Acoustic radiation-induced static strains in solids

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The controversy surrounding the magnitude of the radiation-induced static strain accompanying acoustic wave propagation in solids is resolved by a consideration of the associated Boussinesq radiation stress. Experimental verification of the results is presented for waves propagating along the pure mode directions of crystalline silicon.

The radiation stress associated with finite amplitude acoustic waves propagating in a nonlinear medium has been a subject of considerable controversy for a large part of the present century. As pointed out by Beyer,[1] "It might be said that (acoustic) radiation pressure is a phenomenon that the observer thinks he understands—for short intervals, and only every now and then." No less controversial are the static strains resulting from such radiation stresses. Although Brillouin[2] found the acoustic radiation stress in solids and "laterally confined" fluids to be nonzero, his theory leads to a zero value of the radiation-induced static strain. Gol'dberg[3] argued that the radiation stress in "laterally unconfined" fluids is zero and, inferentially, so is the static strain. Thurston and Shapiro[4] predicted the existence of a nonzero static strain from a method-of-characteristics solution to the governing nonlinear differential equation in material coordinates, but their solution differs from that of Thompson and Tiersten[5] who used an iterative approximation approach in solving the same equation. Chu and Apfel[6] identified an "acoustic straining" associated with the radiation pressure in laterally confined fluids and calculated a resulting "coefficient of acoustic expansion." Yost and Cantrell[7] showed the existence of a radiation stress and an associated radiation-induced static strain in crystalline solids corresponding to each propagation mode of the crystal. The purpose of this paper is to show that the difference between the magnitude of the static strains predicted by Thurston and Shapiro,[4] and that predicted by Thompson and Tiersten,[5] can be resolved by a consideration of the Boussinesq radiation stresses derived by Cantrell.[7] We conclude with experimental verification of the results by measuring the acoustic static strains generated along the pure mode propagation directions of single-crystal silicon.

We consider the propagation of an elastic wave in a lossless semi-infinite solid of arbitrary crystalline symmetry. The nonlinear equations of motion along a given propagation direction may be transformed into the form[7]

$$\frac{\partial^2 P_\epsilon}{\partial t^2} = C_\epsilon^2 \left[1 - \beta_\epsilon \frac{\partial P_\epsilon}{\partial a} \right] \frac{\partial^2 P_\epsilon}{\partial a^2} , \tag{1}$$

where $\epsilon = j,N$ is a mode index representing a wave of polarization $j = 1,2,3$ and direction of propagation N, "a" is the Lagrangian coordinate transformed such that it is always along the direction of wave propagation, t is time, P_ϵ is the particle displacement for mode ϵ, β_ϵ is the corresponding modal nonlinearity parameter of the solid, and C_ϵ is the "linear" wave speed. Thompson and Tiersten[5] use an asymptotic iteration procedure to solve Eq. (1) subject to the boundary condition

$$P_\epsilon = \xi \cos\omega t \quad \text{at } a = 0 \tag{2}$$

They obtain as part of their solution in the first iterate ($\beta_\epsilon = -\beta_3$ in notation of Ref. 5)

$$\frac{\partial}{\partial a} \left[\frac{\partial A}{\partial a} - \frac{1}{4} \beta_\epsilon \kappa^2 \xi^2 \right] = 0 , \tag{3}$$

where $\kappa = \omega / C_\epsilon$ and $\partial A / \partial a$ is the radiation-induced static strain. They point out that the expression in parentheses in Eq. (3) is proportional to the (static) acoustic radiation stress in the solid and integrate Eq. (3) under the assumption that the static stress, which is proportional to the constant of integration, is zero. They thus predict a static strain in the solid given by

$$\frac{\partial A}{\partial a} = \frac{1}{4} \beta_\epsilon \kappa^2 \xi^2 = \frac{1}{2} \frac{\beta_\epsilon}{\mu_\epsilon} E , \tag{4}$$

where $E = \frac{1}{2} \mu_\epsilon \kappa^2 \xi^2$ is the average density of the propagating wave and $\mu_\epsilon = \rho_0 C_\epsilon^2$ (ρ_0 is the mass density of the unperturbed medium).

Equation (4) differs from the results of Thurston and Shapiro,[4] who solve Eq. (1) using the method of characteristics. Their solution subject to the boundary and initial conditions

$$\begin{aligned} &P_\epsilon(t,a) = \text{const} = \xi \cos\omega t_0 , \quad t \le t_0 , \\ &P_\epsilon(t,0) = \xi \cos\omega t , \quad t \ge t_0 \end{aligned} \tag{5}$$

is

$$\frac{\partial A}{\partial a} = \frac{1}{4} \frac{\beta_\epsilon}{\mu_\epsilon} E . \tag{6}$$

Equations (4) and (6) differ by a factor of 2. According to Thompson and Tiersten,[5] this difference is explained by considering Eq. (6) to be a solution to the initial value problem without loss while Eq. (4) is the steady-state solution without regard to initial conditions. They argue that their steady-state solution is justified by the existence of a

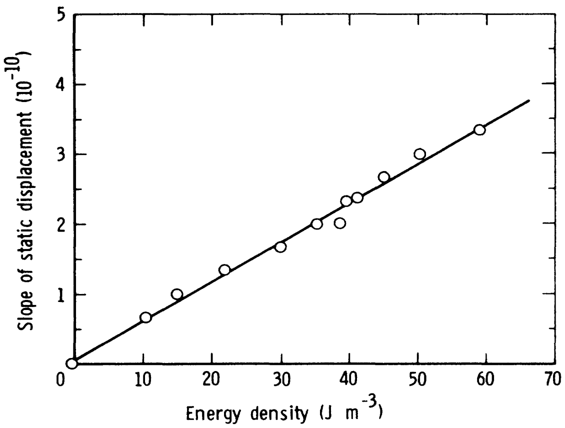

FIG. 1. Plot of the slope of the radiation-induced static displacement pulse along the [110] direction in silicon as a function of the energy density of the acoustic wave.

small but finite loss in a real solid which damps out the transient terms contributing to the results of Eq. (6). They point out that the manner in which the static strain passes from its initial value to its steady-state value can only be determined by solving the initial value problem with loss. In order to avoid the difficulties of such a calculation they assume that the outcome would be equivalent to that of associating a vanishing radiation stress with the steady-state solution. The assumption of a vanishing radiation stress demands a zero value of the integration constant resulting from the integration of Eq. (3) and leads directly to Eq. (4).

According to the derivation of Cantrell,[7] however, the acoustic radiation stress $\langle \tau_{\epsilon 1} \rangle$ in a solid is nonzero and is related to the average energy density of the propagating wave as

$$\langle \tau_{\epsilon 1} \rangle = -\tfrac{1}{4}\beta_\epsilon E \ . \tag{7}$$

Integrating Eq. (3) and setting the constant of integration equal to $\mu_\epsilon^{-1}\langle \tau_{\epsilon 1} \rangle$ now leads directly to Eq. (6) in agreement with Thurston and Shapiro.[4]

In associating a nonvanishing acoustic radiation stress with the initial value problem without loss, Thompson and Tiersten[5] explain that "we cannot state whether the time needed to establish (the steady-state condition with vanishing radiation stress) is consistent with other experimental constraints such as available sample length and attenuation. Measurements of (the static strain) with sufficient precision to detect the difference (between the two theoretical results) appears to be a formidable experimental challenge."

We now present experimental verification that Eq. (6) is, indeed, correct by measuring the acoustic radiation-induced static strains in single-crystal silicon. The present experimental technique has both the precision and the accuracy to measure the static strain well within the factor of 2 that distinguishes the two theoretical predictions. The experimental arrangement is identical to that described in Ref. 7 except that in the present work a signal averager is used following amplification of the received fundamental (driving wave) and static wave forms. Use of the signal averager allows one to work at lower acoustic-drive amplitudes where the theory is expected to be more accurate. As pointed out in Ref. 7, our capacitive receiving transducer provides a measurement of particle displacements rather than strains in the solid. Hence, the static wave form measured is that obtained by spatially integrating Eq. (6). The resulting wave form for an acoustic tone burst propagating through the crystal is a static displacement pulse having the shape of a right-angled triangle whose slope is exactly that given by the static strain Eq. (6).

TABLE I. Comparison of nonlinearity parameters β_ϵ measured in crystalline silicon using acoustic radiation-induced static strains (present work), harmonic generation, and stress derivative techniques.

Propagation direction	Present work β_ϵ	Harmonic generation[a] β_ϵ	Stress derivative[b] β_ϵ
[111]	3.9±0.6	3.8±0.5	3.4±1.6
[110]	4.3±0.7	4.7±0.6	4.7±0.7
[100]	2.1±0.4	2.0±0.2	2.0±0.1

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[b]H. J. McSkimin and P. Andreatch, Jr., J. Appl. Phys. **35**, 3312 (1961).

Equation (6) predicts that the slope of the static displacement pulse is a linear function of the average acoustic energy density E. This linear function itself has a slope depending on $(\beta_\epsilon/4\mu_\epsilon)$. A typical plot of the measured slope of the static displacement pulse as a function of the energy density E is shown in Fig. 1 for acoustic compressional waves propagating along the [110] direction in crystalline silicon. We find a linear relationship as predicted. Measurement of the slope of the curve in Fig. 1 together with a calculation of $\mu_\epsilon = \rho_0 C_\epsilon^2$ allows us to determine the value of the nonlinearity parameter β_ϵ.

Values of β_ϵ obtained using this procedure for each of the pure mode propagation directions in silicon are shown in Table I together with independent measurements of β_ϵ obtained from harmonic generation and stress derivative techniques. We note that for each propagation-mode agreement among the independent measurements is within the stated experimental errors. We infer from these results the existence of modal acoustic radiation-induced static strains in crystalline solids whose behavior is consistent with the predictions of Eq. (6). We also infer from the considerations of Ref. 5 that, at least for the samples measured, the acoustic wave propagation distance and attenuation are not sufficiently large that the "static stress passes from its initial value. . . to its steady-state value."

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