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Weak localization of photons and backscattering from finite systems

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We solve exactly the equation for diffusing photons in a slab and calculate the backscattering intensity which results from weak localization interference. It is shown that the shape of the backscattering peak depends strongly on the finiteness of the system. Our analysis establishes the relationship between the shape of the backscattering peak and the photon loop size.

The phenomenon of weak localization of electrons[1] was recently applied[2] to photons. Weak localization of electrons leads to quantum interference effects which result[3] in new transport phenomena. The quantum interference effects are caused because of the wavelike character of the electron wave functions. It is therefore expected that these interference effects will show up even in classical systems and, in particular, in light scattering. The first observation[2] of weak localization of photons was for polystyrene spheres suspended in water. A coherent backscattering peak of a triangular shape was observed and theoretically explained.[4] Very recently it was found[5,6] that the weak localization effects of photons in a disordered *solid* are richer than for a liquid. The backscattering interference pattern results in amplitude fluctuations similar to the phenomena of universal fluctuations[7] in the electrical conductance of a disordered metal. These intensity fluctuations are characterized by unique statistics[6] which change with the degree of disorder. The problem of ensemble average of conductance fluctuations seems to have a photonic analog.[5,6] Performing an ensemble average of the scattered intensity fluctuations for a disordered solid results[5,6] in a coherent narrow backscattering peak, similar to the observed peak for disordered fluids. The shape of this peak is triangular, in agreement with the theory of Akkermans, Wolf, and Maynard.[4]

In this Rapid Communication, we study finite-size effects on weak localization of photons. We solve the equation for diffusing photons in a finite slab with *two* boundary absorbing planes and find that the shape of the backscattering peak depends strongly on the width of the slab. We also calculate the width of the backscattering peak as a function of the length and the width of the system. It is shown that finite-size effects eliminate the triangular shape present for infinite systems.

Our present analysis establishes the relationship between the classical photon trajectories and the coherent backscattering peak. The removal of large loops results in a wider peak. The importance of this effect is that the as yet unobserved factor of 2 in the backscattering intensity enhancement, which is predicted by the scalar theory of weak localization, will now be easier to test experimentally, even with a milliradian resolution. We use the diffusion approximation[4,6] to describe the photon motion in the disordered medium and study the finite-size effects by incorporating in the theory *two* boundary conditions. The backscattering peak is caused by interference of two photon trajectories. Each is the time-reversed trajectory of the other. The finite slab removes large loop-size trajectories and therefore rounds off the sharp peak.

In Fig. 1, we plot the backscattering intensity $I(\theta,z_0)$ as a function of the backscattering angle θ for different values of the width of the slab z_0 and the wavelength λ. Curve *a* corresponds to $z_0 \to \infty$, a case calculated by Akkermans *et al.*[4] Curves $b-d$ correspond to $z_0/\lambda = 400$, 80, and 16, respectively. It is clear from the figure that the effect of the finiteness of the slab is to eliminate the triangular shape. We find that the half-width of the rounded portion of the peak is given by

$$\delta\theta = A(\lambda/z_0) \ , \tag{1}$$

where $A = 3/2\pi$.

In Fig. 2, we plot the backscattering peak $I(\theta,L)$ as a function of L (the size length of the surface). Similar effects are found. The half-width of the rounded portion of the peak is given by

$$\delta\theta = B(\lambda/L) \ , \tag{2}$$

where $B = 2/\pi$.

The backscattered peak from a disordered medium is

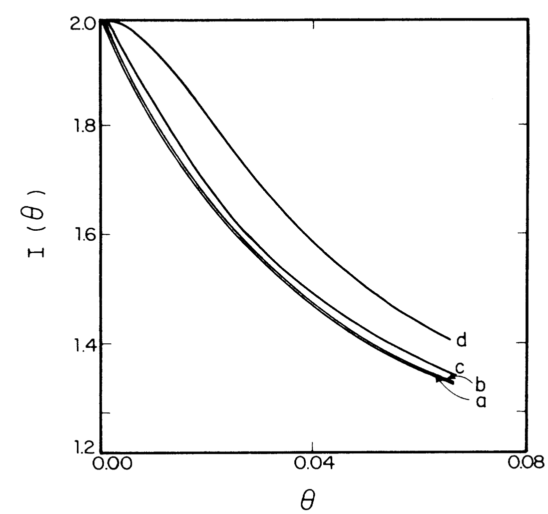

FIG. 1. The backscattering intensity $I(\theta,z_0)$ as a function of the backscattering angle θ for (a) $z_0 \to \infty$, (b) $z_0/\lambda = 400$, (c) $z_0/\lambda = 80$, and (d) $z_0/\lambda = 16$.

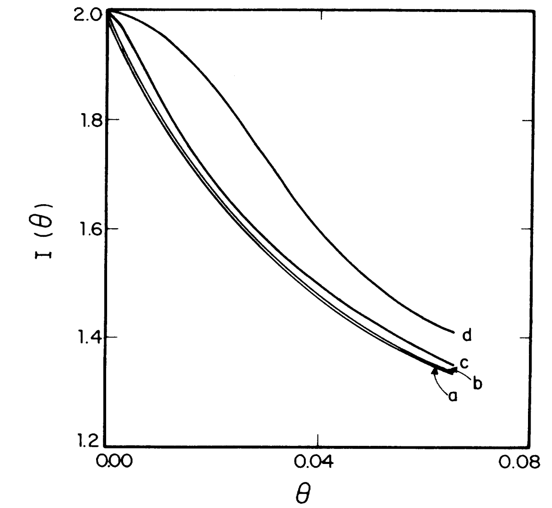

FIG. 2. The backscattering intensity $I(\theta,L)$ as a function of L for (a) $L/\lambda=400$, (b) $L/\lambda=80$, and (c) $L/\lambda=16$.

given by[4,6]

$$I(\theta)=I_0\sum_{l,m}P_{lm}[1+\cos(\mathbf{k}_i+\mathbf{k}_f)(\mathbf{r}_l-\mathbf{r}_m)] \ , \tag{3}$$

where $\mathbf{k}_i$ and $\mathbf{k}_f$ are, respectively, the incident and scattered photon wave vectors, $\mathbf{r}_l$ and $\mathbf{r}_m$ are the random positions of the scatterers, θ is defined by $q=(2\pi/\lambda)\sin\theta$ where $\mathbf{q}=\mathbf{k}_i+\mathbf{k}_f$, and P_{lm} is the probability that a photon which enters at $\mathbf{r}_l$ is emitted at $\mathbf{r}_m$. From (3) it is evident that $I(\theta)$ possesses a peak around $\theta\simeq 0$. This peak results from elastic multiple scattering. First order (single scattering) contributes insignificantly to the scattered background in a system with strong multiple scattering. The shape of the peak, however, depends on the assumptions made for the probability P_{lm}. We use the continuum approximation[4] which leads to a normalized intensity

$$I_N(\theta)=C\int[1+\cos(\mathbf{q}\cdot\boldsymbol{\rho})]P(\rho)d^2\rho \ , \tag{4}$$

where $C^{-1}=\int P(\rho)d^2\rho$ and we have used the fact that $P_{lm}=P(|\mathbf{r}_l-\mathbf{r}_m|)$. The probability $P(\rho)$ is given by

$$P(\rho)=\int_0^\infty dt\, n(\rho,t) \ , \tag{5}$$

where $n(\rho,t)$ is a solution of the diffusion equation

$$D\nabla^2 n(\mathbf{r},t)=\frac{\partial n(\mathbf{r},t)}{\partial t} \ . \tag{6}$$

For a finite slab we need the solution of (6) with the following boundary conditions:

$$\begin{aligned} &n(\mathbf{r},0)=\delta(z-a)\delta(x)\delta(y) \ , \\ &n(\rho,z=0,t)=0 \ , \\ &n(\rho,z=z_0,t)=0 \ , \end{aligned} \tag{7}$$

where we impose *two* absorbing planes at $z=0$ and $z=z_0$ and assume[4] that both the incident and emitted photons are at the same plane $z=a$. The parameter a is of the order of the elastic mean-free path l and is given[4] by $a=1.7l$.

The exact solution of (6) with the conditions given by (7) is

$$n(\rho,t)=(2\pi Dtz_0)^{-1}\exp(-\rho^2/4Dt) \times\sum_{n=1}^{\infty}\sin^2\left[\frac{n\pi a}{z_0}\right]\exp\left[-D\left[\frac{n\pi}{z_0}\right]^2 t\right] \ . \tag{8}$$

Inserting (8) into (5) leads to

$$P(\rho)=(\pi D z_0)^{-1}\sum_{n=1}^{\infty}\sin^2\left[\frac{n\pi a}{z_0}\right]K_0\left[\frac{n\pi\rho}{z_0}\right] \ , \tag{9}$$

where K_0 is the Bessel function. Inserting (9) into (4) yields the final result for $I_N(\theta,z_0)$,

$$I_N(\theta,z_0)=2C(z_0/D) \times\sum_{n=1}^{\infty}\sin^2\left[\frac{n\pi a}{z_0}\right]\left[\frac{1}{n^2\pi^2}+\frac{1}{q^2z_0^2+n^2\pi^2}\right] \ . \tag{10}$$

This leads to

$$I_N(\theta,z_0)=1+[2qa(1-a/z_0)]^{-1} \times\{\coth(qz_0)[1-\cosh(2qa)]+\sinh(2qa)\} \ . \tag{11}$$

We may distinguish between two limiting cases, the first being $qz_0\gg 1$ $(\theta\gg\lambda/2\pi z_0)$ and the second being $qz_0\ll 1(\theta\ll\lambda/2\pi z_0)$. The first case yields

$$I_N(\theta,z_0\gg q^{-1})=\{1+(2qa)^{-1}[1-\exp(-2qa)]\} \ , \tag{12}$$

which is an *exact* result for $z_0\to\infty$ and coincides with the solution [Eq. (4)] of Akkermans *et al.*[4] for an infinite medium. The other limit is given by

$$I_N(\theta,z_0)=\left[2-\frac{z_0-2a}{3(1-a/z_0)}q^2a+\cdots\right] \quad q<z_0^{-1} \ . \tag{13}$$

Comparing (13) with (12) leads to the width of the peak as given by (1).

We now turn to finite-size effects in the (x,y) plane by taking the upper limit in (4) to be $\rho=L$. The normalized intensity $I_N(\theta,L)$ is given by

$$I_N(\theta,L)=\left[2-\frac{3L-4a}{24(1-a/2L)}q^2a+\cdots\right] \ . \tag{14}$$

Comparing (14) with (12) leads to the width of the peak that is given by (2). Figures 1 and 2 demonstrate that finite-size effects eliminate the triangular shape obtained[4] for an infinite medium. It should be noted that within the scalar theory presented here the maximum increase of $I_N(\theta,z_0)$ and $I_N(\theta,L)$ is always 2, *independent* of z_0 or L. This result disagrees with a diagrammatic approach by Tsang and Ishimura,[8] who were the first to deal with effects of finite size on multiple scattering. It was recently shown[9-11] that such diagrammatic analyses must lead to an artificial singularity in $I_N(\theta, z_0 \to \infty)$ as $\theta \to 0$. When the right geometry[11] ("half-space" geometry rather than "infinite-space" geometry) is taken into account the singularity is removed and this fact leads to results in agreement with the diffusion approach.[4] Experiments[2] support the fact that the maximum increase of $I_N(\theta)$ is at most a factor of 2. A slight reduction from the maximum enhancement of factor 2 may be due to polarization effects.[11]

We hope that our results will encourage experimental efforts in this field to analyze the coherent backscattering peak for finite slabs or for absorbing media. In the last case of absorbing media, inelastic processes take place. In analogy with electrons[1] one can define an *inelastic* diffusion length $L_i = (D\tau_i)^{1/2}$ in which a photon travels between inelastic scattering events which occur during a time τ_i.

Thus, the maximum photon trajectory in which the time reversal is properly preserved is of order L_i. Thus, L_i acts as a cut-off length and replaces z_0 or L in Eq. (13) or (14) when $L_i < z_0$ or $L_i < L$, correspondingly. Under these conditions one can deduce L_i from Eq. (1) or (2) and extract the photon inelastic scattering time τ_i.

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