ising model,[13] but one of these points has unphysical parameters and can never be reached. A stable fixed point which is not reached because of a first order transition was found for the Potts model in $d=6-\epsilon$ dimensions by Pytte.[15]

There, for certain parameter values, he found that there exists a (single) stable fixed point which, however, is not accessible as a first-order transition must occur before this point can be reached. This is precisely what we expect to occur when the system is within the region of attraction (in the renormalization-group sense) of the alternate stable fixed points given in lines (b) of Table II. Unlike the Potts model case, however, we find another fixed point which describes the LTP critical behavior and which, as shown in Fig. 3(b) and 3(c) has its own region of attraction on the critical surface. This latter point, unlike the alternate one, *is* thermodynamically stable.

An additional aspect of our study was to consider the special LTP, relevant to describing dynamical critical behavior at an ordinary critical point in one less spatial dimension, in the more general space required for a general LTP. We find that this special fixed point also describes the general LTP when n, the number of components of the order parameter, is greater than $n_c=4+O(\epsilon)$. For $n<n_c$, the special point is relevant to $(d-1)$ critical dynamics only.

Finally, we return to the possibility of finding a LTP experimentally. From the scaling relation[2]

$$2-\alpha=m\nu_{\parallel}+(d-m)\nu_{\perp}\ , \tag{37}$$

it follows that the specific-heat exponent at LTP is given by

$$\alpha=\tfrac{1}{2}+\tfrac{1}{4}(2+3a)\epsilon+O(\epsilon^2)\ , \tag{38}$$

where $(a\epsilon)$ are the quantities given in Table IV for different (m,n). In all cases of experimental interest $a>0$ and thus the coefficient of ϵ in (38) is larger than $\frac{1}{2}$. This immediately implies that the specific heat exponent α is anomalously large compared with the usual tricritical value[1] $\alpha=\frac{1}{2}$. The large coefficient of ϵ indicates that the $O(\epsilon)$ truncation in (38) does not give reliable numerical predictions for α (unphysical values larger than unity are obtained by naive substitution). Nevertheless, we expect a LTP to be characterized by a particularly large specific heat anomaly.

TABLE IV. Critical exponent $[(\nu_{\perp})^{-1}-2]$ for the LTP.

	n			
m	1	2	3	4
1	0.286ϵ	0.464ϵ	0.558ϵ	0.486ϵ
2	0.300ϵ	0.586ϵ	0.634ϵ	0.522ϵ
3	0.316ϵ	0.609ϵ	0.608ϵ	0.495ϵ
4	0.333ϵ	0.538ϵ	0.509ϵ	0.417ϵ

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APPENDIX

The coefficients X_i, $i=1,\ldots,6$ all arise from the first diagram in Fig. 1. In the following, we set $u=0$, and symmetrize the y_1 term in Eq. (2c):

$$V_4(\mathbf{q}_1,\mathbf{q}_2,\mathbf{q}_3,\mathbf{q}_4)=\tfrac{1}{4}y_1(k_1^2+k_2^2+k_3^2+k_4^2)+y_2(\mathbf{k}_1+\mathbf{k}_2)^2\ . \tag{A1}$$

In the initial step of the renormalization-group iteration, we expand Z in powers of V_4 and integrate over $\phi_{\mathbf{q}}$ with $1/a^4<k^4+p^2<1$. To second order in V_4, the contribution to the renormalized V_4 is

$$\begin{aligned}\Delta V_4(\mathbf{q}_1,\mathbf{q}_2,\mathbf{q}_3,\mathbf{q}_4)=&-n\int^{>}V_4(\mathbf{q}_1,\mathbf{q}_2,\mathbf{q},\mathbf{q}_1+\mathbf{q}_2-\mathbf{q})V_4(\mathbf{q},\mathbf{q}_1+\mathbf{q}_2-\mathbf{q},\mathbf{q}_3,\mathbf{q}_4)G_0(\mathbf{q})G_0(\mathbf{q}_1+\mathbf{q}_2-\mathbf{q})\\&-4\int^{>}V_4(\mathbf{q}_1,\mathbf{q}_2,\mathbf{q},\mathbf{q}_1+\mathbf{q}_2-\mathbf{q})V_4(\mathbf{q},\mathbf{q}_1+\mathbf{q}_2-\mathbf{q},\mathbf{q}_3,\mathbf{q}_4)G_0(\mathbf{q})G_0(\mathbf{q}_1+\mathbf{q}_2-\mathbf{q})\\&-4\int^{>}V_4(\mathbf{q}_1,\mathbf{q}_3,\mathbf{q},\mathbf{q}_1+\mathbf{q}_3-\mathbf{q})V_4(\mathbf{q},\mathbf{q}_1+\mathbf{q}_3-\mathbf{q},\mathbf{q}_2,\mathbf{q}_4)G_0(\mathbf{q})G_0(\mathbf{q}_1+\mathbf{q}_3-\mathbf{q})\ ,\end{aligned} \tag{A2}$$

where

$$G_0(\mathbf{q})=1/(k^4+p^2)\ , \tag{A3}$$

and the integral $\int^{>}$ is over $1/a^2<k^4+p^2<1$, with $\mathbf{q}\equiv(\mathbf{k},\mathbf{p})$. We next set $\mathbf{p}_i=0$, expand $G_0(\mathbf{q}_1+\mathbf{q}_2-\mathbf{q})$ in powers of $(\mathbf{q}_1+\mathbf{q}_2)$, take pair products of V_4's, and keep only terms quadratic in the $\mathbf{k}_i$'s. The results involve the integrals

$$I_l=\int^{>}\frac{k^{4l-2}}{(k^4+p^2)^{l+1}};\quad 1/a^2<k^4+p^2<1\ , \tag{A4}$$

with $l=1,2,3$. We now introduce polar coordinates[16]

$$y=|\mathbf{k}|^2,\quad x=|\mathbf{p}|,\quad z=(x^2+y^2)^{1/2},\quad \theta=\tan^{-1}(y/z)\ ,$$

obtaining

$$I=\tfrac{1}{2}K_mK_{d-m}\int_{1/a^2}^{1}z^{d-4-m/2}dz\int_0^{\pi/2}(\sin\theta)^{2l-2+m/2}(\cos\theta)^{d-m-1}d\theta\ . \tag{A5}$$

For $d = d_u = 3 + m/2$, this becomes

$$I_l = \tfrac{1}{4} K_m K_{d-m} B \left[\frac{m+4l-2}{4}, \frac{6-m}{4} \right] \ln a^2 , \tag{A6}$$

where B is the beta function.[16] Thus

$$I_1 = \tfrac{1}{2} I \ln b, \quad I_2 = \frac{m+2}{8} I_1, \quad I_3 = \frac{(m+2)(m+6)}{96} I_1 , \tag{A7}$$

with I given in Table I. Collecting the terms in the expansion, using the symmetrizing transformation from Eq. (2c) to (A1), and multiplying by the factor in Eq. (7a) yields the coefficients in Table I. The analysis of the other diagrams in Fig. 1 is similar, and involves the same momentum integrals given above.

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