Critical electronic wave functions on quasiperiodic lattices: Exact calculation of fractal measures

Bill Sutherland
Department of Physics, University of Utah, Salt Lake City, Utah 84112
(Received 12 December 1986)

We examine two wave functions which have recently been found for electrons in quasiperiodic systems, and which can be shown to satisfy an exact self-similarity relation—closely related to the self-similarity of the quasiperiodic lattice itself. The first wave function is for the ground state of an electron on a two-dimensional Penrose lattice, and the second is for the center of the spectrum of the Hamiltonian for an electron on a one-dimensional Fibonacci lattice. We calculate exactly the fractal measure of the singularities of these wave functions, as reflected in the exponent $\alpha(\gamma)$, defined by $\sum_{|\mathbf{x}|\le R} |\Psi(\mathbf{x})|^{\gamma} \sim R^{\alpha(\gamma)}$.

I. INTRODUCTION

There has been much recent interest in the electronic properties of quasiperiodic systems. These systems are intermediate between the completely periodic perfect crystals, and the random or disordered amorphous solids. In particular, one tries to answer delicate questions about the spectrum and eigenstates. Is the spectrum absolutely continuous, pointlike or singular continuous; or equivalently, are the states extended, localized or critical?

Of special interest are the critical states. A solid-state physicist is familiar with extended and localized states, but the critical states represent a new phenomena. One way of characterizing the states is by their growth through an exponent $\alpha(\gamma)$, defined by

$$\sum_{|\mathbf{x}|\le R} |\Psi(\mathbf{x})|^{\gamma} \sim R^{\alpha(\gamma)} . \tag{1}$$

(This quantity is closely related to the participation ratios.) If the state were extended, we would expect $\alpha(\gamma)=d$, where d is the dimensionality of the lattice, while if the state were exponentially localized, we would expect $\alpha(\gamma)=0$. On the other hand, if the wave function should grow as a power, so that $|\Psi(\mathbf{x})| \sim R^{\lambda}$, then we would expect $\alpha(\gamma)=\lambda\gamma+d$, i.e., a linear dependence. [In any case, we have $\alpha(0)=d$.]

In this context then, two recent papers[1,2] which present exact results for electronic wave functions or quasiperiodic lattices are of special interest. Both systems are tight-binding models, in which the electrons sit on the sites of a quasiperiodic lattice with some local potential energy, hopping along bonds to nearest-neighbor sites. The first paper[1] is by Sutherland, and he finds exactly the ground state of an electron on a Penrose lattice ($d=2$). The second paper[2] is by Kohmoto, Sutherland, and Tang, and they examine the state at the center of the spectrum for an electron on a Fibonacci lattice ($d=1$).

Both wave functions are critical, and exhibit scaling or self-similarity of the following sort: If we consider the wave function over two identical regions of the lattice—and by a theorem of Conway identical regions do occur with a spacing of the order of the diameter of the region—then these two portions of the wave function are identical up to normalization, or equivalently a multiplicative scale factor. Further, since both lattices exhibit self-similarity through the so-called inflation transformation, the wave functions also exhibit self-similarity. This self-similarity was exploited in the two papers to set up an equation whose solution gives the scaling limit of the wave function, and hence the exponent $\alpha(\gamma)$. Although in the two papers the equations were only investigated numerically, the results were quite accurate, and clearly ruled out a linear dependence for $\alpha(\gamma)$. However, certain important qualitative features, such as the existence of singularities, were not resolved by the numerical investigations.

It is the purpose of the present paper to present an exact analytic calculation of the exponent $\alpha(\gamma)$. We refer to Refs. 1 and 2 for the original derivation of the equations used as a starting point in this investigation, as well as for references to earlier work.

Before we begin the calculations, however, we wish to place the exponent $\alpha(\gamma)$ and the whole calculation in a more general framework. Such a framework has been supplied by the paper of Halsey, Jensen, Kadanoff, Procaccia, and Shraiman on fractal measures and their singularities. There, the authors consider the measure p on a fractal set, which is expected to scale as

$$p^{\gamma} \sim l^{\gamma x} , \tag{2}$$

for small l. On the other hand, the number of times x takes a value between x' and $x'+dx'$ is expected to be of the form

$$dx' \rho(x') l^{-f(x')} , \tag{3}$$

where $f(x')$ is a continuous function.

The correspondence with our work is simply that we consider the wave function $|\Psi|$ as such a measure on the region $|\mathbf{x}| \le R$, and thus $p=|\Psi|$. It was found that for a region inflated k times, where $R \sim \phi^k$ and ϕ is the golden mean equal to $(1+\sqrt{5})/2$, the wave function scales as $|\Psi| \sim \phi^{-k\alpha}$. This gives the identification $l=\phi^{-k}$.

Halsey *et al.* then proceed to evaluate a quantity $\chi(\gamma)$, defined as

$$\chi(\gamma)=\int dx\,\rho(x)l^{\gamma x-f(x)}\,, \tag{4}$$

by the saddle-point method. It is seen that with the identifications we have made so far, this quantity is precisely

$$\sum_{|\mathbf{x}|\le R}|\Psi(\mathbf{x})|^{\gamma}\sim R^{\alpha(\gamma)}\,. \tag{5}$$

The authors then focus their attention on a set of dimensions D_γ introduced by Hentschel and Procaccia,[3]

$$D_\gamma=\lim_{l\to 0}([1/(\gamma-1)]\{\ln[\chi(\gamma)]/\ln l\})\,. \tag{6}$$

Then clearly we have the final correspondence

$$\alpha(\gamma)=(1-\gamma)D_\gamma\,. \tag{7}$$

[Our $\alpha(\gamma)$ is identical to the $-\tau(q)$ of Ref. 4.]

In the following two sections, we calculate the exponent $\alpha(\gamma)$ exactly. Since we refer back to the original papers for the derivation of the equations we use, we stay as close as possible to the original notation, sacrificing, however, some consistency in the process.

II. GROUND STATE OF THE PENROSE LATTICE

The ground-state wave function of an electron on the Penrose lattice is determined by $F(n\mid k)$ and $T(n\mid k)$, the number of fat and thin tiles, respectively, at an arrow distance of n after k (double) inflations, since the wave function at a site with an arrow distance n is proportional to e^{-n}. (The general form would be $e^{-n\theta}$, but we take the constant θ to be unity.) After these k inflations, the lattice size is of order ϕ^{4k}. The quantity we wish to eventually evaluate is

$$\sum_{|\mathbf{x}|\le R}|\Psi(\mathbf{x})|^{\gamma}\sim R^{\alpha(\gamma)}\,. \tag{8}$$

This we will do by evaluating $P(n\mid k)$, the number of sites with wave function e^{-n} in a lattice inflated k times, which is linearly related to $F(n\mid k)$ and $T(n\mid k)$.

The distribution functions $F(n\mid k)$ and $T(n\mid k)$ in turn are determined by the two inflation equations,

$$\begin{aligned}F(n\mid k+1)&=F(n\mid k)+3F(n-1\mid k)+F(n-2\mid k)\\&\quad+2T(n\mid k)+T(n-1\mid k)\,,\\T(n\mid k+1)&=2T(n-1\mid k)+F(n-1\mid k)+2F(n-2\mid k)\,.\end{aligned} \tag{9}$$

These equations are linear and invariant if we translate by either n or k, so we Fourier transform the equations by

$$\begin{aligned}F(n\mid k)&=(1/2\pi)\int_{-\pi}^{\pi}\exp[i\lambda(n-k)]f(\lambda\mid k)d\lambda\,,\\T(n\mid k)&=(1/2\pi)\int_{-\pi}^{\pi}\exp[i\lambda(n-k)]t(\lambda\mid k)d\lambda\,.\end{aligned} \tag{10}$$

Thus

$$\begin{aligned}f(\lambda\mid k)&=\sum_n\exp[-i\lambda(n-k)]F(n\mid k)\,,\\t(\lambda\mid k)&=\sum_n\exp[-i\lambda(n-k)]T(n\mid k)\,.\end{aligned} \tag{11}$$

If we introduce the variable $\Lambda=e^{i\lambda}$, the inflation equations become

$$\begin{aligned}f(\lambda\mid k+1)&=(3+\Lambda+\Lambda^{-1})f(\lambda\mid k)+(2\Lambda+1)t(\lambda\mid k)\,,\\t(\lambda\mid k+1)&=2t(\lambda\mid k)+(2\Lambda^{-1}+1)f(\lambda\mid k)\,.\end{aligned} \tag{12}$$

Let

$$T(\lambda)\Phi_\pm(\lambda)=\Omega_\pm(\lambda)\Phi_\pm(\lambda)\,, \tag{13}$$

where

$$T(\lambda)=\begin{bmatrix}3+\Lambda+\Lambda^{-1} & 2\Lambda+1\\ 2 & 2\Lambda^{-1}+1\end{bmatrix}. \tag{14}$$

Now we can write

$$\begin{aligned}f(\lambda\mid k)&=f_+(\lambda)\Phi_+^{(1)}\Omega_+^k+f_-(\lambda)\Phi_-^{(1)}\Omega_-^k\,,\\t(\lambda\mid k)&=t_+(\lambda)\Phi_+^{(2)}\Omega_+^k+t_-(\lambda)\Phi_-^{(2)}\Omega_-^k\,.\end{aligned} \tag{15}$$

Then, we have that

$$\det T(\lambda)=\Omega_+(\lambda)\Omega_-(\lambda)=1\,, \tag{16}$$

so

$$\Omega_-(\lambda)=[\Omega_+(\lambda)]^{-1}\,, \tag{17}$$

and

$$\begin{aligned}\mathrm{Tr}T(\lambda)&=\Omega_+(\lambda)+\Omega_-(\lambda)\\&=\Lambda+\Lambda^{-1}+5=2\cos(\lambda)+5\ge 3\,.\end{aligned} \tag{18}$$

The equation for the eigenvalues is then

$$\Omega^2+1-\Omega(\Lambda+\Lambda^{-1}+5)=0\,, \tag{19}$$

with solutions

$$\begin{aligned}\Omega_\pm&=\{\Lambda+\Lambda^{-1}+5\pm[(\Lambda+\Lambda^{-1}+7)(\Lambda+\Lambda^{-1}+3)]^{1/2}\}/2\\&=\cos(\lambda)+\tfrac{5}{2}\pm\{[\cos(\lambda)+\tfrac{7}{2}][\cos(\lambda)+\tfrac{3}{2}]\}^{1/2}\,.\end{aligned} \tag{20}$$

We write these as

$$\begin{aligned}&\Omega_+(\lambda)=e^{\omega(\lambda)}>1,\ \text{ real}\,,\\&0<\Omega_-(\lambda)=e^{-\omega(\lambda)}<1,\ \text{ real}\,,\end{aligned} \tag{21}$$

where $\omega(\lambda)$, the logarithm of $\Omega_+(\lambda)$, is determined by

$$\cosh[\omega(\lambda)]=\cos(\lambda)+\tfrac{5}{2}\,. \tag{22}$$

We see from the expansion of $f(\lambda\mid k)$ and $t(\lambda\mid k)$, that for a large number of inflations k,

$$\begin{aligned}f(\lambda\mid k)&\to f_+(\lambda)\Psi_+^{(1)}\Omega_+^k\,,\\t(\lambda\mid k)&\to t_+(\lambda)\Psi_+^{(2)}\Omega_+^k\ \text{ as } k\to\infty\,.\end{aligned} \tag{23}$$

As a consequence,

$$\begin{aligned}F(n\mid k)&\to F_+(n\mid k)\\&=(1/2\pi)\int_{-\pi}^{\pi}\exp[i\lambda(n-k)+k\omega(\lambda)]\\&\qquad\times f_+(\lambda)\Psi_+^{(1)}d\lambda\,,\end{aligned} \tag{24}$$

as $k \to \infty$. We have a similar expression for $T(n \mid k)$.

We now examine $F_+(n \mid k)$ for large k. Since the function is nonzero only for $n \approx k$, we define a scaled variable x by

$$x = n/k - 1, \quad -1 \le x \le 1 . \tag{25}$$

Then $F_+(n \mid k)$ takes the form

$$F_+(n \mid k) = \int_{-\pi}^{\pi} \exp[kG(\lambda \mid x)] g(\lambda) d\lambda , \tag{26}$$

with

$$G(\lambda \mid x) = i\lambda x + \omega(\lambda) , \quad g(\lambda) = f_+(\lambda)\Psi_+^{(1)}/2\pi . \tag{27}$$

We can evaluate this integral for large k by the saddle-point method, as follows. The contour from $\lambda = -\pi$ to π along the real axis is deformed so that it passes over a saddle point λ_0 of $G(\lambda \mid x)$ in a direction so that the integrand has a maximum at the saddle point. Therefore, the saddle point λ_0 is determined by

$$dG/d\lambda \mid_{\lambda=\lambda_0} = 0 \quad \text{or} \quad d\omega/d\lambda \mid_{\lambda=\lambda_0} = -ix . \tag{28}$$

Thus λ_0 is a function of x.

To continue with the saddle-point method, for λ near λ_0, we write

$$\begin{aligned} &\lambda - \lambda_0 = \rho e^{i\theta}, \quad d\lambda = d\rho\, e^{i\theta} , \\ &G(\lambda \mid x) \approx G(\lambda_0 \mid x) + (\rho e^{i\theta})^2 G''(\lambda_0 \mid x)/2 . \end{aligned} \tag{29}$$

We choose the direction θ so that

$$e^{2i\theta} G''(\lambda_0 \mid x) = -\Gamma ,$$

where Γ is real and positive. Therefore, we have the asymptotic evaluation of $F(n \mid k)$ for large k as

$$\begin{aligned} F(n \mid k) &\to \exp[kG(\lambda_0 \mid x)] g(\lambda_0) e^{i\theta} \int_{-\infty}^{\infty} \exp(-k\Gamma\rho^2/2) \\ &= \exp[kG(\lambda_0 \mid x)][g(\lambda_0) e^{i\theta} (2\pi/k\Gamma)^{1/2}] . \end{aligned} \tag{30}$$

In fact, in this and in all later sections, we are only interested in the leading terms of the distribution functions, in the sense that

$$\lim_{k \to \infty} \{ \ln[F(n \mid k)]/k \} = G(\lambda_0 \mid x) . \tag{31}$$

This we write as

$$F(n \mid k) \sim \exp[kG(\lambda_0 \mid x)] . \tag{32}$$

To return to the equation for the saddle point λ_0, we see from the expression for $\Omega_+(\lambda)$, using

$$\omega(\lambda) = \ln[\Omega_+(\lambda)] , \tag{33}$$

that

$$d\omega/d\lambda = -\sin(\lambda)\{[\cos(\lambda) + \tfrac{7}{2}][\cos(\lambda) + \tfrac{3}{2}]\}^{-1/2} . \tag{34}$$

Thus the saddle point λ_0 is on the imaginary axis, so we write

$$\lambda_0 = -i\gamma , \tag{35}$$

and the final equation for the saddle point is

$$x = -\sinh(\gamma)\{[\cosh(\gamma) + \tfrac{7}{2}][\cosh(\gamma) + \tfrac{3}{2}]\}^{-1/2} . \tag{36}$$

In Fig. 1, we show $G(\lambda_0(x) \mid x) = \omega(-i\gamma(x)) + x\gamma(x)$ as a function of x. As we shall see, it is more useful to leave $x(\gamma)$ in parametric form, than to try to solve explicitly for $\gamma(x)$.

Now, we proceed to evaluate the exponent α as a function of γ, which was the original aim of Ref. 1. There is a linear relation between the wave-function distribution function $P(n \mid k)$ and the tile distribution functions, so that the leading term of $P(n \mid k)$ is also given by

$$P(n \mid k) \sim \exp[kG(\lambda_0 \mid x)] . \tag{37}$$

Thus, considering the quantity we wish to evaluate,

$$D(k \mid \gamma) = \sum_{|x| < R} \Psi^{\gamma} = \int_0^{2k} P(n \mid k) e^{-n\gamma} dn \sim R^{\alpha(\gamma)} , \tag{38}$$

we have the expression

$$k \int_{-1}^{+1} \exp[kG(x) - \gamma k(x+1)] dx = R^{\alpha(\gamma)} . \tag{39}$$

By the saddle-point method, where no deformation of the contour is needed, the leading contribution to the integral comes from near a point x_0 which maximizes the integrand, and hence the exponent of the integrand. The equation which determines this point x_0 is then

$$\begin{aligned} &-\gamma + dG/dx \mid_{x=x_0} \\ &\quad = 0 = -\gamma + d/dx[\omega(-i\gamma'(x)) + x\gamma'(x)] \mid_{x=x_0} \\ &\quad = -\gamma + \gamma' + \{d\lambda/dx(d\omega/d\lambda + ix)\} \mid_{x=x_0} . \end{aligned} \tag{40}$$

But the quantity in parentheses vanishes, since it is exactly the condition which determined the saddle point $\lambda_0 = \lambda(x) = -i\gamma(x)$. Thus we arrive at the result $\gamma = \gamma'$, justifying the use of the same symbol for the two quantities, and

$$\exp\{k[\omega(-i\gamma) - \gamma]\} = R^{\alpha(\gamma)} = \exp[k\alpha(\gamma) \ln\phi] . \tag{41}$$

[The constant ϕ is the golden mean $\phi = (1 + \sqrt{5})/2$.] Our final result for the exponent $\alpha(\gamma)$ is then

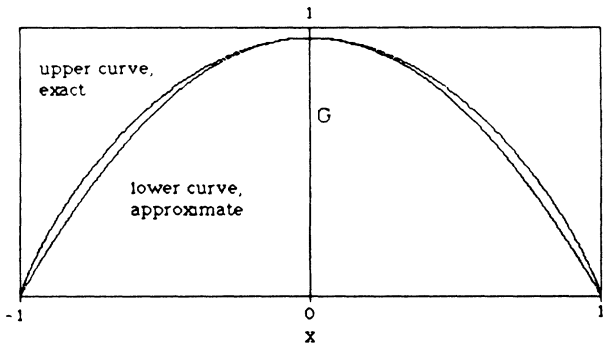

FIG. 1. We show the scaling function G as a function of the scaled variable x, for both the exact result and the quadratic approximation. This is the ground-state wave function on the Penrose lattice.

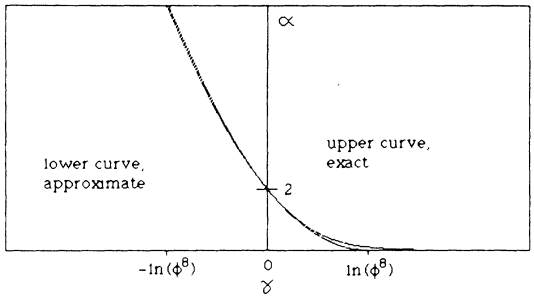

FIG. 2. We show the exponent α as a function of the power γ of the wave function, for both the exact result and the quadratic approximation. This is the ground-state wave function on the Penrose lattice.

$$\alpha(\gamma)=\ln(\cosh(\gamma)+\tfrac{5}{2}+\{[\cosh(\gamma)+\tfrac{7}{2}][\cosh(\gamma)+\tfrac{3}{2}]\}^{1/2}-\gamma)/\ln(\phi^2)\ . \tag{42}$$

We show this function in Fig. 2; for very small γ, $\alpha(\gamma)$ increases linearly to the left.

The previous approximation of Ref. 1, based on numerical evidence, was

$$\alpha(\gamma)=\begin{cases}0, & \gamma\geq\ln(\phi^8)\\ 2[1-\gamma/\ln(\phi^8)]2, & \ln(\phi^8)\geq\gamma\geq-\ln(\phi^8)\\ -\gamma/\ln(\phi), & -\ln(\phi^8)\geq\gamma\ .\end{cases} \tag{43}$$

This function is also shown in Fig. 2.

III. SIX-CYCLE OF FIBONACCI LATTICE

For the six-cycle of the one-dimensional Fibonacci lattice, the evolution equations are more complicated. As discussed in Ref. 2, the equations involve the distribution function $P(q,r,a\mid k)$, which is the number of times the group element $g(q,r)=B^qA^r$, followed by the element $a=A$ or B, occurs in the lattice inflated k times, since the wave function is given by $|\Psi|=e^{\pm r\theta}$. For large k, this lattice has $n=\phi^{3k}$ sites. (Again, ϕ is the golden mean.) The eventual aim is to calculate

$$\sum_{m\leq n}|\Psi(m)|^\gamma=\sum_q\sum_r\sum_a P(q,r,a\mid k)e^{-(k\pm r)\gamma\theta}\sim n^{\alpha(\gamma)}\ . \tag{44}$$

The evolution equations for the distribution functions are

$$\begin{aligned}P(q,r,B\mid k+1)&=P(q-2r-2,r-(-1)^q,B\mid k)+P(q-2r-2,r-(-1)^q,A\mid k)+P(q-2r-1,r,A\mid k)\ ,\\ P(q,r,A\mid k+1)&=P(q-2r,r,A\mid k)+P(q-2r+1,r-(-1)^q,A\mid k)\\ &\quad+P(q-2r-2,r,A\mid k)+P(q-2r,r,B\mid k)+P(q-2r+1,r-(-1)^q,B\mid k)\ .\end{aligned} \tag{45}$$

Since the equations are different in form depending upon whether q and r are even or odd, we must Fourier transform each series independently. (Remember that q takes only the values 0,1,2,3, and so is to be interpreted as modulo 4.) Let $s=(-1)^q$ and $t=(-1)^r$ denote the parities of q and r; they will serve as labels for the four series of Fourier coefficients. Then the distribution functions are given as

$$P(q,r,a\mid k)=(1/8\pi)\sum_j\int_{-\pi}^{\pi}d\lambda\exp[i(r\lambda+\pi qj/2)]\times p_{\rm sta}(j,\lambda\mid k)\ . \tag{46}$$

The index j for the Fourier transform of q only takes the values 0,1,2,3. However, we see that in the evaluation of the exponent $\alpha(\gamma)$, we perform a summation over q of $P(q,r,a\mid k)$, which selects the value $j=0$.

Upon substituting this expression into the evolution equations, and projecting out the equations for $j=0$ and λ, we have the following matrix evolution equation for the resulting eight Fourier coefficients $p_{\rm sta}(0,\lambda\mid k)$, where $s=\pm1$, $t=\pm1$, and $a=B,A$, arranged as a column vector $\eta(k)$:

$$\eta(k+1)=T\eta(k)\ . \tag{47}$$

Once again, we introduce $\Lambda=e^{i\lambda}$, so that the evolution matrix T can be written as

$$T=\begin{pmatrix}0&0&\Lambda^{-1}&0&0&0&\Lambda^{-1}&1\\ 0&0&0&\Lambda&0&0&1&\Lambda\\ \Lambda^{-1}&0&0&0&\Lambda^{-1}&1&0&0\\ 0&\Lambda&0&0&1&\Lambda&0&0\\ 1&\Lambda^{-1}&0&0&2&\Lambda^{-1}&0&0\\ \Lambda&1&0&0&\Lambda&2&0&0\\ 0&0&1&\Lambda^{-1}&0&0&2&\Lambda^{-1}\\ 0&0&\Lambda&1&0&0&\Lambda&2\end{pmatrix}\ . \tag{48}$$

As seen previously, the behavior of all distribution functions for large inflations k, and hence large lattices, will be dominated by the maximum eigenvalue of the evolution matrix T, which we once again denote as $\Omega(\lambda)=e^{\omega(\lambda)}$. Then in fact, we will have

$$p_{\rm sta}(0,\lambda \mid k) \sim e^{k\omega(\lambda)} . \tag{49}$$

However, it is not so easy to explicitly diagonalize an 8×8 matrix. We first observe that the eigenvectors belong to one of two families; either the components are a,b,a,b,c,d,c,d or $a,b,-a,-b,c,d,-c,-d$. The maximum eigenvalue corresponds to the first choice, and this reduces the matrix T to a 4×4 size:

$$T=\begin{bmatrix} \Lambda^{-1} & 0 & \Lambda^{-1} & 1 \\ 0 & \Lambda & 1 & \Lambda \\ 1 & \Lambda^{-1} & 2 & \Lambda^{-1} \\ \Lambda & 1 & \Lambda & 2 \end{bmatrix} . \tag{50}$$

If we examine the matrix $T-I$, we see that the fourth row is Λ times the third, and thus the two rows are not independent, and $T-I$ has a zero eigenvalue. Our original matrix T has an eigenvalue 1, and for the other eigenvectors, the last two components must be related by $d=\Lambda c$. This allows a final reduction of the matrix T to a 3×3 matrix:

$$T=\begin{bmatrix} \Lambda^{-1} & 0 & \Lambda^{-1}+\Lambda \\ 0 & \Lambda & 1+\Lambda^2 \\ 1 & \Lambda^{-1} & 3 \end{bmatrix} . \tag{51}$$

This size is manageable.

Setting the determinant of $T-\Omega I$ equal to zero, we obtain a third-order equation for Ω,

$$\Omega^3-\Omega^2[3+2\cos(\lambda)]+\Omega[1+2\cos(\lambda)] -[3-4\cos^2(\lambda)]=0 . \tag{52}$$

Although we could determine Ω as a function of λ, our previous experience suggests that we instead leave it in this parametric form, or even better determine $\cos(\lambda)$ as a function of Ω. This gives

$$\cos(\lambda)=[\Omega(\Omega-1)-(\Omega^4-6\Omega^3+13\Omega^2-4\Omega+12)^{1/2}]/4 . \tag{53}$$

As before, to evaluate the distribution function itself from the Fourier transform, we employ the saddle-point method. The saddle point is on the imaginary λ axis, and at the saddle points $\Omega\geq 2+\sqrt{5}$. The correct branch of the square root in the expression for $\cos(\lambda)$ has been selected so that the saddle point is obtained for $\Omega\geq 2+\sqrt{5}$.

And as before, evaluation of the exponent $\alpha(\gamma)$—a second saddle-point evaluation—has the effect of selecting the Fourier-transform variable to be imaginary and equal to $-i\gamma\theta$. Thus,

$$\sum_{m\leq n} |\Psi(m)|^{\gamma} = \sum_q \sum_r \sum_a P(q,r,a \mid k) e^{-(k\pm r)\gamma\theta} \sim \exp\{k[\omega(-i\gamma\theta)-\gamma\theta]\} = n^{\alpha(\gamma)} = \phi^{3k\alpha(\gamma)} . \tag{54}$$

Thus, we have as our final expression for the exponent $\alpha(\gamma)$ the parametric form,

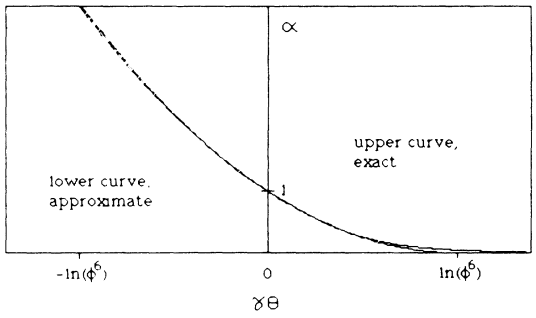

FIG. 3. We show the exponent α as a function of the power γ of the wave function, for both the exact result and the quadratic approximation. This is the wave function at the center of the spectrum on the Fibonacci lattice.

$$\alpha=[\ln(\Omega)-\gamma\theta]/\ln(\phi^3) ,$$
$$\gamma\theta=\cosh^{-1}([\Omega(\Omega-1) -(\Omega^4-6\Omega^3+13\Omega^2-4\Omega+12)^{1/2}]/4) , \tag{55}$$

with $\Omega\geq 2+\sqrt{5}$. The result is shown in Fig. 3 along with the approximation from Ref. 2,

$$\alpha(r)=\begin{cases} 0, & \gamma\theta\geq \ln(\phi^6) \\ [1-\gamma\theta/\ln(\phi^6)]^2, & \ln(\phi^6)\geq\gamma\theta\geq -\ln(\phi^6) \\ -4\gamma\theta/\ln(\phi), & -\ln(\phi^6)\geq\gamma\theta . \end{cases} \tag{56}$$

IV. CONCLUSION

In conclusion, we review the general structure of the calculations, because similar exact results probably can be obtained for other physically interesting problems. The object we seek to calculate is $P_a(n \mid k)$, the number of times that the logarithm of the absolute value of the wave function is $-n$, in a sample inflated k times with linear dimensions ϕ^k. The index a denotes other quantities that must be specified. Then these distribution functions evolve by an equation

$$P_a(n \mid k+1)=\sum_{n'}\sum_{a'} T_{aa'}(n-n')P_{a'}(n' \mid k) . \tag{57}$$

That the evolution matrix depends only on the difference $n-n'$ is a reflection of the self-similarity of the wave function, where the wave functions over identical regions differ only by a multiplicative normalization factor. Because of this dependence, however, it is useful to Fourier transform $P_a(n \mid k)$,

$$P_a(n \mid k)=(1/2\pi)\int_{-\pi}^{\pi} e^{in\lambda} p_a(\lambda \mid k)d\lambda , \tag{58}$$

so that the evolution equation becomes a matrix equation

$$p_a(\lambda \mid k+1)=\sum_{a'} t_{aa'}(\lambda)p_{a'}(\lambda \mid k) . \tag{59}$$

Here $t_{aa'}(\lambda)$ is the Fourier transform of the evolution matrix $T_{aa'}(n)$. The behavior of $p_a(\lambda \mid k)$ will be dominated by the maximum eigenvalue $\Omega(\lambda)=e^{\omega(\lambda)}$ of the matrix $t(\lambda)$,

$$t(\lambda)\Phi=\Omega(\lambda)\Phi\ , \tag{60}$$

so that the asymptotic behavior is

$$p(\lambda \mid k)\sim e^{k\omega(\lambda)}\ . \tag{61}$$

In inverting the Fourier transform for $P_a(n \mid k)$, we can perform a saddle-point evaluation at the saddle point $\lambda_0=-i\gamma$, given as the solution to

$$d\omega/d\lambda \mid_{\lambda=\lambda_0}+in/k=0\ , \tag{62}$$

to find

$$P_a(n \mid k)\sim e^{n\gamma+k\omega(-i\gamma)}\ . \tag{63}$$

We define an exponent α by

$$\sum_n \mid \Psi \mid^{\gamma}=\sum_n e^{-n\gamma}\sim e^{k\alpha(\gamma)}$$
$$=\sum_n e^{-n\gamma+n\gamma'+k\omega(-i\gamma')}\ , \tag{64}$$

so $\alpha(\gamma)=\omega(-i\gamma)$ since $\gamma=\gamma'$.

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