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Hierarchy of current cumulants on a Sierpiński gasket

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By solving exactly the electrical-conductivity problem for a Sierpiński gasket, we calculate the cumulants of all powers of the currents $|i|^a$, the corresponding critical exponents $D(a)$, and scaling factors $2^{D(a)}$, as well as the complete spectrum of singularities. For positive, even integer powers of a, we obtain theoretically the scaling factors and corrections to scaling, and hence conclude that the sequence $2^{D(a)}$ is unlikely to be obtained simply. Finally, we introduce a generalized "grafted" gasket, which exhibits a tunable crossover from a fractal to a classical measure.

Originally introduced as a possible model for percolation systems, the Sierpiński gasket[1] has become, because of its conceptual and calculational simplicity yet great richness of structural details, one of the most frequently studied nonhomogeneous deterministic fractals. Although some scaling properties have long been known[1-3] (e.g., conductivity, diffusion, noise, elasticity, etc.), a complete analysis of the hierarchy of exponents and of their singularity spectrum[4] has not yet, to our knowledge, been done.

I. A FIRST STEP: THE EXACT DISTRIBUTION

In spite of the fact that the Sierpiński gasket displays a remarkably rich distribution of currents, we found that a simple algorithm, which we shall describe elsewhere, allows us to write, by inspection, the complete set of currents. Specifically, we consider the case of the nth iteration of a two-dimensional Sierpiński gasket (hereafter called n-SG), all elementary bonds of which are unit resistors. Unit current is injected into one of the external nodes and flows out of a second, the third being left open. The distribution of currents in the bonds of the gasket is displayed in Fig. 1 for $n=9,10$. With a logarithmic current abscissa, we observe that the distribution consists of a quasicontinuous region as well as spikes at high and low values of the current (Fig. 1).

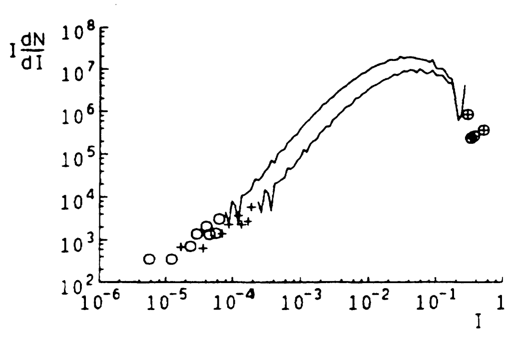

FIG. 1. Histogram of the currents in a 10-SG (upper curve, ○) and a 9-SG (lower, +) showing the distribution function $dN/d(\ln I)$ vs the current amplitudes I. (N.B., the isolated points at left and right correspond to "spikes," separated from the "continuous" curves, which dominate the cumulant behavior for large negative and positive exponents, respectively.)

In the most general case, let $j,k,-(j+k)$ be the currents flowing inward at the three external vertices of an n-SG. We want to compute, for arbitrary exponent a, the cumulant

$$S_n^a(j,k)=\sum_m |i_m|^a ,$$

where j_m is the current flowing in the mth bond and the sum runs over the whole n-SG. We define the scaling factor $\lambda(a)$, independent of j and k, as

$$\lambda(a)=\lim_{n\to\infty} S_{n+1}^a(j,k)/S_n^a(j,k).$$

Equivalently, $S_n^a(j,k)\propto\lambda^n(a)$, for any external boundary condition (j,k fixed). The scaling factor $\lambda(a)$ is related to the fractal scaling exponent, $D(a)$. Since the length L_n of the n-SG is $L_n=2^n$, we have $S_n^a(j,k)\propto L_n^{D(a)}$, where

$$D(a)=\log[\lambda(a)]/\log(2). \quad (1)$$

II. SYMMETRY CONDITIONS

When $a=0$, the sum S_n^0 reduces to the mass (number of bonds) of the n-SG; since there are no dangling ends, $S_n^0=3^n$. Therefore $\lambda(0)=3$, and we have $D(0)=\log(3)/\log(2)$ as the fractal dimension of the gasket.

The case $a=2$ measures the electrical energy dissipated in the gasket. For fixed boundary conditions, this sum gives the scaling of the resistance. $S_n^2(j,k)$ is a second-degree homogeneous polynomial in j and k. Let us write

$$S_n^2(j,k)=A_nj^2+B_njk+C_nk^2.$$

The invariance of S_n^2 under the change $(j,k)\to(k,j)$ (plane symmetry) gives $A_n=C_n$. The threefold symmetry $(j,k)\to(k,-(j+k))$ implies

$$A_nj^2+B_njk+A_nk^2=A_nk^2-B_njk+A_n(j+k)^2,$$

yielding $A_n=B_n$, and thus,

$$S_n^2(j,k)=A_n(j^2+jk+k^2).$$

The most general expression of the energy dissipated must be proportional to this last polynomial. The vector space of second-degree homogeneous polynomials invariant under the symmetries of the gasket is one dimensional. We may now examine the relation between A_n and A_{n-1} for two consecutive generations. Minimizing the dissipative energy of a n-SG with respect to the current flowing through one of the nodes connecting two of the three constitutive $(n-1)$-SG provides us with the recursion relation

$$A_n = (5/3)A_{n-1} \; ; \tag{2}$$

thus $S_n^2(j,k) = (5/3)S_{n-1}^2(j,k) \propto (5/3)^n$, $\lambda(2) = 5/3$, and $D(2) = \log(5/3)/\log(2)$.

The fourth moment of the distribution of intensities $S_n^4(j,k)$, related to the flicker noise,[5] can be handled in a very similar way. There is still only one polynomial satisfying the symmetries so that

$$S_n^4(j,k) = A_n(j^4 + 2j^3k + 3j^2k^2 + 2jk^3 + k^4).$$

Now, going from the $(n-1)$th to the nth generation is straightforward since the currents in the middle of each side are $(j-k)/3$, $(2j+k)/3$, and $(2k+j)/3$. Thus,

$$\begin{aligned} S_n^4(j,k) = {} & S_{n-1}^4((j-k)/3,(2j+k)/3) \\ & + S_{n-1}^4((2j+k)/3,(2k+j)/3) \\ & + S_{n-1}^4((2k+j)/3,(k-j)/3), \end{aligned} \tag{3}$$

giving

$$S_n^4(j,k) = (\tfrac{11}{9})S_{n-1}^4(j,k), \tag{4}$$

so that $\lambda(4) = \frac{11}{9}$ and $D(4) = \log(\frac{11}{9})/\log(2)$.

The results $\lambda(0) = 3$, $\lambda(2) = \frac{5}{3}$, and $\lambda(4) = \frac{11}{9}$ have been previously derived[3] by other methods. Above $a = 4$ (a an even integer), $\lambda(a)$ does not keep such a simple form. The scaling relation, simple in (2) and (4), becomes more complex as corrections to scaling appear.

For $a = 6$, the symmetry conditions select a two-dimensional vector space of homogeneous polynomials. We can choose, for instance, the basis

$$P_6(j,k) = j^6 + 3j^5k - 5j^3k^3 + 3jk^5 + k^6,$$

$$P_6'(j,k) = j^4k^2 + 2j^3k^3 + j^2k^4.$$

Thus,

$$S_n^6(j,k) = A_nP_6(j,k) + B_nP_6'(j,k)$$

and the recursion formula (3), valid for any value of a, allows us to derive the evolution of (A_n, B_n) in a matrix form:

$$\begin{pmatrix} A_n \\ B_n \end{pmatrix} = 3^{-5}\begin{pmatrix} 85 & 240 \\ 24 & 225 \end{pmatrix}\begin{pmatrix} A_{n-1} \\ B_{n-1} \end{pmatrix}.$$

This matrix has two eigenvalues, $\lambda(6) = [155 + 2(2665)^{1/2}]/243$ and $\mu(6) = [155 - 2(2665)^{1/2}]/243$, giving the dominant scaling factor and a first correction to scaling. This last result casts serious doubts on the possible existence of a simple expression yielding the whole sequence $\lambda(a)$ in a closed form.

Higher terms can be treated in a very similar fashion: First, select a basis of polynomials satisfying the required symmetries, say P_α^a ($\alpha = 1, \ldots, m$), then, according to the recursion formula (3) compute the matrix $M_{\alpha\beta}$, such that

$$P_\alpha^a((j-k)/3,(2j+k)/3) + P_\alpha^a((2j+k)/3,(2k+j)/3) + P_\alpha^a((2k+j)/3,(k-j)/3) = M_{\alpha\beta}P_\beta^a(j,k).$$

The scaling factor $\lambda(a)$ will then be the largest eigenvalue of the matrix M. Thus $\lambda(a)$ will be a root of an mth-degree polynomial (the characteristic polynomial of M), and m increases with a (i.e., $a = 8$ still has $m = 2$, but $a = 10$ or 12 has $m = 3$, etc.). The expression for $\lambda(a)$ thus appears algebraically more and more complex with increasing order. The calculation becomes more tedious but does not present any fundamental difficulty.

This exact analytical treatment can, however, only be carried out for positive, even integer values of a. For other cases, we can proceed by way of the exact distribution known from our numerical algorithm. In this way, we derive the asymptotic form

$$\lambda(a) \simeq 1 + 4(2^{-a}), \quad a \to +\infty. \tag{5}$$

For $a < 0$, we find that the ratio $S_{n+1}^a(1,0)/S_n^a(1,0)$ oscillates quite strongly as a function of n. Thus, only more approximate values of $\lambda(a)$ can be derived. Since, however, the asymptotic value for $a \to -\infty$ depends only on the weakest current spike of the distribution, we can obtain the form

$$\lambda(a) \simeq (1.609)3^{-a}, \quad a \to -\infty. \tag{6}$$

III. THE SPECTRUM OF SINGULARITIES

We can now inscribe our results in the framework of the spectrum-of-singularities approach, recently developed by Halsey *et al.*, and also illustrated experimentally,[4] for systems with self-similarity properties which need to be described by an infinite set of critical exponents.

The reader is referred to Ref. 4 for full details. In Halsey *et al.*, the authors suggest the use of a normalized probability distribution p_j as the fundamental measure of the system, and then study the behavior of the cumulants of various powers of this probability:

$$\chi(q) = \sum_j p_j^q \simeq L^{-\tau_q},$$

where $\tau_q = -\log\chi(q)/\log L$ thus plays the role of a "fractal dimension" describing the scaling of the cumulant. In terms of τ_q, they then define a singularity strength $\alpha_q = d\tau_q/dq$, and the density of the singularity spectrum $f(\alpha_q) = q\alpha_q - \tau_q$. τ_q is also related to the dimensions D_q (initially introduced by Hentschel and Procaccia[6]), defined by $\tau_q = (q-1)D_q$.

To apply their formalism to our particular problem we use as a natural definition of the measure p_j, the fraction

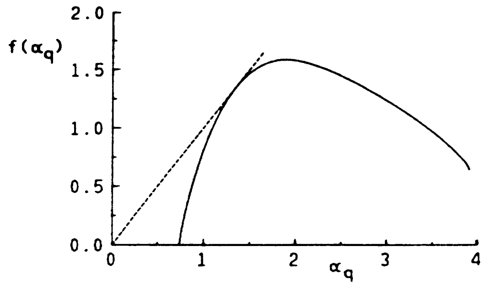

FIG. 2. Spectrum, $f(a_q)$ vs a_q, of the singularities of the Sierpiński gasket distribution. The left end of the curve corresponds to $q=+\infty$, the right one to $q=-\infty$. At these points the value of f is, respectively, the fractal dimension of the hottest, and coldest, bonds. The apex value for $q=0$ gives the fractal dimension of the SG. [The tangency of the dashed line $f(a)=a$, to the spectrum for $q=1$, and the derivative relation $q=df(a_q)/da_q$, are generic properties coming from the normalization of the measure.]

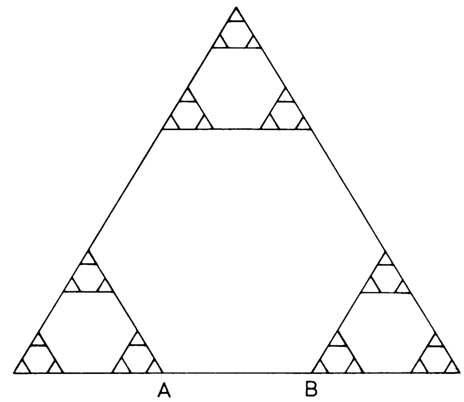

FIG. 3. The grafted nth-generation Sierpiński gasket (n-GSG) is obtained by connecting three $(n-1)$-GSG by means of linear elements, like AB, of length y^n. (Here $n=3$.) The classical SG results if $y=0$.

of the total energy dissipated in the jth bond of the n-SG, i.e., $p_j=i_j^2/\sum_m i_m^2$. This definition (which differs from other approaches to the problem[7]) has the merit of using a fundamental physical quantity which also preserves normalization as an essential feature of the analysis. Thus,

$$\chi(q)=\sum_j i_j^{2q}\Big/\left(\sum_m i_m^2\right)^q$$

and, using the same analysis as for Eq. (1), we derive

$$\tau_q=\{q\log[\lambda(2)]-\log[\lambda(2q)]\}/\log 2=qD(2)-D(2q),$$

where a_q and $f(a_q)$ are trivially obtained from their definitions above.

Then, using the exactly calculated values of the current distributions for all $n\leq 10$, we are able to obtain numerically the entire curve of $f(a_q)$ corresponding to all values of q (Fig. 2). We note in particular that from the asymptotic forms (5) and (6), the limiting values $a_{+\infty}=\log\lambda(2)/\log(2)$, $f(a_{+\infty})=0$, $a_{-\infty}=\log(15/3)/\log(2)$, and $f(a_{-\infty})\simeq 0.686$ result. These correspond to the physical situation that, for a very large n-SG, we find exactly four bonds that carry the highest current ($q\to+\infty$), each with a current of 0.5, while there is a fractal set of lowest-current-carrying bonds ($q\to-\infty$) each carrying a current $3^{-(n+1)}$. In addition to the limits just stated, the function $f(a_q)$ has the standard[4] properties: $df/da_q=q$, $f(a_1)=a_1$, etc. We find indeed that the range of a and f characterizes in very compact form the singularity properties of the current-carrying n-SG.

IV. THE "GRAFTED" SIERPIŃSKI GASKET

The grafted Sierpiński gasket (GSG) is a fractal structure which is a generalization of the ordinary SG. The n-GSG (nth generation) is constructed out of three $(n-1)$-GSG's connected by three linear elements of length y^n (Fig. 3) where y is a free parameter. All the symmetries of the usual gasket are preserved. The scaling behavior of the various moments of the distribution of currents exhibits some curious features. The length of a n-GSG L_n follows the recursion formula

$$L_n=2L_{n-1}+y^n,$$

whereas the mass M_n obeys

$$M_n=3M_{n-1}+3y^n.$$

If $y<2$, then $L_n\propto 2^n$ and $M_n\propto 3^n$, giving the usual SG fractal dimensionality $d_f=D(0)=\log(3)/\log(2)$. If $2<y<3$, then $L_n\propto y^n$, $M_n\propto 3^n$, and thus $d_f=\log(3)/\log(y)$. And for $y>3$: $L_n\propto y^n$, $M_n\propto y^n$, and $d_f=1$. Thus, varying y from 2 to 3, the fractal dimensionality can be tuned continuously from that of the SG to 1. This last dimension refers to that of the linear elements and there the mass, length, resistance, noise, etc., all scale the same way. Thus it has only one exponent, namely, 1.

We can proceed and compute the resistance R_n of an n-GSG. An argument very similar to the one described in Sec. II (the symmetries are identical) gives

$$R_n=\tfrac{5}{3}R_{n-1}+\tfrac{2}{3}y^n;$$

therefore, $R_n\propto\Lambda(2)^n$ with a scaling factor $\Lambda(2)=\max(\frac{5}{3},y)$.

The critical index $\log[\Lambda(2)]/\log[\max(y,2)]$ varies continuously from that of the SG [$\Lambda(2)=\lambda(2)$ for $y<\frac{5}{3}$] to that of the one-dimensional elements for $y<2$.

This feature is true for all moments. The scaling law obeyed by S_n^a is always controlled by the highest of y and $\lambda(a)$.

For any given x, we can construct a GSG with $y=\lambda(x)$. Then the scaling factor of order a, $\Lambda(a)$, will be $\Lambda(a)=\lambda(a)$ (that of the SG) if $a<x$, and $\Lambda(a)=\lambda(x)=y$ if $a>x$.

The series of factors and thus of critical exponents is cut at order x and remains constant thereafter. The spectrum of singularities (described in Sec. III) of the GSG is only a subpart of the SG spectrum. It stops for a value $a=x$.

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[7]L. de Arcangelis, S. Redner, and A. Coniglio (unpublished) introduce a similar scaling approach for a hierarchical fractal. One should beware of the fact that our definitions of α_q and $f(\alpha_q)$ follow Ref. 4 and thus differ somewhat from similar quantities used in this reference.